Chapter 21 (21.6-21.7 excluded)

NONLINEAR OPTICS II

Coupled-wave theory of three-wave mixing

Second-order nonlinear medium: $\nabla^2\mathcal{E} - \frac{1}{c^2}\frac{\partial^2\mathcal{E}}{\partial t^2} = -\mathcal{S}$

$$\mathcal{S} = -\mu_o \frac{\partial^2 \mathcal{P}_{\mathrm{NL}}}{\partial t^2}, \quad \mathcal{P}_{\mathrm{NL}}(t) = 2\mathrm{d}\mathcal{E}^2$$

$$\mathcal{E}(t) = \sum_{q=\pm1,\pm2,\pm3} \tfrac{1}{2}E_q \exp(j\omega_q t) \Rightarrow \mathcal{P}_{\mathrm{NL}}(t) = 2\mathrm{d}\cdot\tfrac{1}{4}\sum_{q,r=\pm1,\pm2,\pm3} E_q E_r \exp\left[j(\omega_q+\omega_r)t\right]$$

$$\Rightarrow \mathcal{S} = \tfrac{1}{2}\mu_o \mathrm{d} \sum_{q,r=\pm1,\pm2,\pm3} (\omega_q+\omega_r)^2 E_q E_r \exp\left[j(\omega_q+\omega_r)t\right]$$

For distinct frequencies satisfying $\omega_1+\omega_2=\omega_3$, we separate the wave equations:

$$\begin{aligned}
(\nabla^2+k_1^2)E_1 &= -S_1 = -2\mu_o\omega_1^2\mathrm{d}\,E_3E_2^* \\
(\nabla^2+k_2^2)E_2 &= -S_2 = -2\mu_o\omega_2^2\mathrm{d}\,E_3E_1^* \\
(\nabla^2+k_3^2)E_3 &= -S_3 = -2\mu_o\omega_3^2\mathrm{d}\,E_1E_2.
\end{aligned}$$

For *collinear waves* within the *slowly varying envelope* approximation, we obtain

$$\frac{d\mathfrak{a}_1}{dz} = -j\mathfrak{g}\mathfrak{a}_3\mathfrak{a}_2^*\exp(-j\Delta k\,z)$$

$$\frac{d\mathfrak{a}_2}{dz} = -j\mathfrak{g}\mathfrak{a}_3\mathfrak{a}_1^*\exp(-j\Delta k\,z)$$

$$\frac{d\mathfrak{a}_3}{dz} = -j\mathfrak{g}\mathfrak{a}_1\mathfrak{a}_2\exp(j\Delta k\,z)$$

$$E_q = \sqrt{2\eta\hbar\omega_q}\,\mathfrak{a}_q\exp(-jk_qz)$$

$$\phi_q = \frac{I_q}{\hbar\omega_q} = |\mathfrak{a}_q|^2$$

$$\mathfrak{g}^2 = 2\hbar\omega_1\omega_2\omega_3\eta^3\mathrm{d}^2$$

$$\Delta k = k_3 - k_2 - k_1 \quad \leftarrow \text{mismatch}$$

Second-harmonic generation

A degenerate case: $\omega_1 = \omega_2 = \omega, \quad \omega_3 = 2\omega$ and $\mathbf{k}_3 = 2\mathbf{k}_1$.
For collinear waves with perfect phase matching, $\Delta k = 0$, we have

$$\begin{cases} \dfrac{d\mathfrak{a}_1}{dz} = -j\mathfrak{g}\mathfrak{a}_3\mathfrak{a}_2^* \exp(-j\Delta k\, z) \\ \dfrac{d\mathfrak{a}_2}{dz} = -j\mathfrak{g}\mathfrak{a}_3\mathfrak{a}_1^* \exp(-j\Delta k\, z) \\ \dfrac{d\mathfrak{a}_3}{dz} = -j\mathfrak{g}\mathfrak{a}_1\mathfrak{a}_2 \exp(j\Delta k\, z) \end{cases} \Rightarrow \begin{cases} \dfrac{d\mathfrak{a}_1}{dz} = -j\mathfrak{g}\mathfrak{a}_3\mathfrak{a}_1^* \\ \dfrac{d\mathfrak{a}_3}{dz} = -j\dfrac{\mathfrak{g}}{2}\mathfrak{a}_1\mathfrak{a}_1 \\ \hline \mathfrak{a}_1^2(z) + 2|\mathfrak{a}_3(z)|^2 = \mathfrak{a}_1^2(0) \end{cases}$$

