Aalto University
School of Science

Department of Computer Science

Combinatorics of Efficient Computations

Approximation Algorithms

Lecture 9: An Approximation Scheme for Euclidean TSP

Joachim Spoerhase

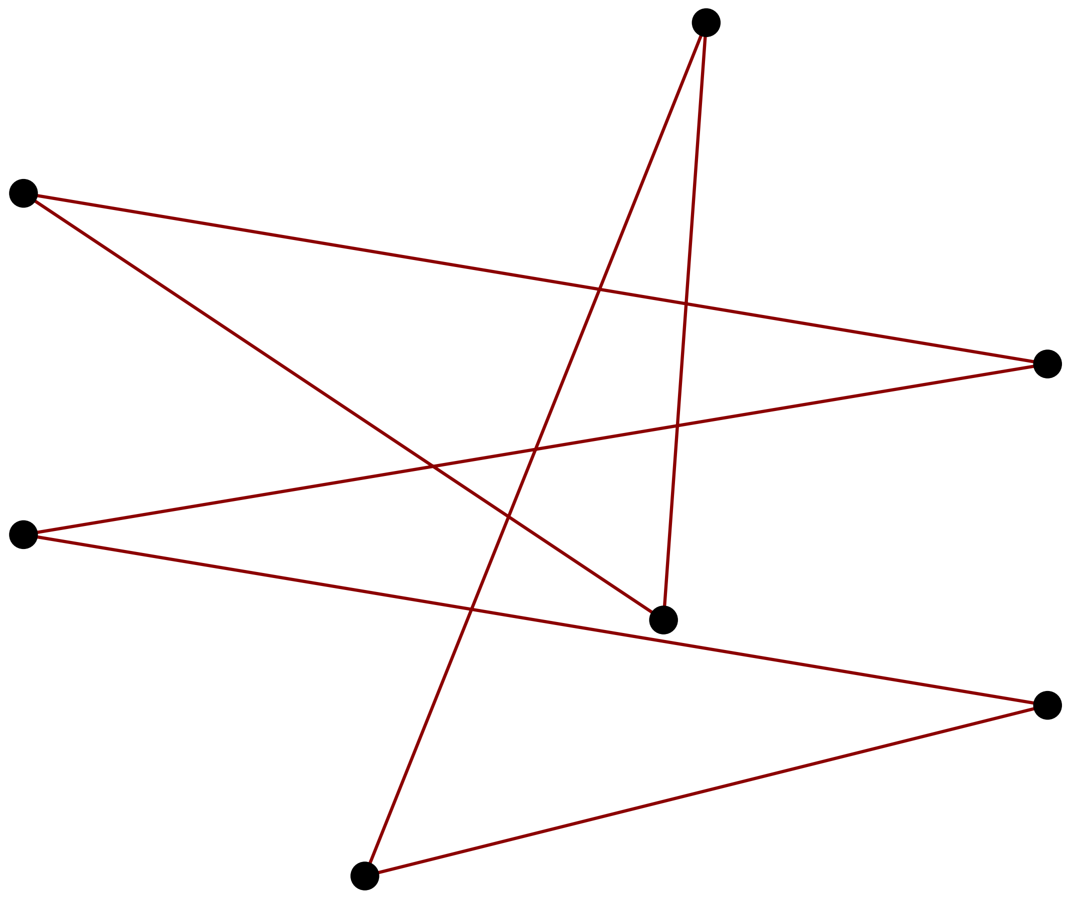
2019

Euclidean TSP

Given: A set of n points in $\mathbb{R}^2$.

The distance between two points is the euclidean distance.

Find: A Hamiltonian cycle (tour) of minimum length.



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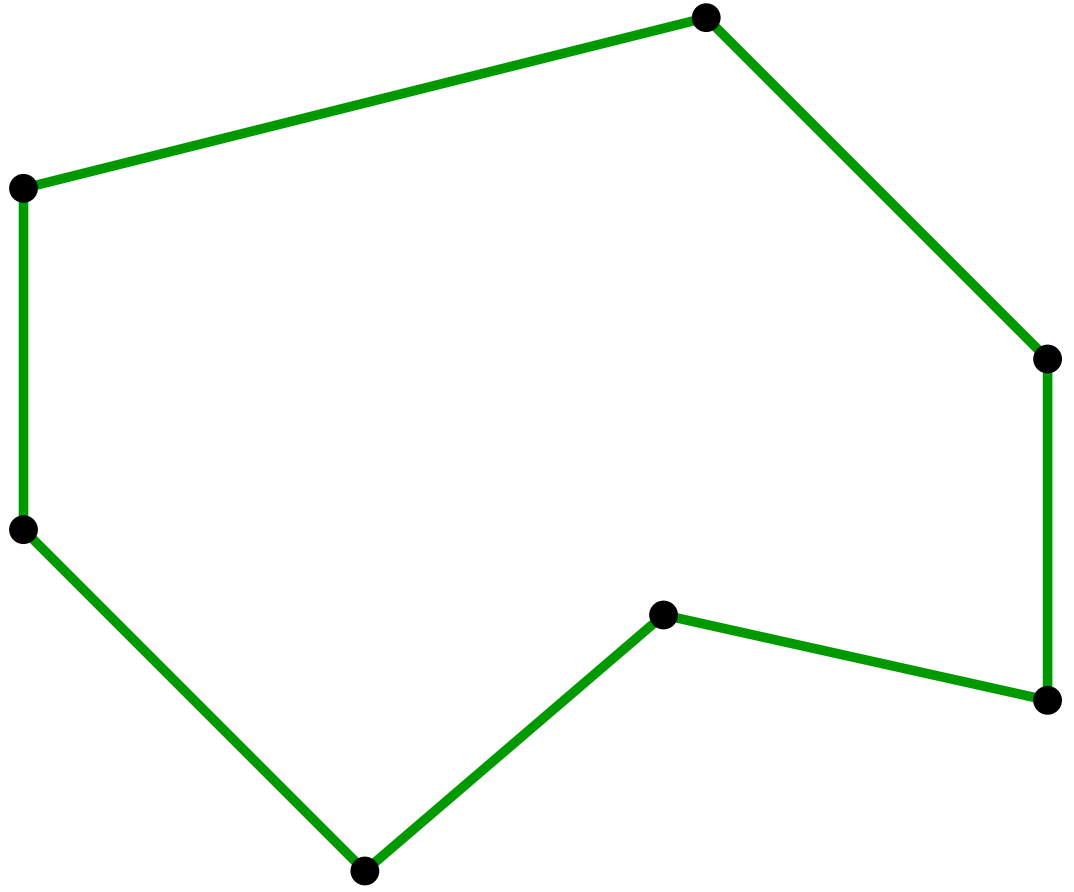

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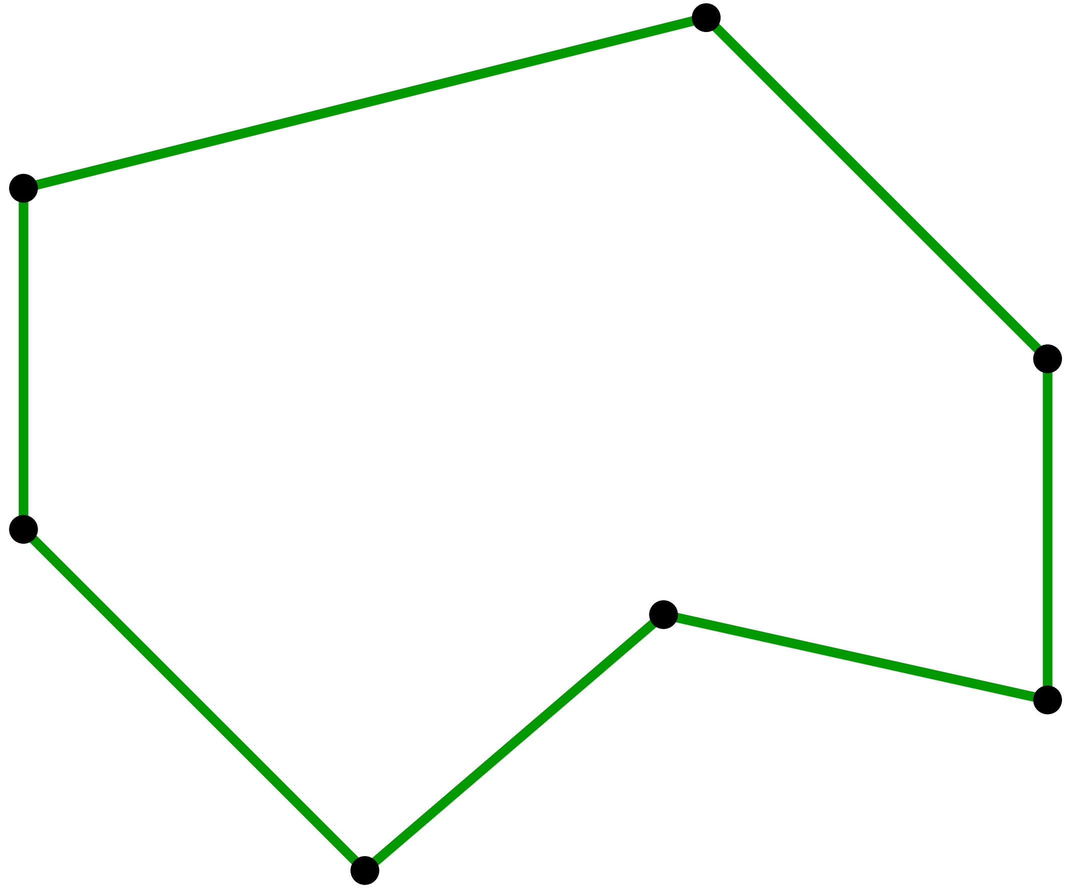

Goal:$(1 + \epsilon)$-Approximation!

