Aalto University
School of Science

Department of Computer Science

Combinatorics of Efficient Computations

Approximation Algorithms

Lecture 7: Min. Degree Spanning Trees via Local Search

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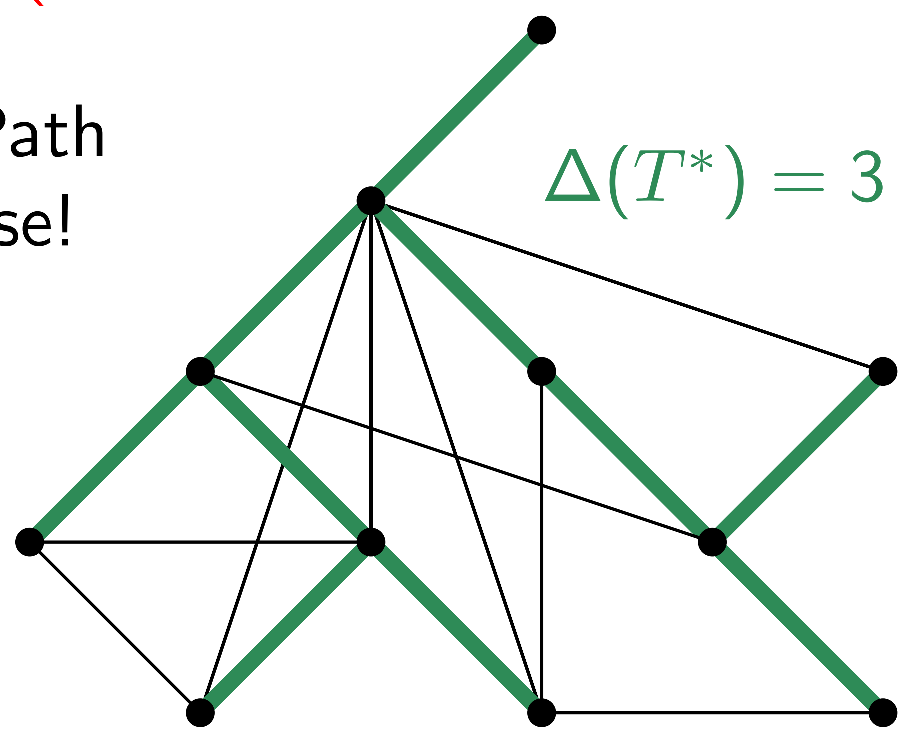
2019