The solutions:

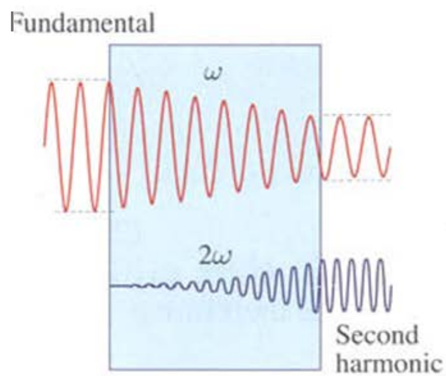
$$\begin{cases} \mathfrak{a}_1(z) = \mathfrak{a}_1(0)\,\mathrm{sech}\left(\frac{1}{\sqrt{2}}\mathfrak{g}\mathfrak{a}_1(0)z\right) \\ \mathfrak{a}_3(z) = -\dfrac{j}{\sqrt{2}}\,\mathfrak{a}_1(0)\tanh\left(\frac{1}{\sqrt{2}}\mathfrak{g}\mathfrak{a}_1(0)z\right) \end{cases} \Rightarrow \begin{cases} \phi_1(z) = \phi_1(0)\,\mathrm{sech}^2\dfrac{\gamma z}{2} \\ \phi_3(z) = \frac{1}{2}\phi_1(0)\tanh^2\dfrac{\gamma z}{2} \end{cases}$$

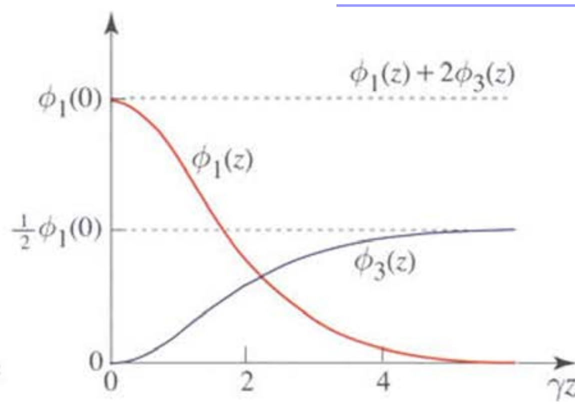
$$\gamma/2 = \mathfrak{g}\mathfrak{a}_1(0)/\sqrt{2}$$

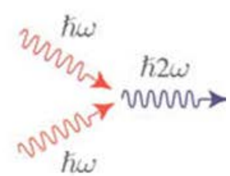
SHG efficiency:

$$\eta_{\mathrm{SHG}} \equiv \frac{I_3(L)}{I_1(0)}$$

$$\eta_{\mathrm{SHG}} = \tanh^2\frac{\gamma L}{2}$$

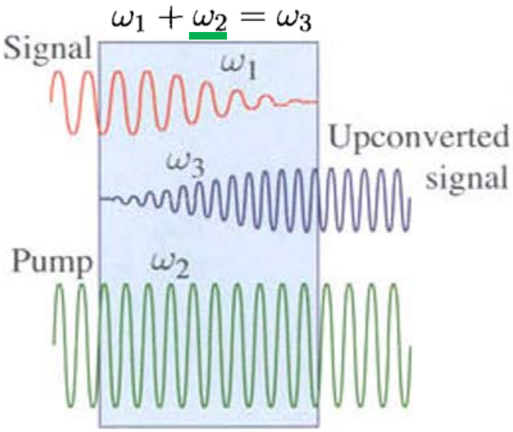


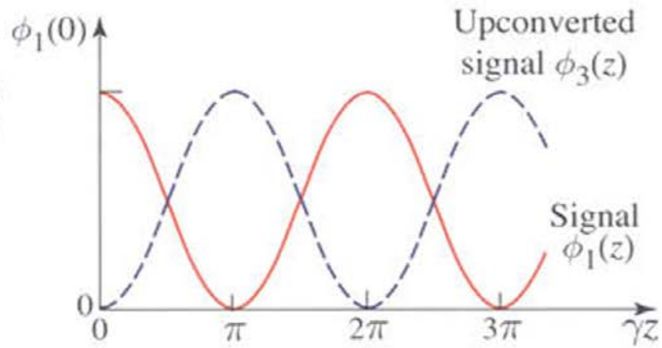


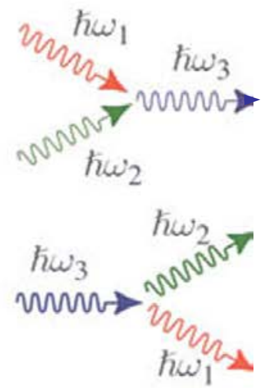


If $\Delta k \neq 0$, η_{SHG} is reduced by $\mathrm{sinc}^2(\Delta k\, L/2\pi)$.

Frequency conversion







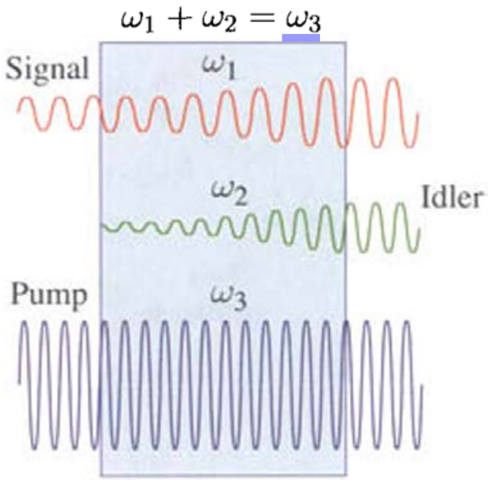
$$\frac{d\mathfrak{a}_1}{dz} = -j g \mathfrak{a}_3 \mathfrak{a}_2^* \exp(-j\Delta k\, z)$$
$$\frac{d\mathfrak{a}_2}{dz} = -j g \mathfrak{a}_3 \mathfrak{a}_1^* \exp(-j\Delta k\, z) \Rightarrow \begin{cases} \dfrac{d\mathfrak{a}_1}{dz} = -j\dfrac{\gamma}{2}\mathfrak{a}_3 \\ \dfrac{d\mathfrak{a}_3}{dz} = -j\dfrac{\gamma}{2}\mathfrak{a}_1 \\ \text{when } \mathfrak{a}_2(z) \approx \mathfrak{a}_2(0) \end{cases} \Rightarrow \begin{cases} \mathfrak{a}_1(z) = \mathfrak{a}_1(0)\cos\dfrac{\gamma z}{2} \\ \mathfrak{a}_3(z) = -j\mathfrak{a}_1(0)\sin\dfrac{\gamma z}{2} \end{cases}$$
$$\frac{d\mathfrak{a}_3}{dz} = -j g \mathfrak{a}_1 \mathfrak{a}_2 \exp(j\Delta k\, z)$$

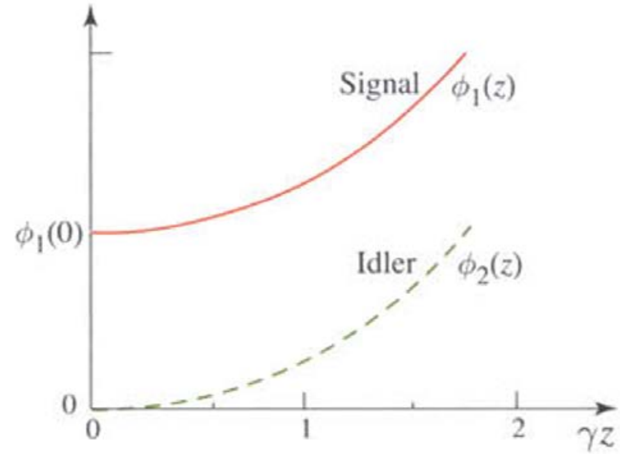
$$\Rightarrow \begin{cases} \phi_1(z) = \phi_1(0)\cos^2\dfrac{\gamma z}{2} \\ \phi_3(z) = \phi_1(0)\sin^2\dfrac{\gamma z}{2}. \end{cases}$$