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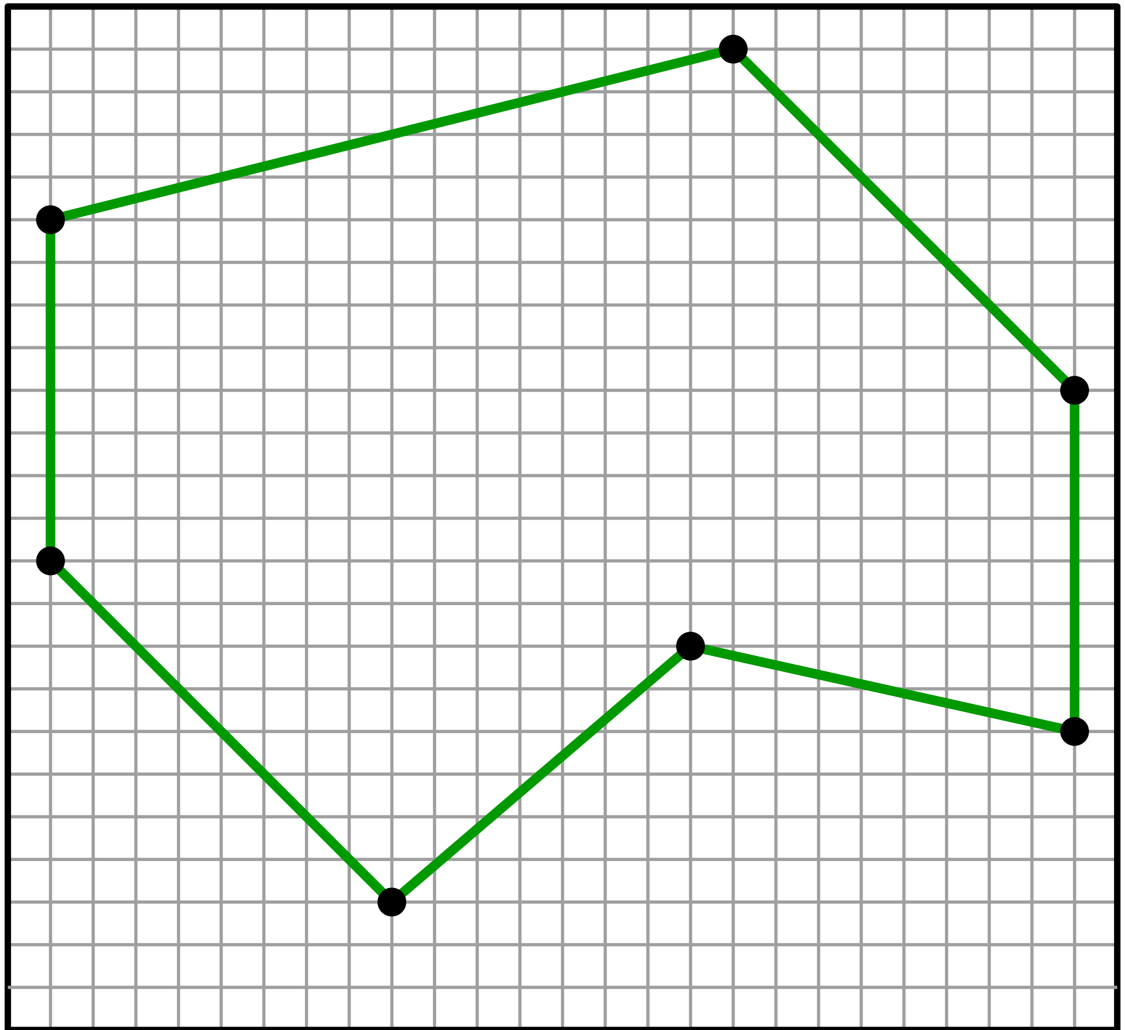

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Simplifying Assumptions

- points belong to an $L \times L$-square
- $L := 4n^2 = 2^k$; $k = 2 + 2\log_2 n$
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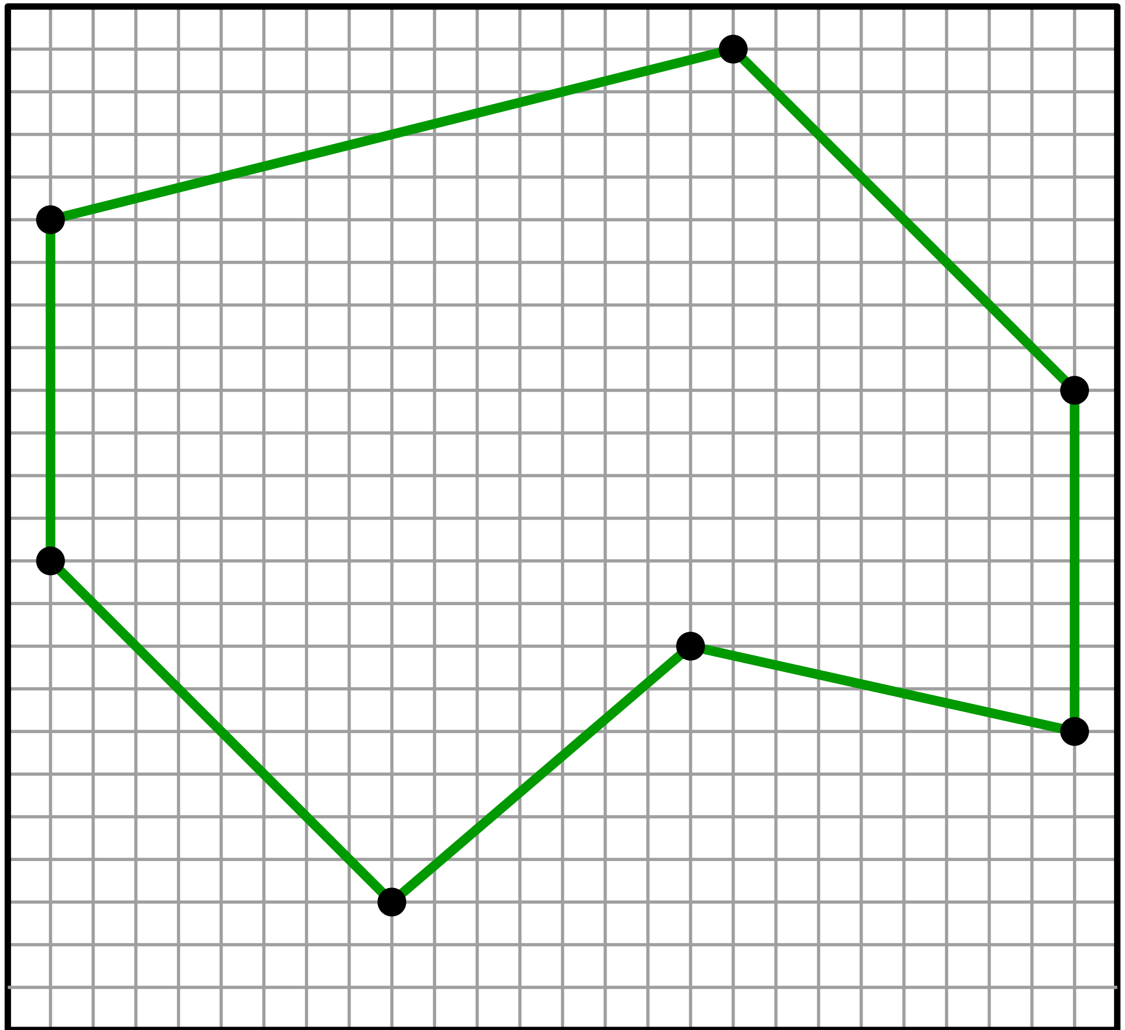