MinimumDegreeSpanningTree

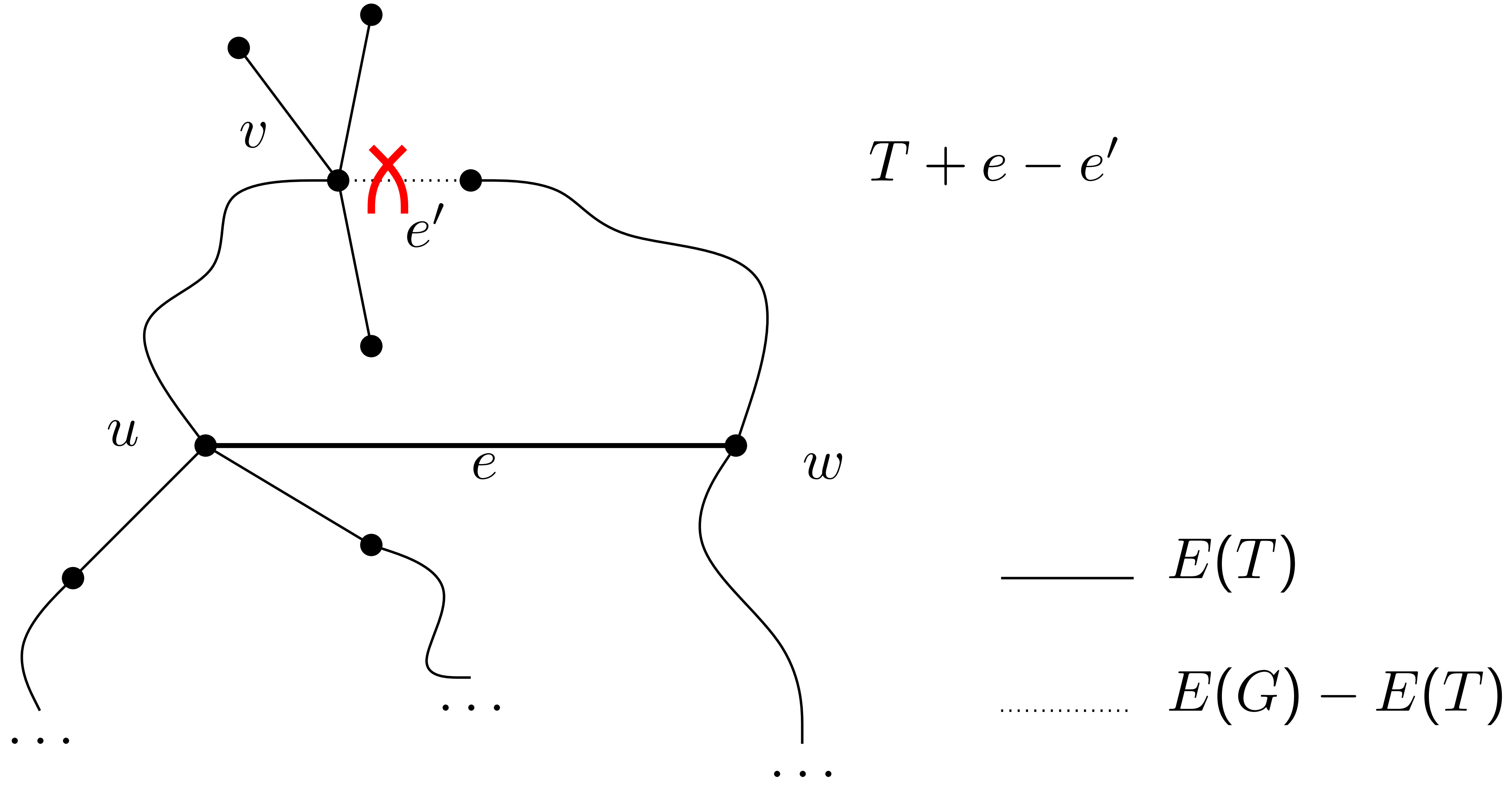
Given: A connected graph Graph $G = (V, E)$.
Find: A spanning tree T which has the minimum maximum degree $\Delta(T)$ among all spanning trees of G.

NP-hard :-(
Why?
Hamiltonian Path
is a special case!