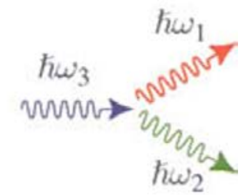
The efficiency of up-conversion:

$$\eta_{\mathrm{OFC}} = \frac{I_3(L)}{I_1(0)} = \frac{\omega_3}{\omega_1}\sin^2\frac{\gamma L}{2} \xrightarrow[\gamma L \ll 1]{} \eta_{\mathrm{OFC}} = C^2 \frac{L^2}{A} P_2, \quad C^2 = 2\omega_3^2 \eta_o^3 \frac{\mathrm{d}^2}{n^3}$$

Optical parametric amplification (OPA)







$$
\begin{aligned}
\frac{d\mathfrak{a}_1}{dz} &= -jg\mathfrak{a}_3\mathfrak{a}_2^* \exp(-j\Delta k\, z) \\
\frac{d\mathfrak{a}_2}{dz} &= -jg\mathfrak{a}_3\mathfrak{a}_1^* \exp(-j\Delta k\, z) \\
\frac{d\mathfrak{a}_3}{dz} &= -jg\mathfrak{a}_1\mathfrak{a}_2 \exp(j\Delta k\, z)
\end{aligned}
\Rightarrow
\begin{cases}
\dfrac{d\mathfrak{a}_1}{dz} = -j\dfrac{\gamma}{2}\mathfrak{a}_2^* \\
\dfrac{d\mathfrak{a}_2}{dz} = -j\dfrac{\gamma}{2}\mathfrak{a}_1^* \\
\text{when } \mathfrak{a}_3(z) \approx \mathfrak{a}_3(0)
\end{cases}
\Rightarrow
\begin{cases}
\mathfrak{a}_1(z) = \mathfrak{a}_1(0)\cosh\dfrac{\gamma z}{2} \\
\mathfrak{a}_2(z) = -j\mathfrak{a}_1^*(0)\sinh\dfrac{\gamma z}{2}
\end{cases}
$$

$$
\Rightarrow
\begin{cases}
\phi_1(z) = \phi_1(0)\cosh^2\dfrac{\gamma z}{2} \\
\phi_2(z) = \phi_1(0)\sinh^2\dfrac{\gamma z}{2}.
\end{cases}
$$

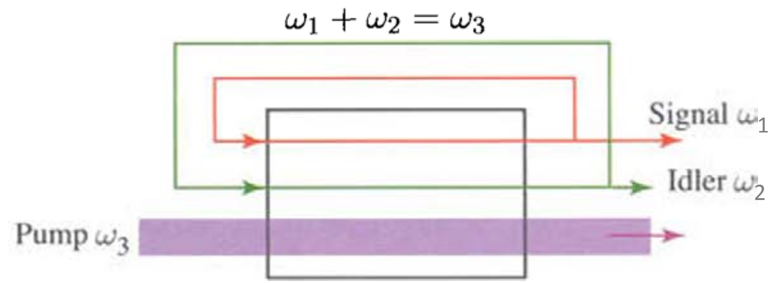
The gain coefficient:

$G = \phi_1(L)/\phi_1(0) = \cosh^2(\gamma L/2) \approx e^{\gamma L}/4$, if $\gamma L \gg 1$.

Here, $\gamma = 2C\sqrt{I_3(0)} = 2C\sqrt{P_3/A}$ and $C^2 = 2\omega_1\omega_2\,\eta_o^3\dfrac{d^2}{n^3}$.

Optical parametric oscillator (OPO)

Singly and doubly resonant oscillators:



Threshold of SRO: One round-trip loss is equal to gain $\Rightarrow$ $a_1(L)\, \mathrm{r}_1^{\,2} = a_1(0)$ **identical mirrors**

$$\Rightarrow \mathrm{r}_1^2 \cosh(\gamma L/2) = 1 \Rightarrow \mathcal{R}_1^2 \cosh^2(\gamma L/2) = 1, \text{ where } \mathcal{R}_1 = \mathrm{r}_1^2.$$