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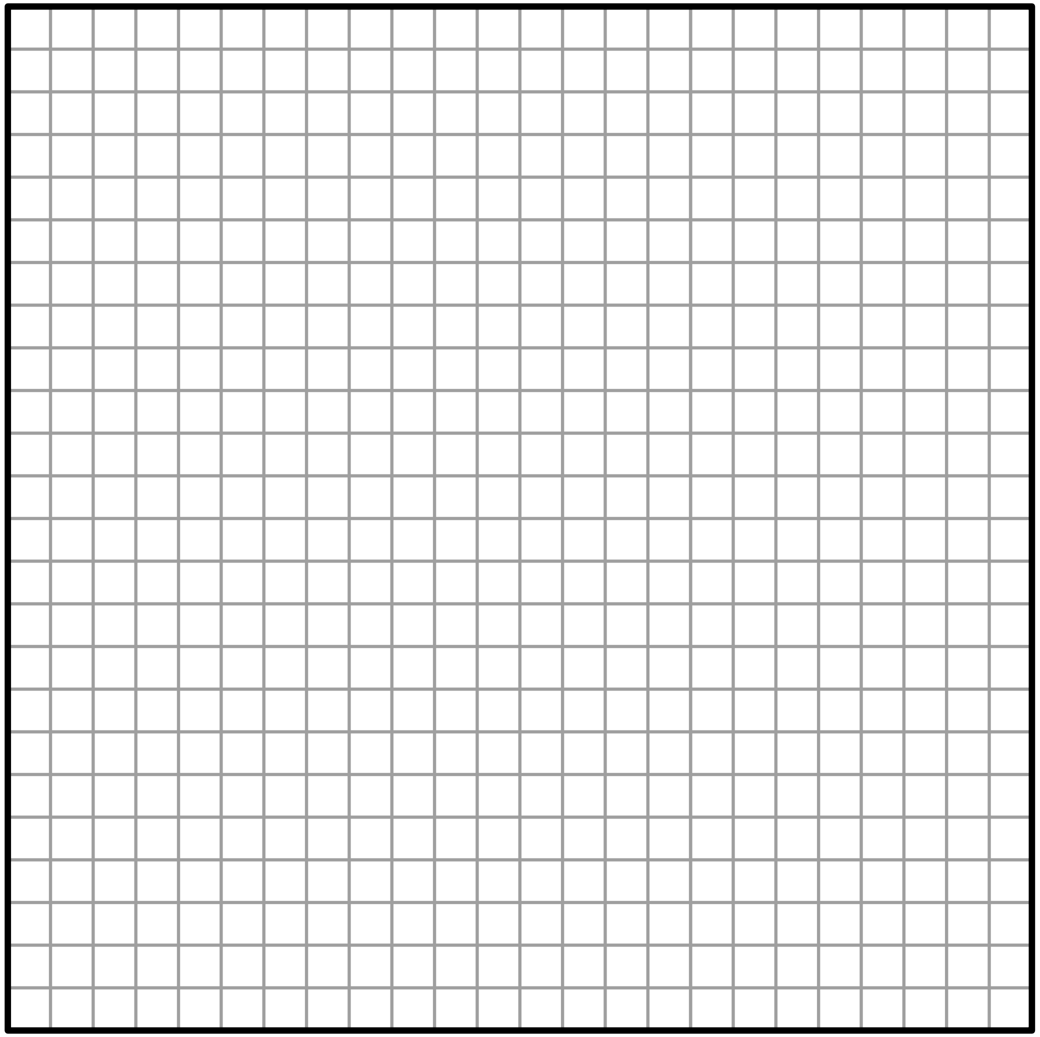
"justification" $\rightsquigarrow$ exercise

Basic Dissection



$$L = 2^k$$

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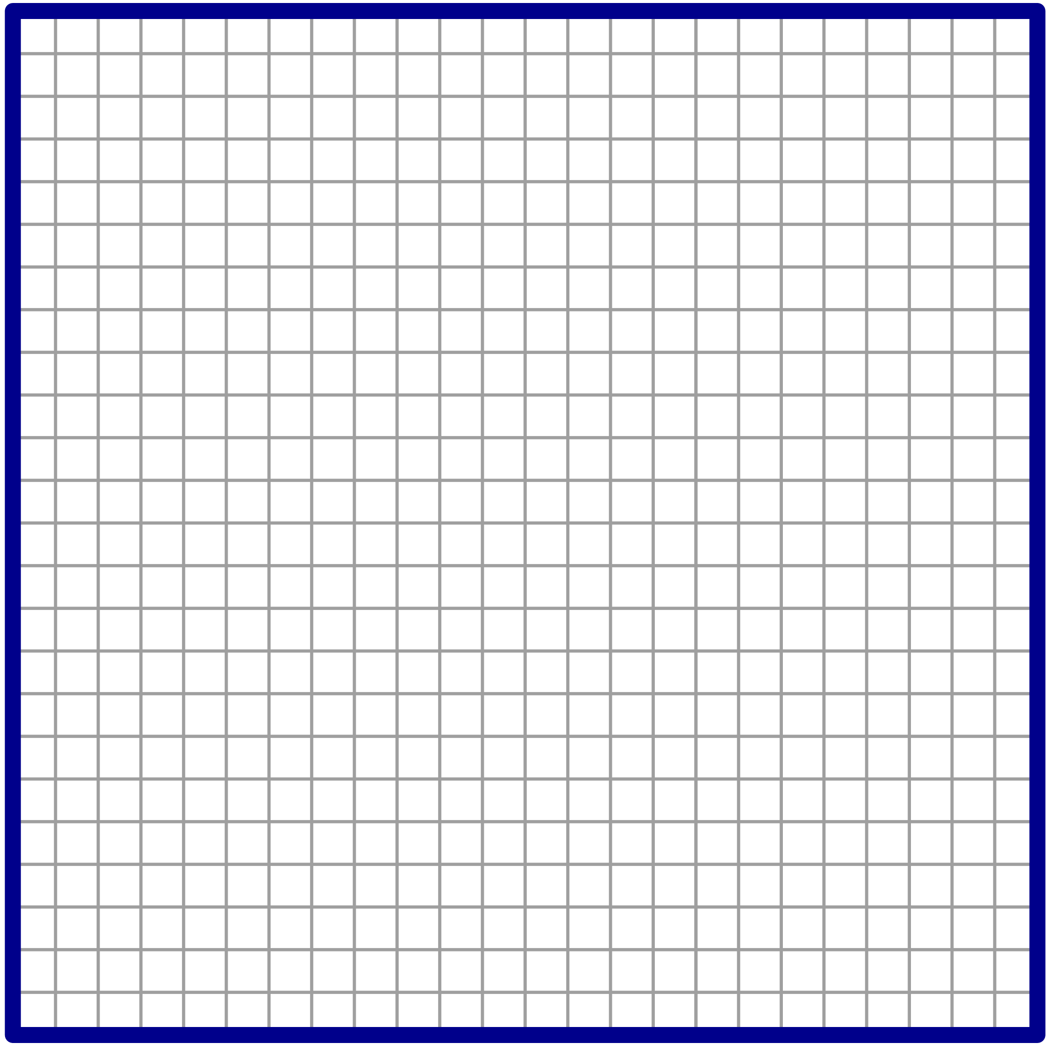

Level 0

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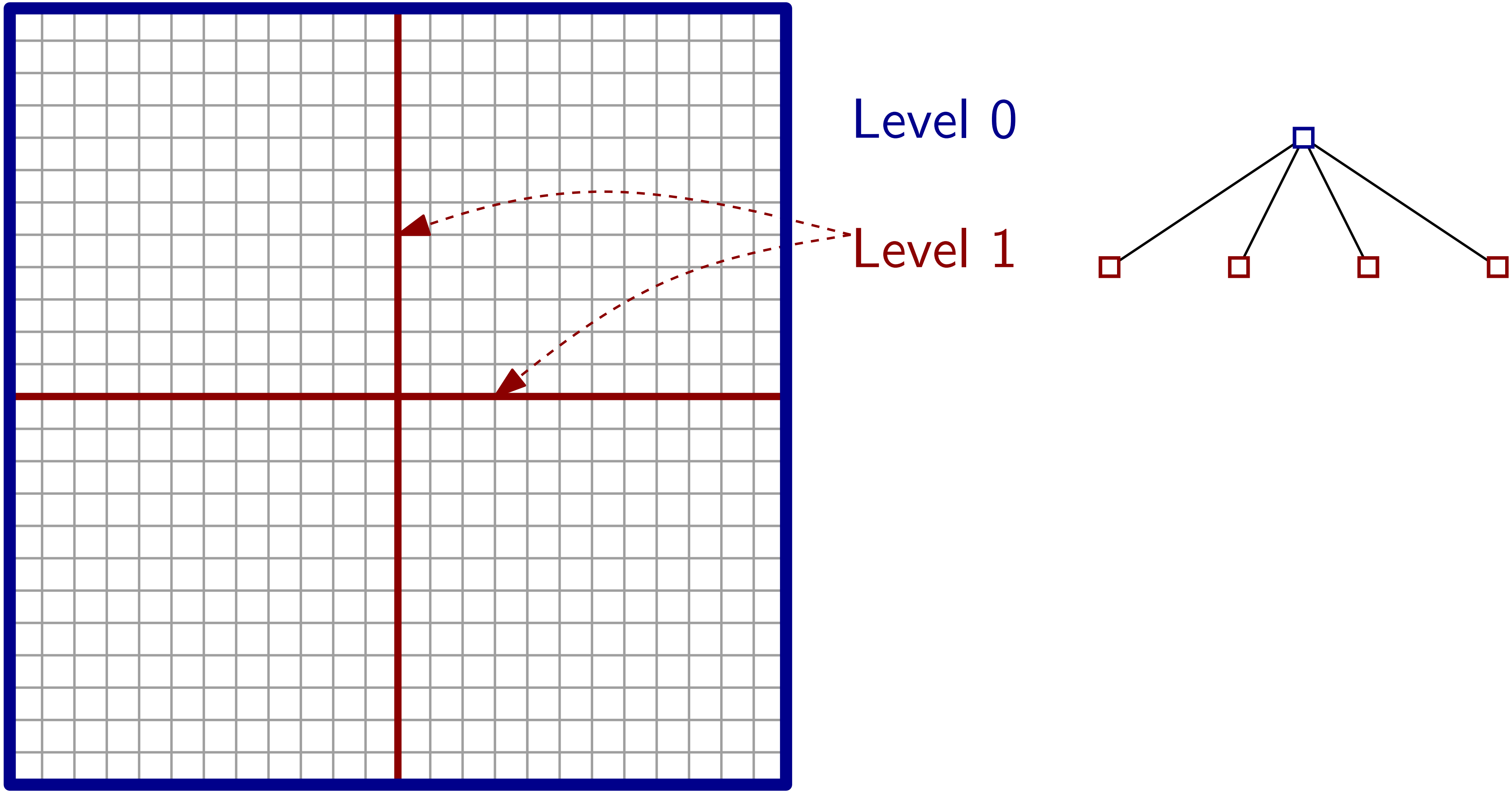