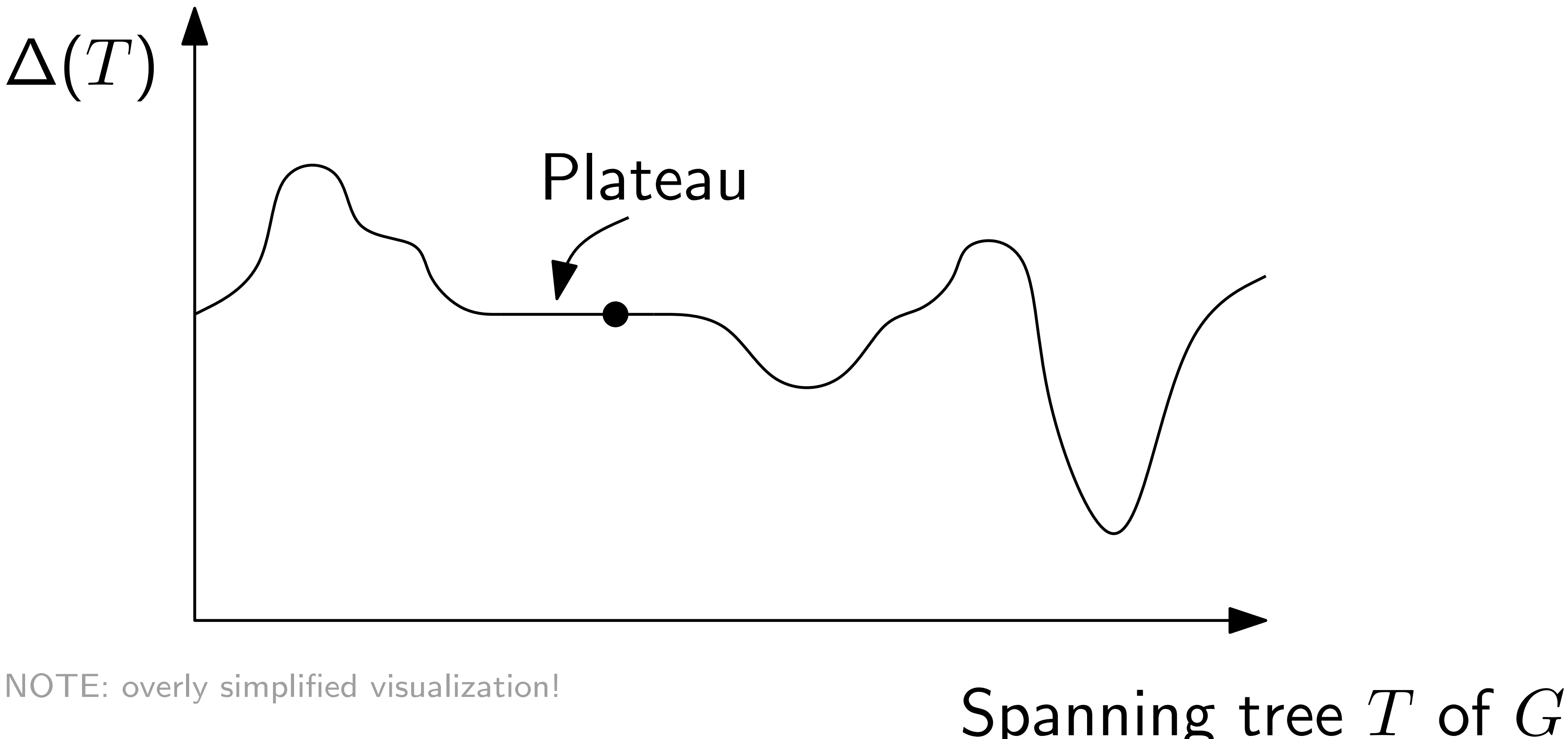
Local Adjustment via Edge Flips

Improvement when $\mathsf{deg}_T(v) - 1 > \mathsf{max}\{\mathsf{deg}_T(u), \mathsf{deg}_T(w)\}$



Local Search

- Start from any spanning tree T of G
- Perform edge flips until no flip improves the solution.