At small x, $\cosh^2(x) \approx 1 + x^2 \Rightarrow (\gamma L/2)^2 \approx (1 - \mathcal{R}_1^2)/\mathcal{R}_1^2$. As $\gamma = 2C\sqrt{P_3/A}$,

$$P_3|_{\text{threshold}}(0) \approx \frac{1}{C^2}\,\frac{A}{L^2}\,\frac{1 - \mathcal{R}_1^2}{\mathcal{R}_1^2},$$

$$C^2 = 2\omega_1\omega_2\,\eta_o^3 \mathrm{d}^2/n^3.$$

Threshold of DRO: $a_1(L)\, \mathrm{r}_1^{\,2} = a_1(0)$ and $a_2(L)\, \mathrm{r}_2^{\,2} = a_2(0)$. These conditions lead to

$$\tanh^2(\gamma L/2) = (1 - \mathcal{R}_1)(1 - \mathcal{R}_2)/(\mathcal{R}_1\mathcal{R}_2) \approx (\gamma L/2)^2$$

$$\Rightarrow P_3|_{\text{threshold}}(0) \approx \frac{1}{C^2}\,\frac{A}{L^2}\,\frac{(1 - \mathcal{R}_1)(1 - \mathcal{R}_2)}{\mathcal{R}_1\mathcal{R}_2}$$ << than for SRO

Coupled-wave theory of four-wave mixing

In third-order nonlinear medium, $\mathcal{P}_{\rm NL} = 4\chi^{(3)}\mathcal{E}^3$ and $\mathcal{E}(t) = \sum_{q=\pm1,\pm2,\pm3,\pm4} \frac{1}{2}E_q \exp(j\omega_q t)$.

$$\Rightarrow \mathcal{S} = -\mu_o \partial^2 \mathcal{P}_{\rm NL}/\partial t^2 = \frac{1}{2}\mu_o\chi^{(3)} \sum_{q,p,r=\pm1,\pm2,\pm3,\pm4} (\omega_q+\omega_p+\omega_r)^2 E_q E_p E_r \exp[j(\omega_q+\omega_p+\omega_r)t].$$

The three coupled Helmholtz equations are

$$(\nabla^2 + k_q^2)E_q = -S_q, \qquad q = 1,2,3,4$$

Example of $\omega_1 + \omega_2 = \omega_3 + \omega_4$:

$$\begin{cases} S_1 = \mu_o\omega_1^2\chi^{(3)}\{6E_3E_4E_2^* + 3E_1[|E_1|^2 + 2|E_2|^2 + 2|E_3|^2 + 2|E_4|^2]\} \\ S_2 = \mu_o\omega_2^2\chi^{(3)}\{6E_3E_4E_1^* + 3E_2[|E_2|^2 + 2|E_1|^2 + 2|E_3|^2 + 2|E_4|^2]\} \\ S_3 = \mu_o\omega_3^2\chi^{(3)}\{6E_1E_2E_4^* + 3E_3[|E_3|^2 + 2|E_2|^2 + 2|E_1|^2 + 2|E_4|^2]\} \\ S_4 = \mu_o\omega_4^2\chi^{(3)}\{6E_1E_2E_3^* + 3E_4[|E_4|^2 + 2|E_1|^2 + 2|E_2|^2 + 2|E_3|^2]\} \end{cases}$$

$$\Rightarrow \boxed{\begin{aligned} S_q &= \overline{S}_q + (\omega_q/c_o)^2\Delta\chi_q E_q \\ \overline{S}_q &= 6\mu_o\omega^2\chi^{(3)}E_l\,E_n E_m^* \\ \Delta\chi_q &= 6\frac{\eta}{\epsilon_o}\chi^{(3)}(\underset{\substack{\uparrow \\ \text{total}}}{2I} - I_q) \end{aligned}} \Rightarrow \boxed{\begin{aligned} &(\nabla^2 + \overline{k}_q^2)E_q = -\overline{S}_q \\ &\overline{k}_q = \bar{n}_q\frac{\omega_q}{c_o}, \quad n_2 = \frac{3\eta_o}{\epsilon_o n^2}\chi^{(3)} \\ &\bar{n}_q \approx n + n_2(2I - I_q) \end{aligned}}$$