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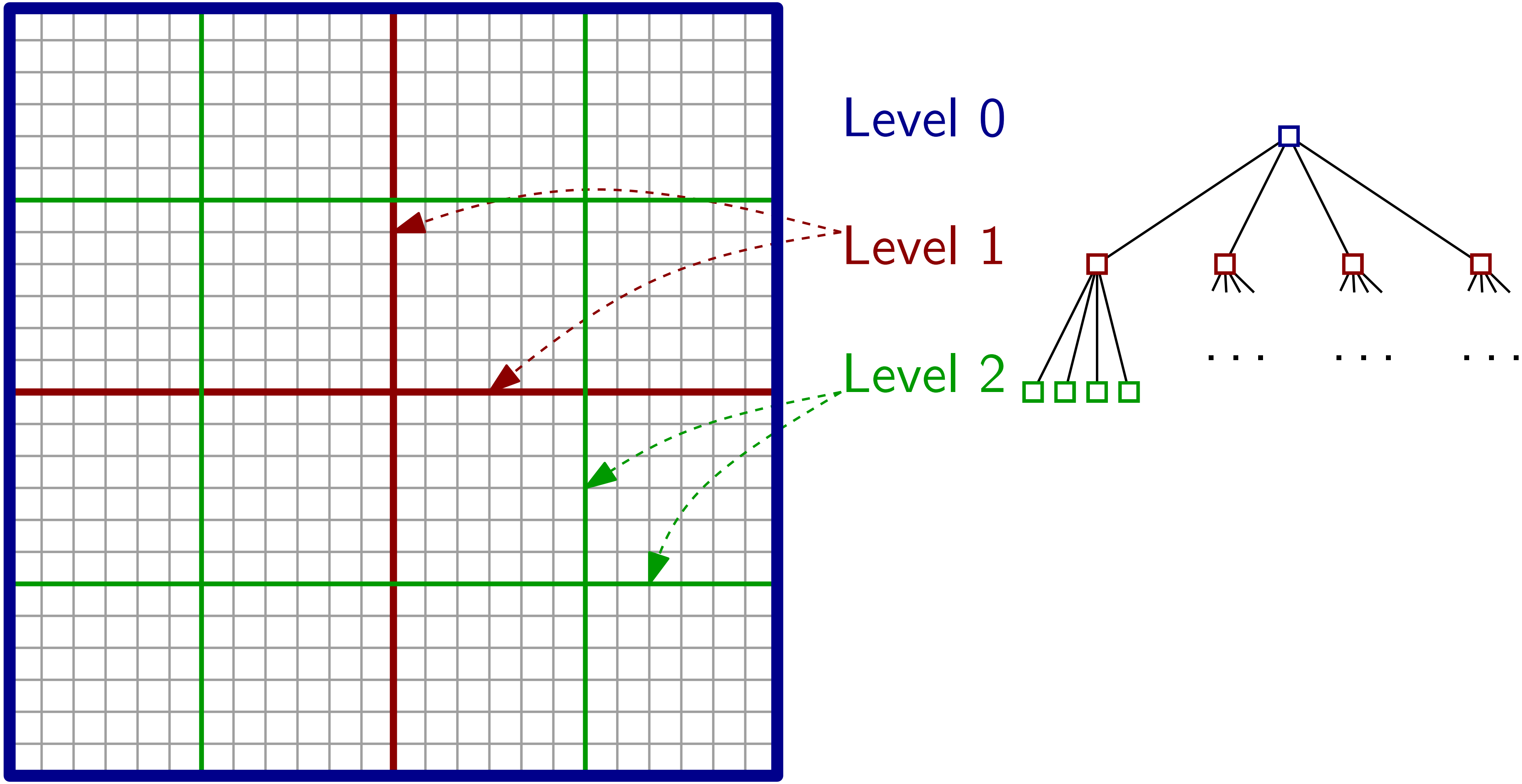

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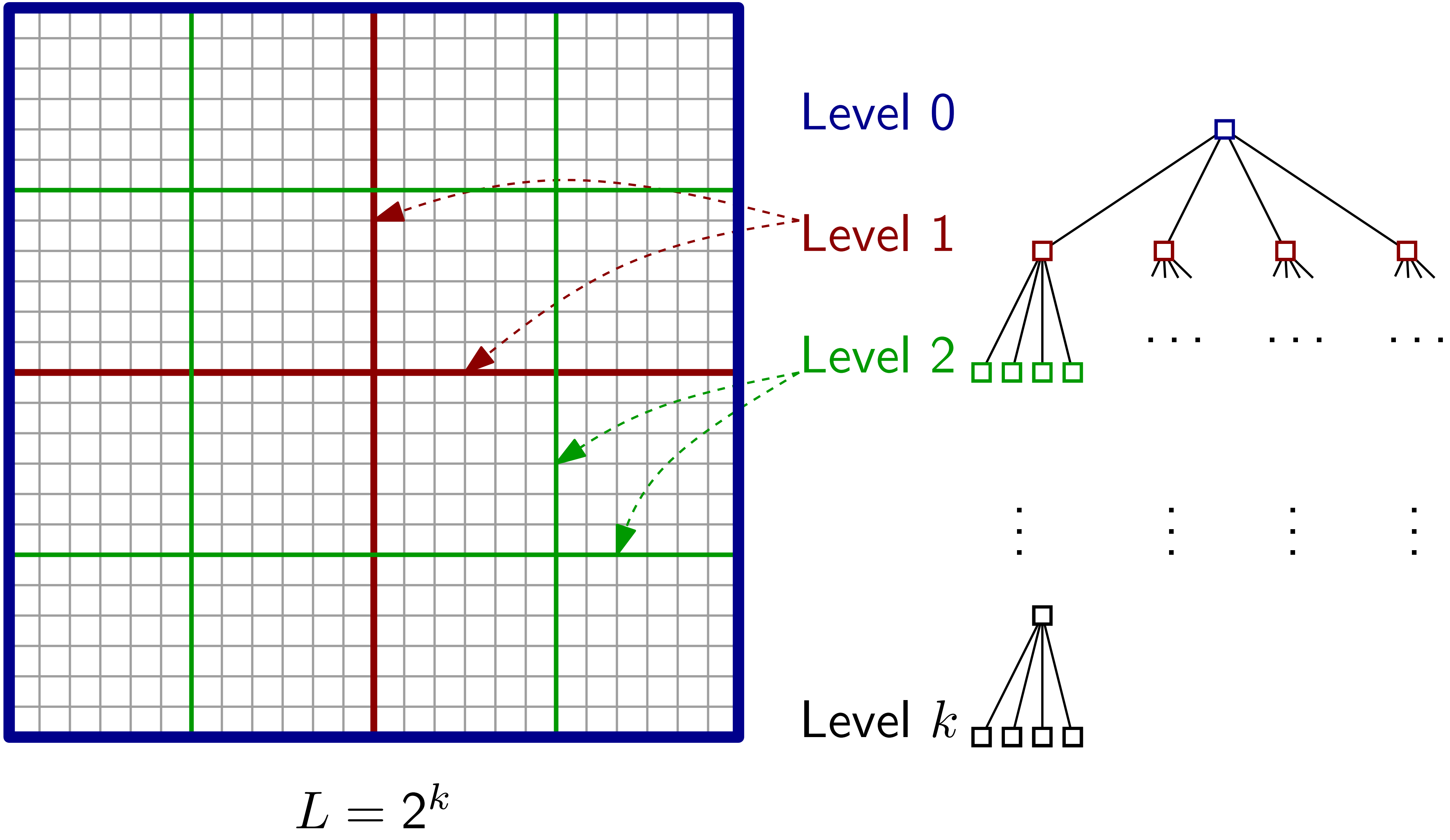
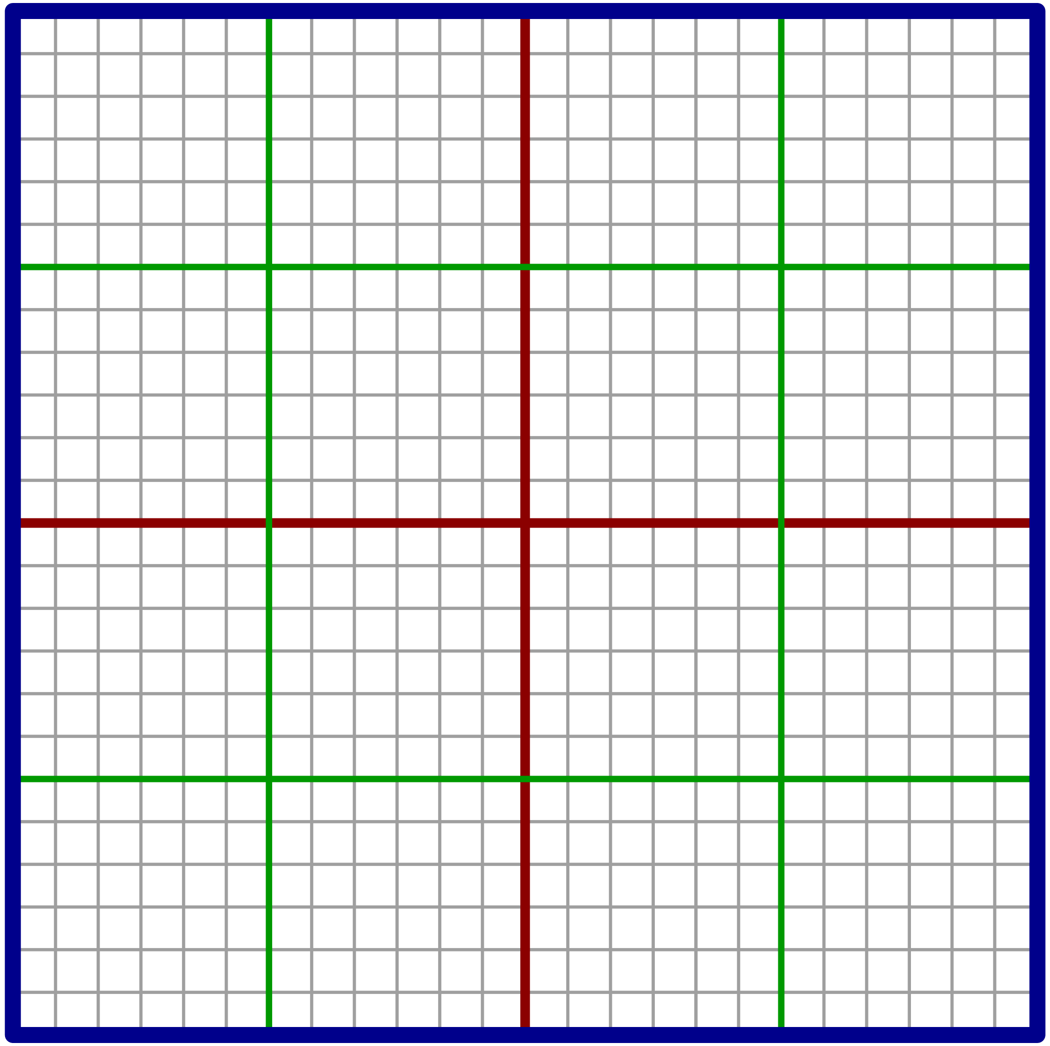