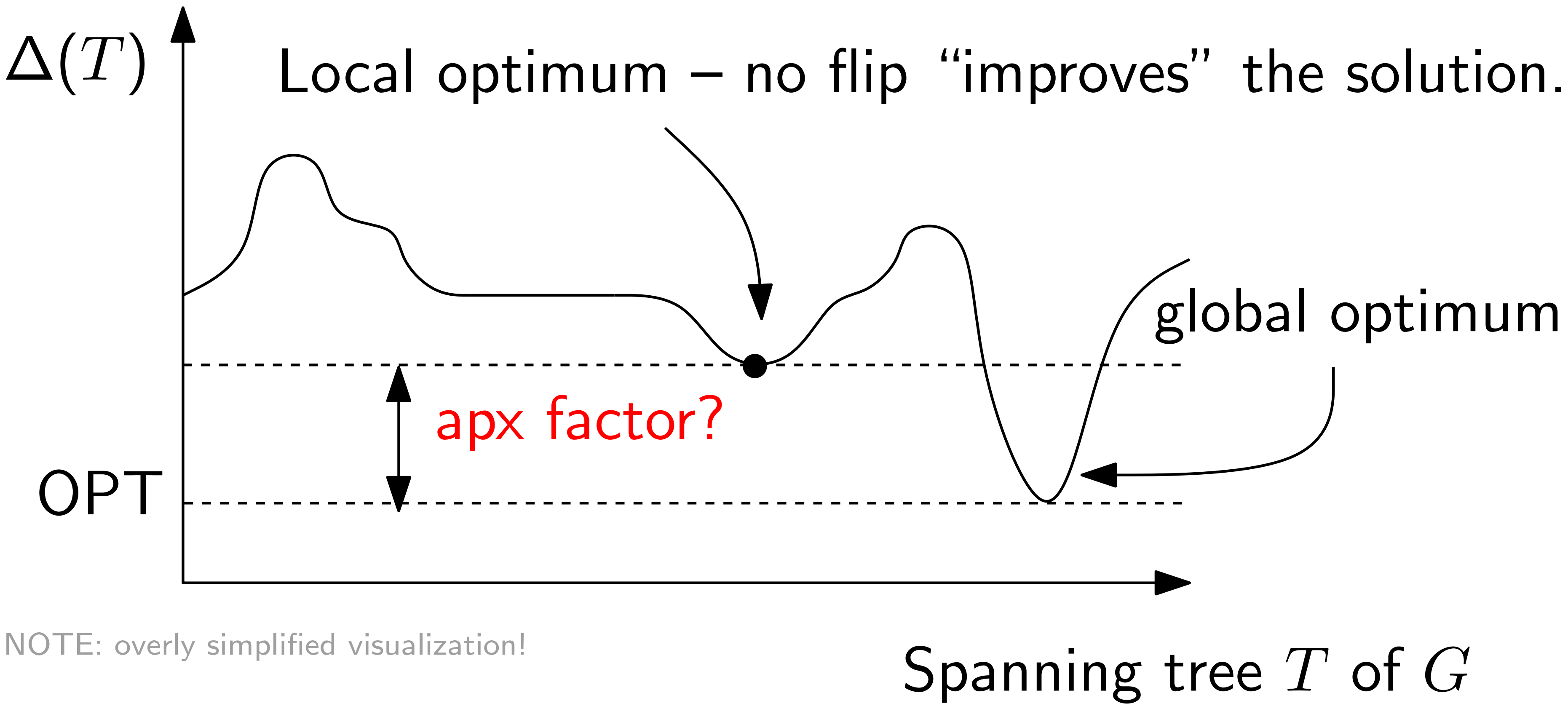


NOTE: overly simplified visualization!

Flips don't always improve $\Delta(T)$!!

Local Search

- Start from any spanning tree T of G
- Perform edge flips until no flip improves the solution.



How to handle plateaus? What is the runtime?

Local Search