Optical Kerr effect

Three-wave mixing

If $\omega_3 = \omega_4 = \omega_0$, then $\omega_1 + \omega_2 = 2\omega_0$ and

$$\begin{cases} S_1 = \mu_o \omega_1^2 \chi^{(3)} \left\{ 3E_0^2 E_2^* + 3E_1 \left[|E_1|^2 + 2|E_2|^2 + 2|E_0|^2 \right] \right\} \\ S_2 = \mu_o \omega_2^2 \chi^{(3)} \left\{ 3E_0^2 E_1^* + 3E_2 \left[|E_2|^2 + 2|E_1|^2 + 2|E_0|^2 \right] \right\} \\ S_0 = \mu_o \omega_0^2 \chi^{(3)} \left\{ 6E_1 E_2 E_0^* + 3E_0 \left[|E_0|^2 + 2|E_1|^2 + 2|E_2|^2 \right] \right\} \end{cases}$$

$$(\nabla^2 + k_q^2)E_q = -S_q \quad \Rightarrow \quad (\nabla^2 + k_q^2)[A_q \exp(-jk_q z)] \approx -j2k_q (dA_q/dz) \exp(-jk_q z)$$

Introducing $\mathfrak{g} = \hbar\omega_0(\omega_0/c_o)n_2$, we obtain

$$\begin{cases} \dfrac{d\mathfrak{a}_1}{dz} = -j\mathfrak{g} \left[\mathfrak{a}_0^2 \mathfrak{a}_2^* \exp(-j\Delta k\, z) + \mathfrak{a}_1 \left(|\mathfrak{a}_1|^2 + 2|\mathfrak{a}_2|^2 + 2|\mathfrak{a}_0|^2 \right) \right] \\ \dfrac{d\mathfrak{a}_2}{dz} = -j\mathfrak{g} \left[\mathfrak{a}_0^2 \mathfrak{a}_1^* \exp(-j\Delta k\, z) + \mathfrak{a}_2 \left(|\mathfrak{a}_2|^2 + 2|\mathfrak{a}_1|^2 + 2|\mathfrak{a}_0|^2 \right) \right] \\ \dfrac{d\mathfrak{a}_0}{dz} = -j\mathfrak{g} \left[2\mathfrak{a}_1 \mathfrak{a}_2 \mathfrak{a}_0^* \exp(j\Delta k\, z) + \mathfrak{a}_0 \left(|\mathfrak{a}_0|^2 + 2|\mathfrak{a}_1|^2 + 2|\mathfrak{a}_2|^2 \right) \right] \end{cases}$$

$$\phi_q = \frac{I_q}{\hbar\omega_q} = |\mathfrak{a}_q|^2$$

Undepleted pump approximation, $a_0(z) \approx const$, and perfect phase matching, $\Delta k = 0$:

$$\begin{cases} \dfrac{d\mathfrak{a}_1}{dz} = -j\gamma(\mathfrak{a}_2^* + 2\mathfrak{a}_1) \\ \dfrac{d\mathfrak{a}_2}{dz} = -j\gamma(\mathfrak{a}_1^* + 2\mathfrak{a}_2) \end{cases} \Rightarrow \begin{cases} \mathfrak{a}_1(z) = [(1 - j\gamma z)\mathfrak{a}_1(0) - j\gamma z \mathfrak{a}_2^*(0)] \exp(-j\gamma z) \\ \mathfrak{a}_2(z) = [-j\gamma z \mathfrak{a}_1^*(0) + (1 - j\gamma z)\mathfrak{a}_2(0)] \exp(-j\gamma z). \end{cases}$$

If $a_2(0) = 0$, we have $|a_1(z)|^2 = \phi_1(z) = (1 + \gamma^2 z^2)\phi_1(0)$.

$\gamma = \mathfrak{g}\mathfrak{a}_0^2$

Otherwise, the amplification depends on $\Delta\varphi(0) = \varphi_1 - \varphi_2$.