Portals



$$L := 4n^2 = 2^k.$$

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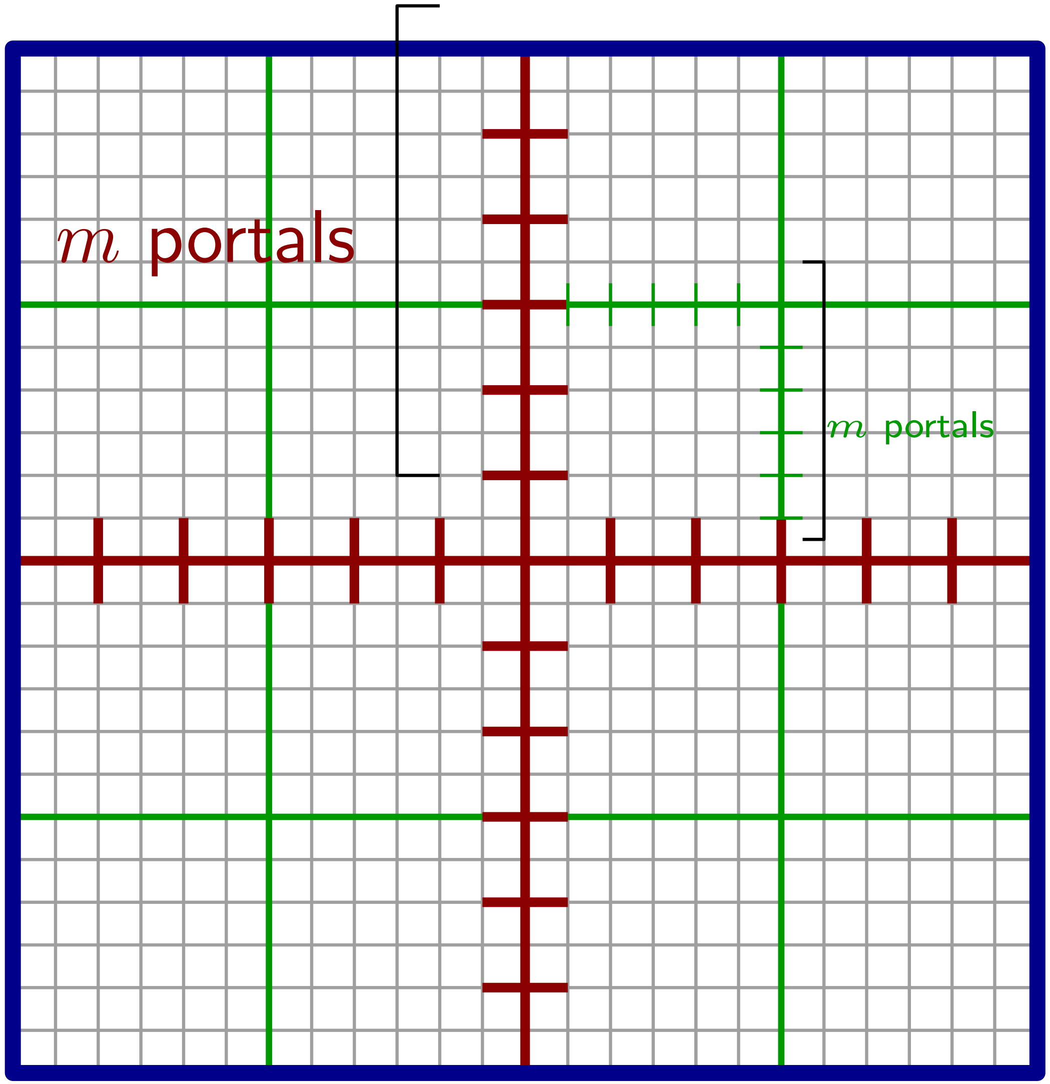
Portals



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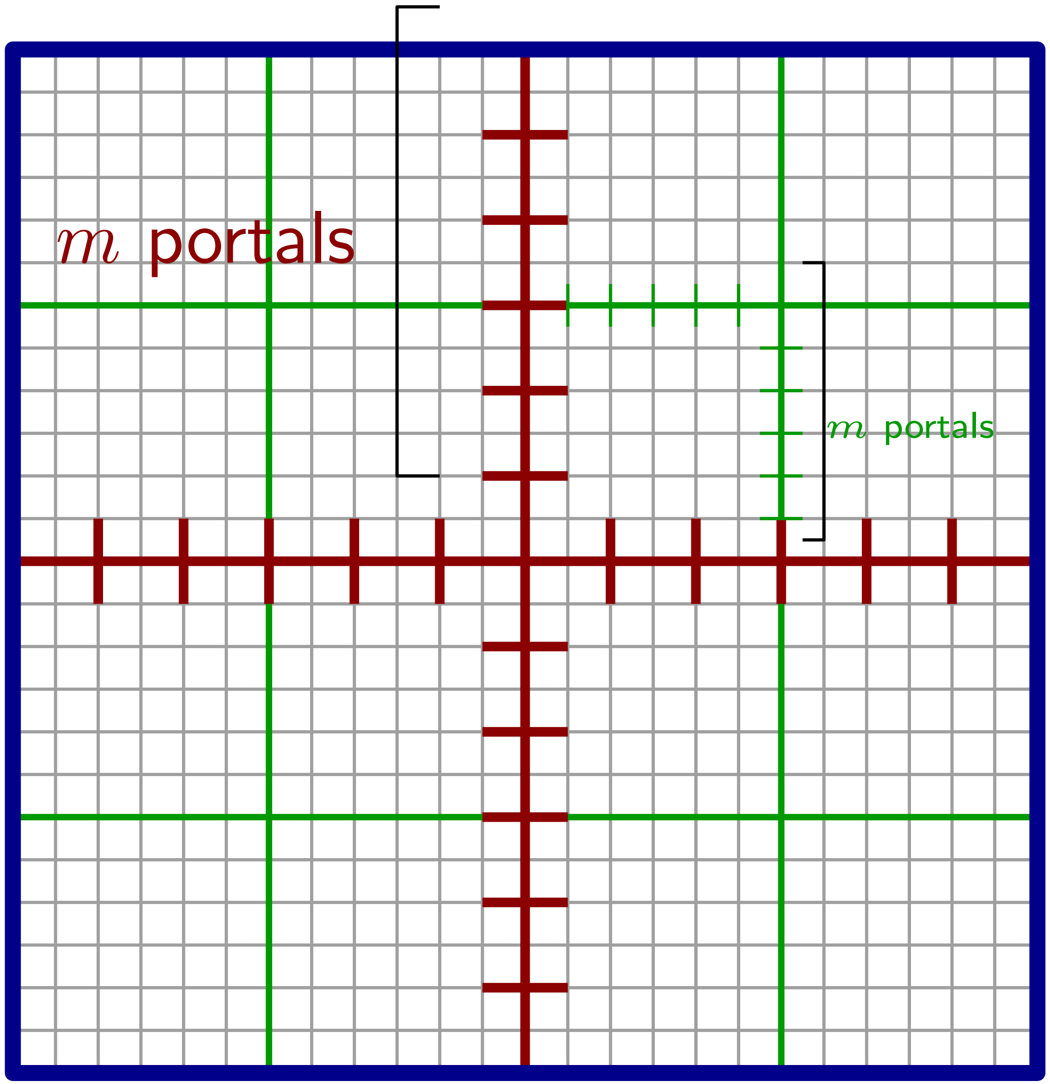