Algorithm MinDegSTLocalSearch(T)
while there is an "improving flip" (*) in T for a vertex v with $d_T(v) \geq \Delta(T) - \ell$ **do**
  perform the flip.

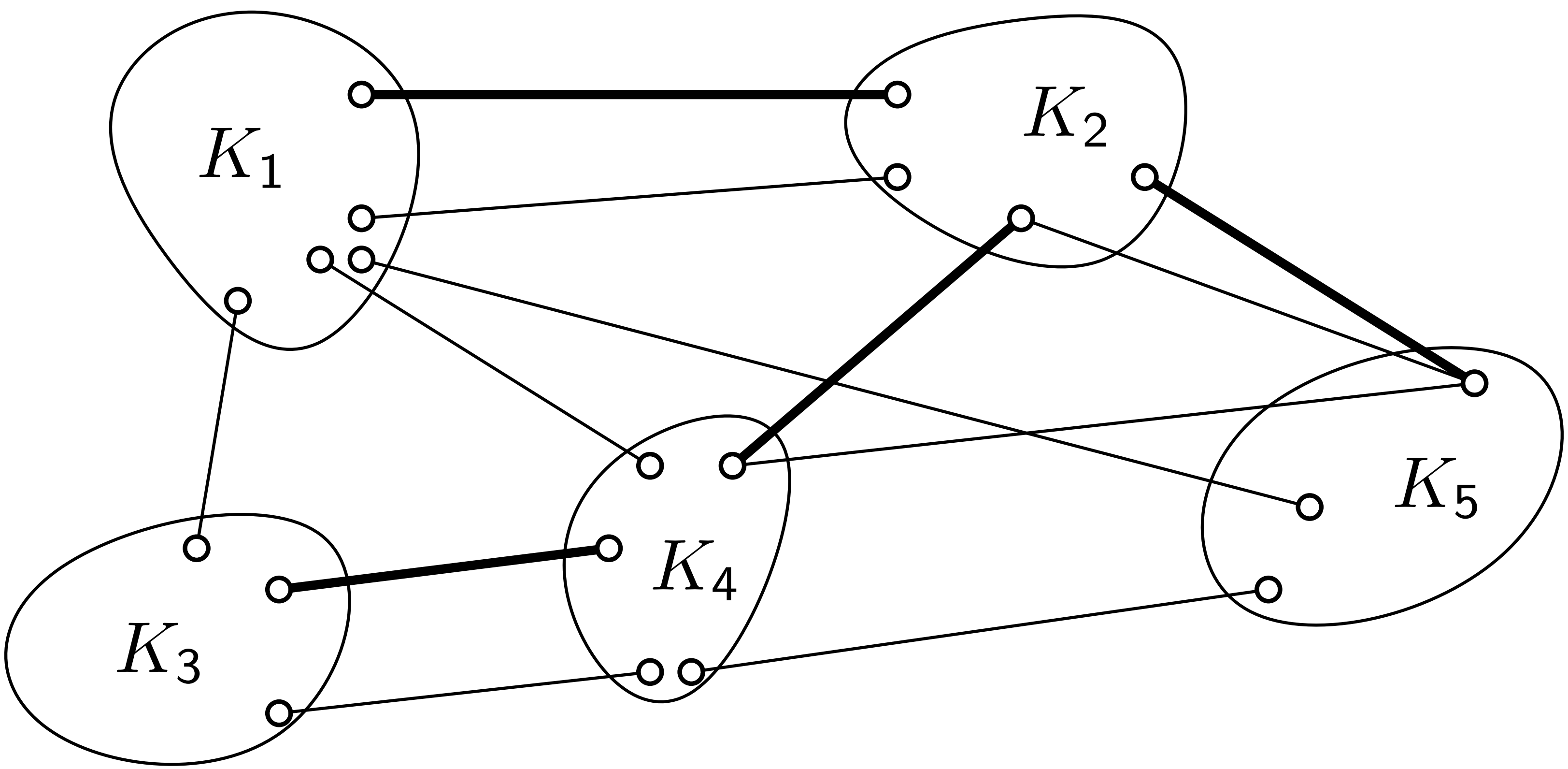
(*) $uw \in E(G) \setminus E(T)$ with $d_T(v) - 1 > \max\{d_T(u), d_T(w)\}$ such that $T \cup \{uw\}$ forms a cycle containing v.