Third-harmonic generation

If $\omega_1 = \omega_2 = \omega_4 = \omega$ and $\omega_3 = \omega_1 + \omega_2 + \omega_4 = 3\omega$, we obtain two Helmholtz equations:

$$(\nabla^2 + k_q^2)E_q = -S_q,$$

where

$$S_1 = \mu_o \omega_1^2 \chi^{(3)} \left\{3E_3 E_1^* E_1^* + 3E_1 \left[|E_1|^2 + 2|E_3|^2\right]\right\}$$

$$S_3 = \mu_o \omega_3^2 \chi^{(3)} \left\{E_1^3 + 3E_3 \left[|E_3|^2 + 2|E_1|^2\right]\right\}.$$

The *undepleted-pump* and *the slowly varying envelope approximations* yield

$$\frac{d\mathfrak{a}_3}{dz} = -j\mathfrak{g}\mathfrak{a}_1^3 \exp(-j\Delta k z),$$

where $A_q = \sqrt{2\eta\hbar\omega_q}\,\mathfrak{a}_q$, $g = \hbar\omega_1^{3/2}\omega_3^{1/2}\eta^3\chi^{(3)}$ and $\Delta k = 3k_1 - k_3$.

The solution is

$$a_3 = -g a_1^3 \frac{1 - e^{-j\Delta k z}}{\Delta k}.$$

The photon flux density, $\phi_3 = |a_3|^2$, is

$$\phi_3 = g^2 \phi_1^3 z^2 \operatorname{sinc}^2\left(\frac{\Delta k z}{2}\right).$$

For perfect phase matching, the photon flux grows quadratically: $\phi_3 = g^2 \phi_1^3 z^2$.

Optical phase conjugation (OPC)

Degenerate four-wave mixing: $\omega_1 = \omega_2 = \omega_3 = \omega_4 = \omega$.
The two pump waves, 3 and 4, are counter-propagating: $\mathbf{k}_3 = -\mathbf{k}_4$.
Assuming that the pump intensities are much higher than those of the signal waves, we obtain

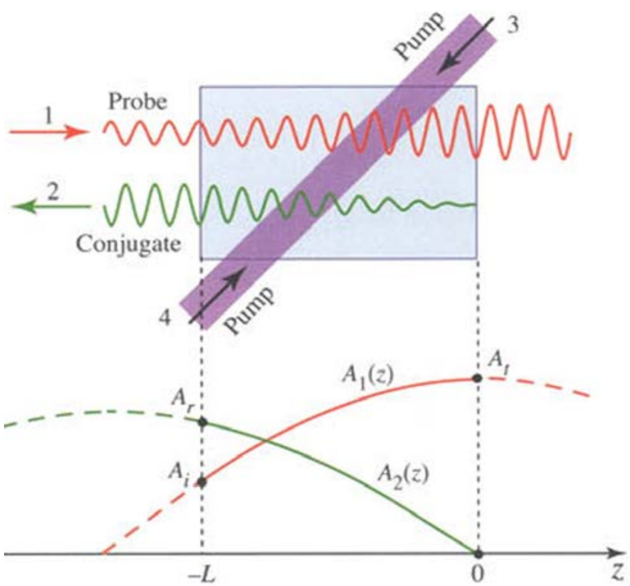
$$\begin{cases} (\nabla^2 + k^2)E_1 = -\xi E_2^* \\ (\nabla^2 + k^2)E_2 = -\xi E_1^*, \end{cases}$$

where

$$\xi = 6\mu_o\omega^2\chi^{(3)}E_3E_4 = 6\mu_o\omega^2\chi^{(3)}A_3A_4$$

$$k = \bar{n}\omega/c_o, \qquad \bar{n} \approx n + 2n_2 I$$

Phase conjugation: As required by the phase-matching condition, $\mathbf{k}_1 = -\mathbf{k}_2$.



The slowly varying envelope approximation yields

$$\begin{cases} \dfrac{dA_1}{dz} = -j\gamma A_2^* \\ \dfrac{dA_2}{dz} = j\gamma A_1^*, \end{cases} \Rightarrow \begin{cases} A_1(z) = \dfrac{A_i}{\cos\gamma L}\cos\gamma z \\ A_2(z) = j\dfrac{A_i^*}{\cos\gamma L}\sin\gamma z. \end{cases}$$

$$A_r = -jA_i^*\tan\gamma L, \qquad |\gamma| = 2C\sqrt{I_3I_4},$$

$$A_t = \frac{A_i}{\cos\gamma L}. \qquad C = 3\omega\eta_o^2\frac{\chi^{(3)}}{\bar{n}^2}$$