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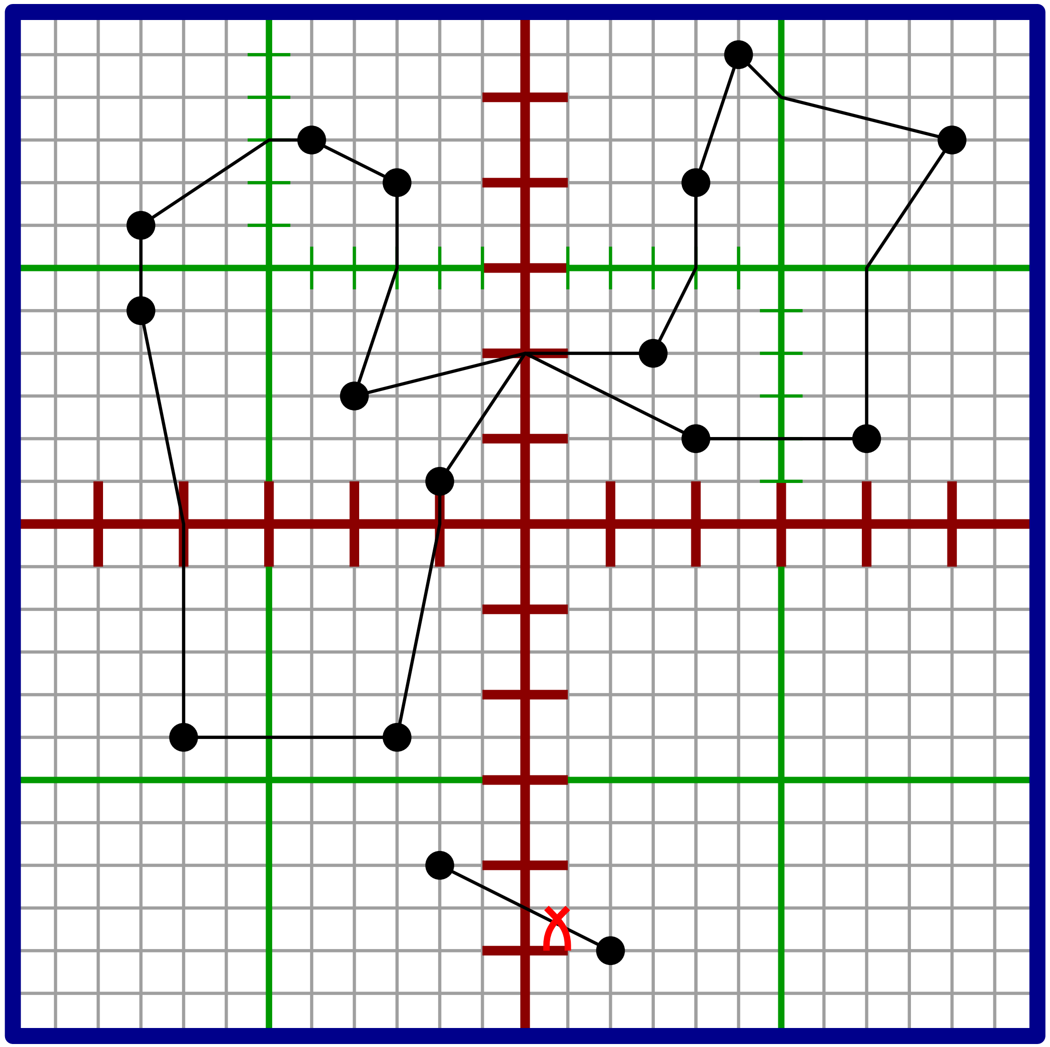
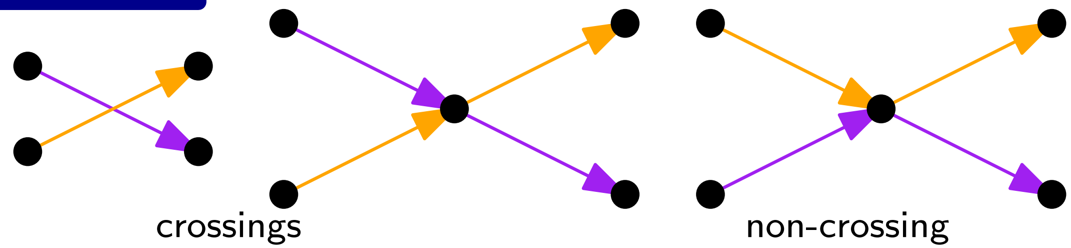
- **Portals** on a level-i-line have distance $L/(2^i m)$
- each level-i-square has at most $4m$ boundary portals

Well Behaved Tours

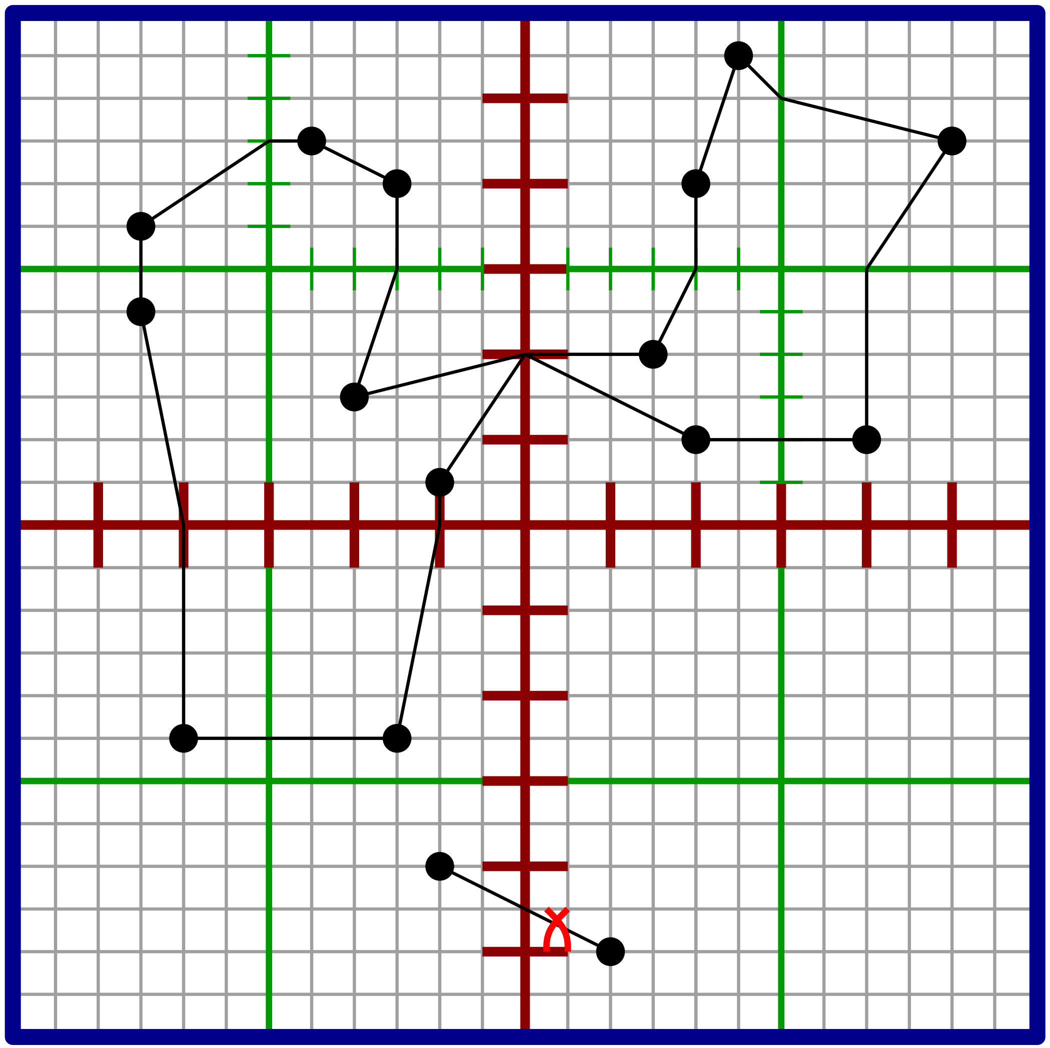


A tour is **well behaved**, when

- it involves all points and a subset of the portals,
- no edge of the tour crosses a line of the basic dissection
- it is crossing-free.