- unclear whether it completes in polynomial time ...
- idea: flip only when the degree of v with $\deg(v) \geq \Delta(T) - \ell$ is reduced where $\ell := \lceil \log_2 n \rceil$
- first the approximation factor, then the runtime

Approximation Factor

Thm. If T is a locally optimal spanning tree, then $\Delta(T) \leq 2 \cdot \mathrm{OPT} + \ell$, where $\ell = \lceil \log_2 n \rceil$.

Let E' be the edges of G between distinct components ($K_i \neq K_j$).

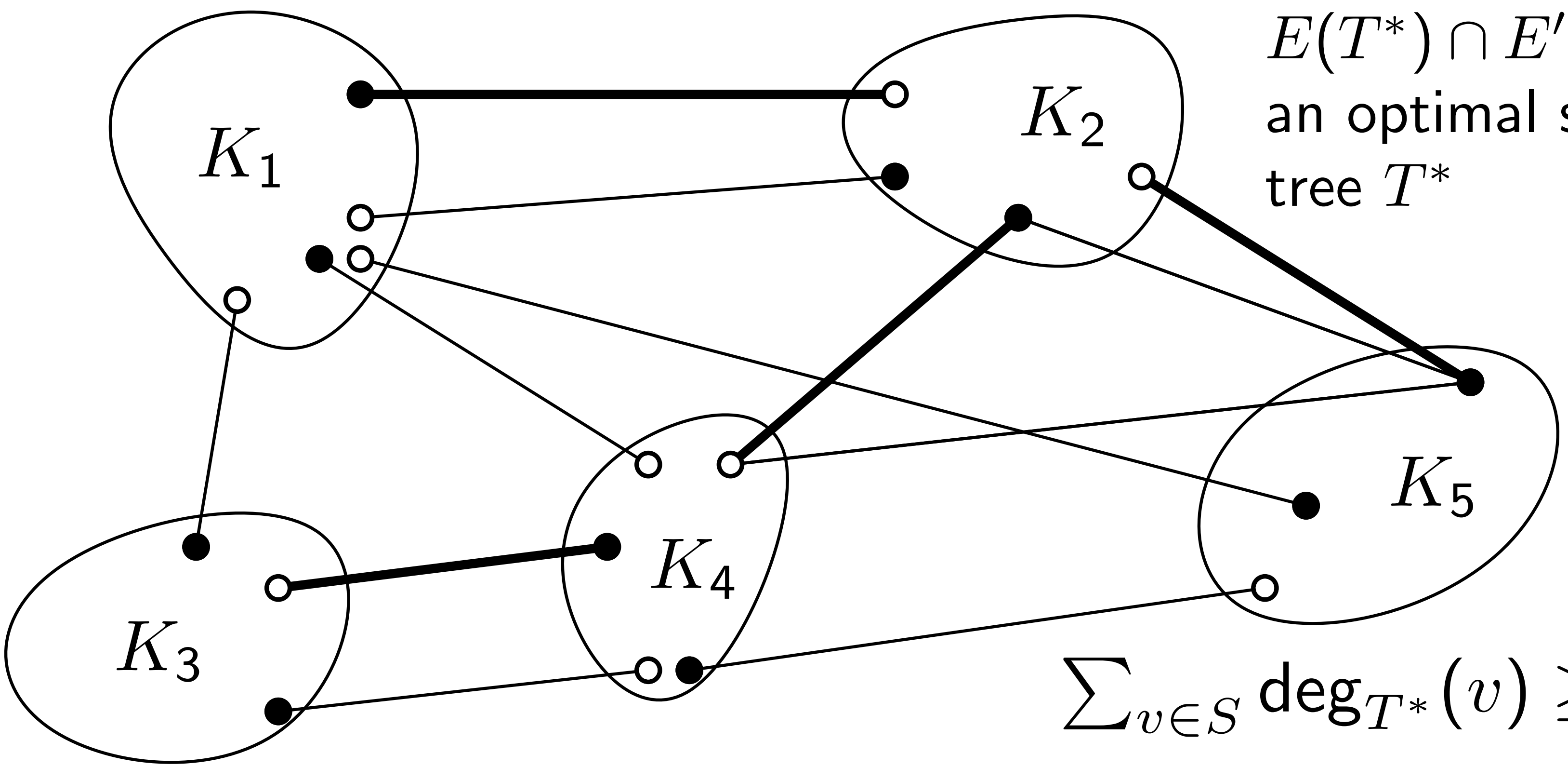


Approximation Factor

Thm. If T is a locally optimal spanning tree, then $\Delta(T) \leq 2 \cdot \mathrm{OPT} + \ell$, where $\ell = \lceil \log_2 n \rceil$.

Proof. **Part 1:** Lower bound on OPT $\boxed{\mathrm{OPT} \geq k/|S|}$

Let E' be the edges of G between distinct components ($K_i \neq K_j$).



$E(T^*) \cap E' \geq k$ for an optimal spanning tree T^*

$\sum_{v \in S} \deg_{T^*}(v) \geq k$

Vertex Cover S of E'

Approximation Factor

Thm. If T is a locally optimal spanning tree, then $\Delta(T) \leq 2 \cdot \mathrm{OPT} + \ell$, where $\ell = \lceil \log_2 n \rceil$.

Proof. **Part 1:** OPT $\geq k/|S|$
Part 2: Applying the bound.

Let S_i be the nodes in T with $d_T(v) \geq i$.
Let E_i be the edges of T incident to S_i.
Claim 1: For $i \geq \Delta(T) - \ell + 1$,
(i) $|E_i| \geq (i-1)|S_i| + 1$,
(ii) Each $e \in E(G) \setminus E_i$ connecting distinct components of $T \setminus E_i$ is incident to a node of S_{i-1}.

Claim 2: There is an i such that $|S_{i-1}| \leq 2|S_i|$.

By Part 1, and Claims 1 & 2 ... how do we choose k and S?

$$\mathrm{OPT} \geq \frac{(i-1)|S_i|+1}{|S_{i-1}|} \geq \frac{(i-1)|S_i|+1}{2|S_i|} > \frac{i-1}{2} \geq \frac{\Delta(T)-\ell}{2} \quad \square$$

Runtime

Thm. The algorithm finds a local optimal in polynomial time.

Proof. Our potential function: $\Phi(T) = \sum_{v \in V(G)} 3^{d_T(v)}$

Via potenial function $\Phi(G, T)$. $\rightsquigarrow$ a function measuring the value of a solution where, e.g., :

- each iteration decreases the potential of a solution.

Lemma: each iteration $\Phi(T') \leq (1 - \frac{2}{27n^3})\Phi(T)$.

- the function is bounded both from above and below.

For any spanning tree T, $\Phi(T) \in [3n, n3^n]$.

- executing $f(n)$ iterations would execeed this lower bound.

Let $f(n) = \frac{27}{2}n^4 \cdot \ln 3$. How does $\Phi(T)$ change?

shrinks by: $(1 - \frac{2}{27n^3})^{f(n)} \leq (e^{-\frac{2}{27n^3}})^{f(n)} = 3^{-n}$ (i.e., $e^{-n \ln 3}$)

Goal $\rightsquigarrow$ after $f(n)$ iterations $\Phi(T) = n < 3n$ $\square$

Extensions

Cor. For a constant $b > 1$, and $\ell = \lceil \log_b n \rceil$, the local search algorithm runs in polynomial time and produces a spanning tree T where $\Delta(T) \leq b \cdot \mathrm{OPT} + \lceil \log_b n \rceil$.

Proof. Similar to before. $\square$

Next Class:

Approximation Schemes:

$(1+\epsilon)$**-approximation**