Well Behaved Tours



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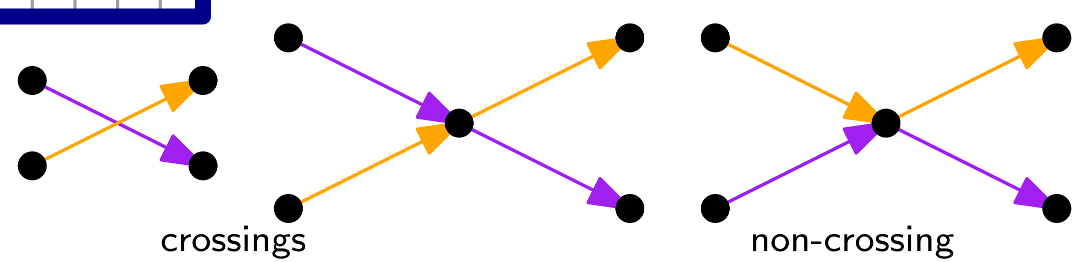
Without loss of generality (exercise): no portal is visited more than twice.



Computing a well behaved tour

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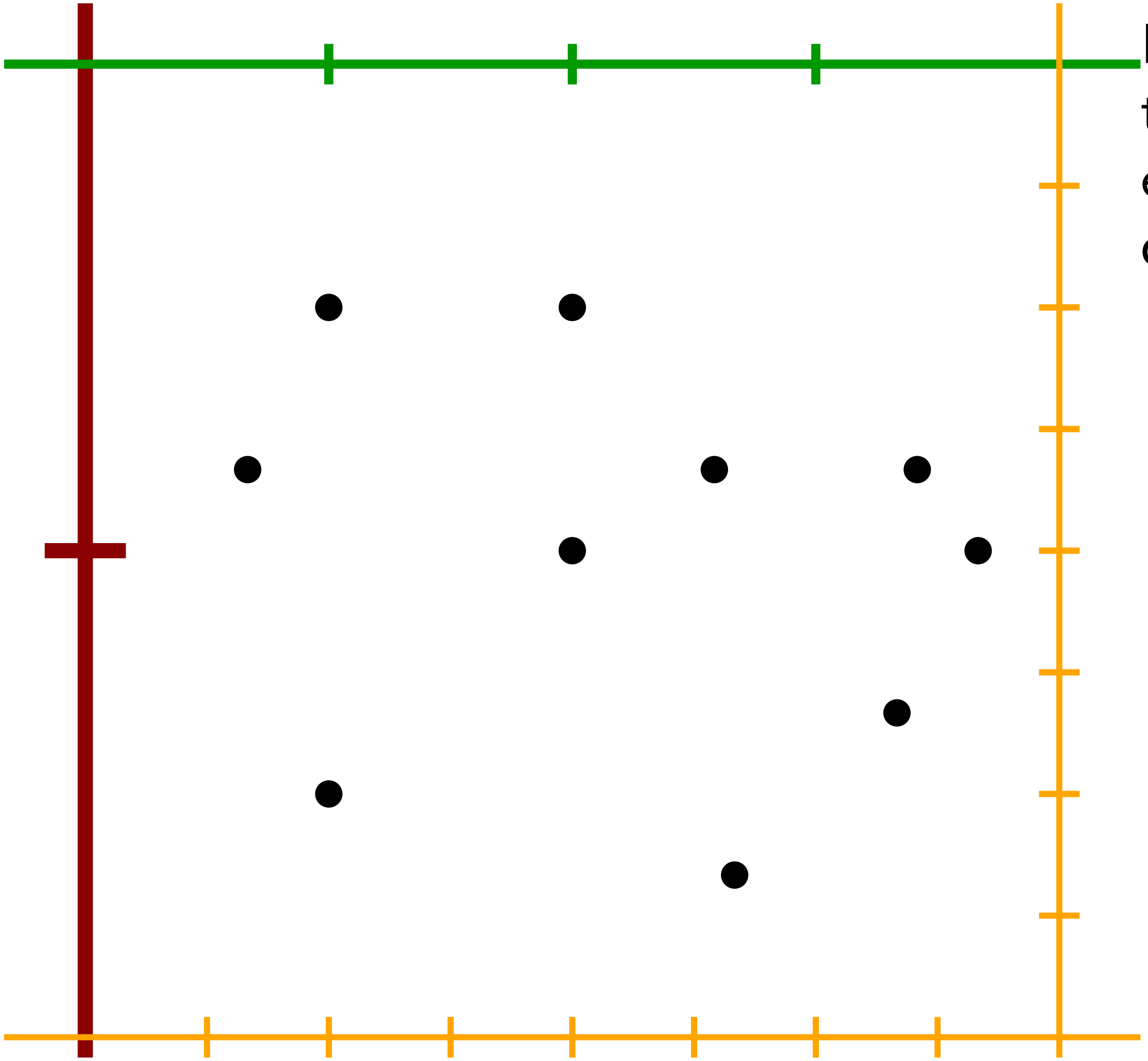
Proof.

- Dynamic Program.
- Compute sub-structure of an optimal tour for each square in the dissection tree.
- These solutions can be efficiently propagated bottom-up through the dissection tree.

Dynamic Program (I)



Each global well behaved tour induces the following in each square Q of the dissection:

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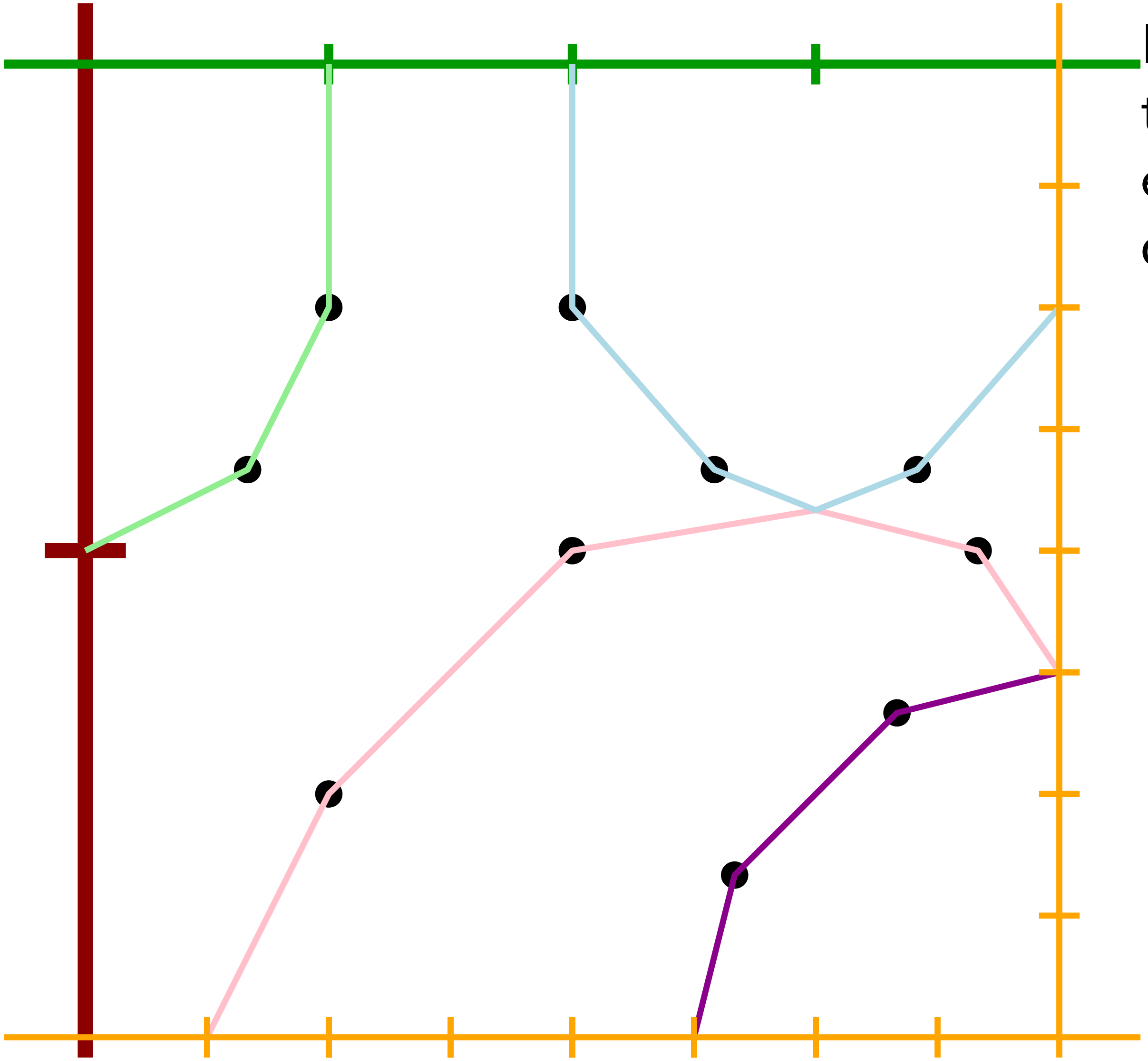

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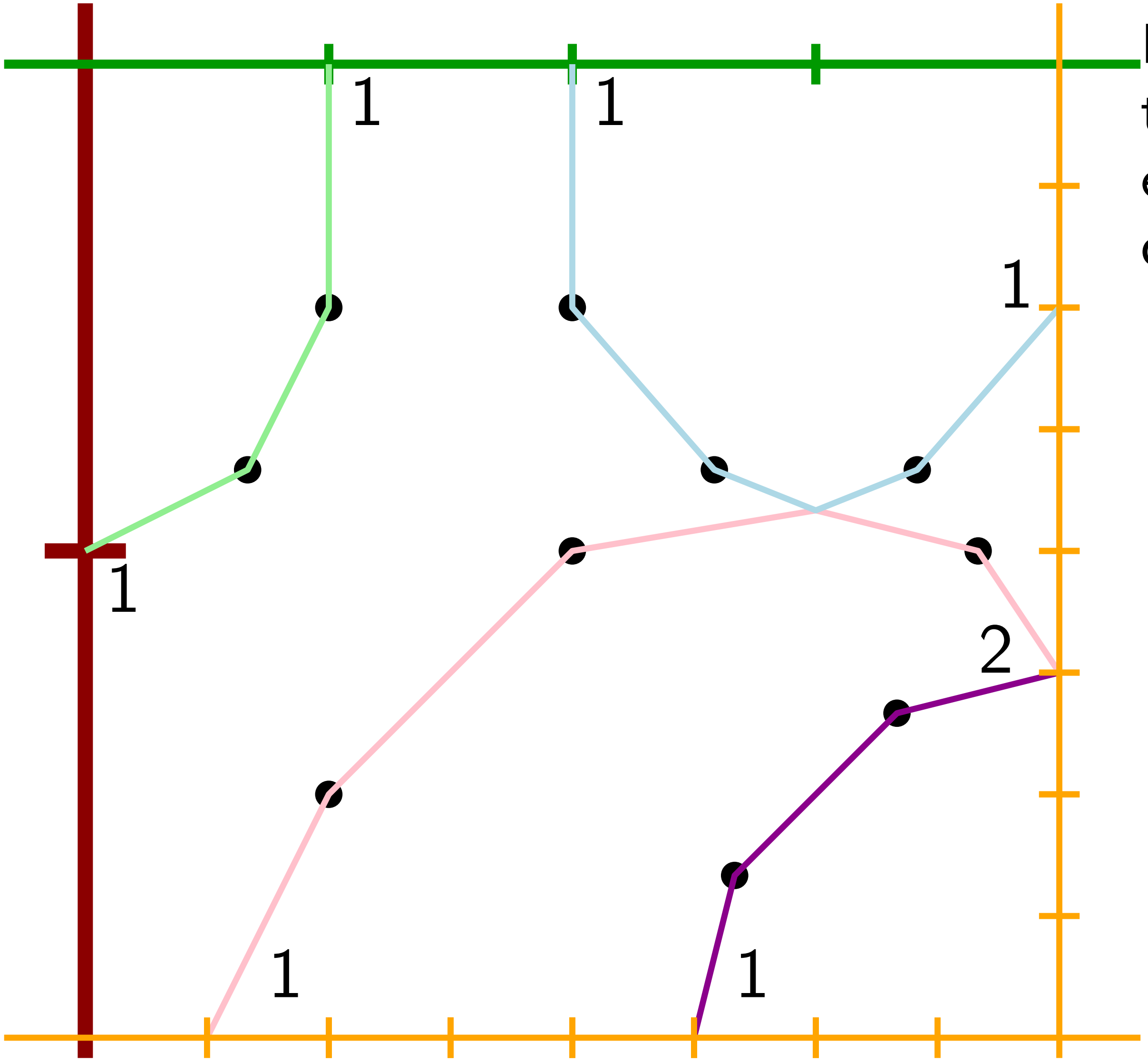

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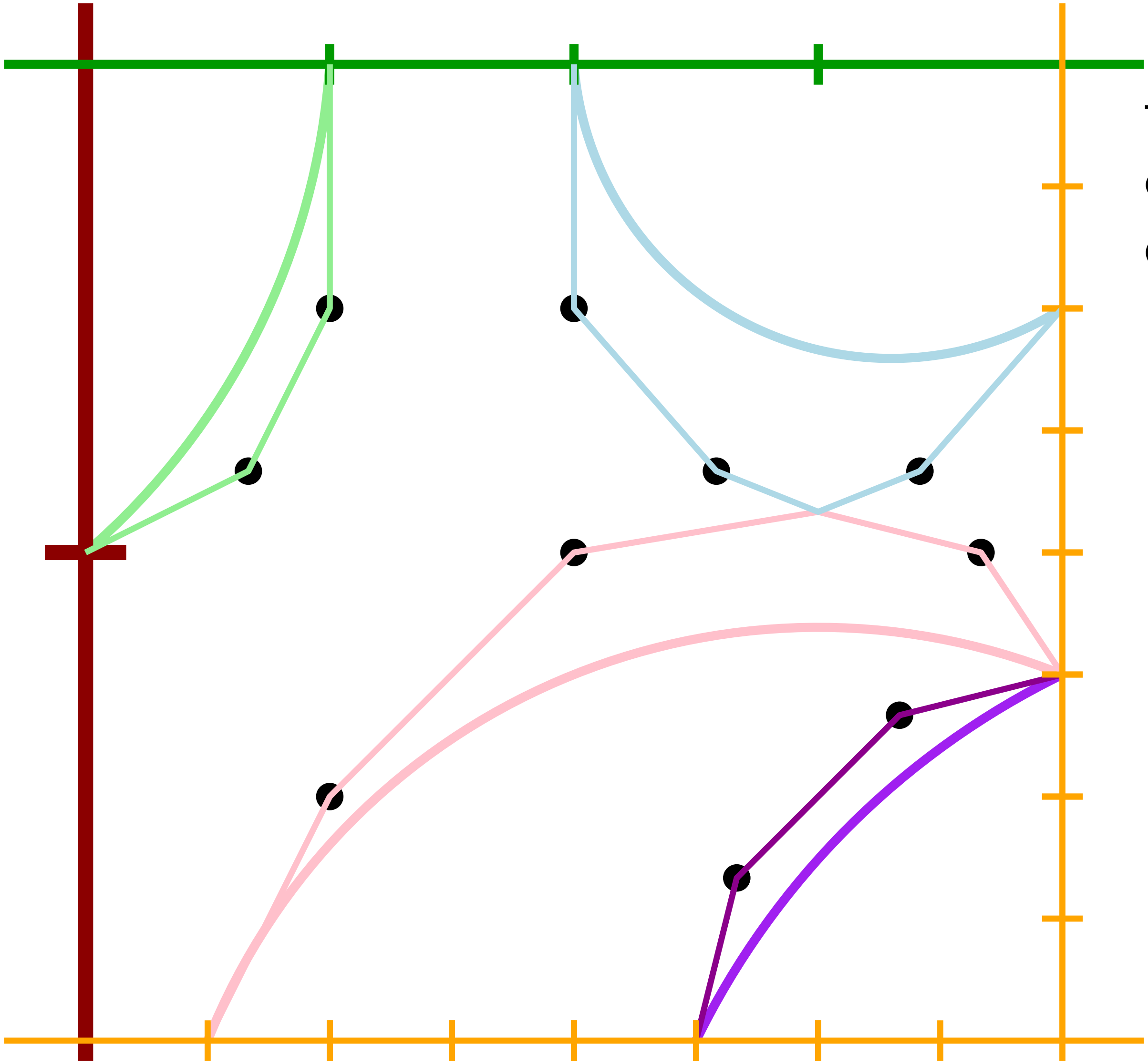
- a path cover on the vertices in Q
- each portal of Q is visted 0,1 or 2 times by this path cover.

max. $3^{4m} = 3^{O(\log n/\epsilon)} = n^{O(1/\epsilon)}$ possibilities

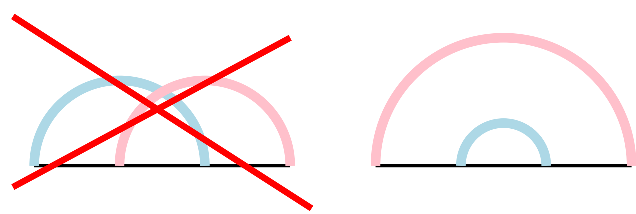
$m \in [k/\epsilon, 2k/\epsilon]$

Dynamic Program (I)



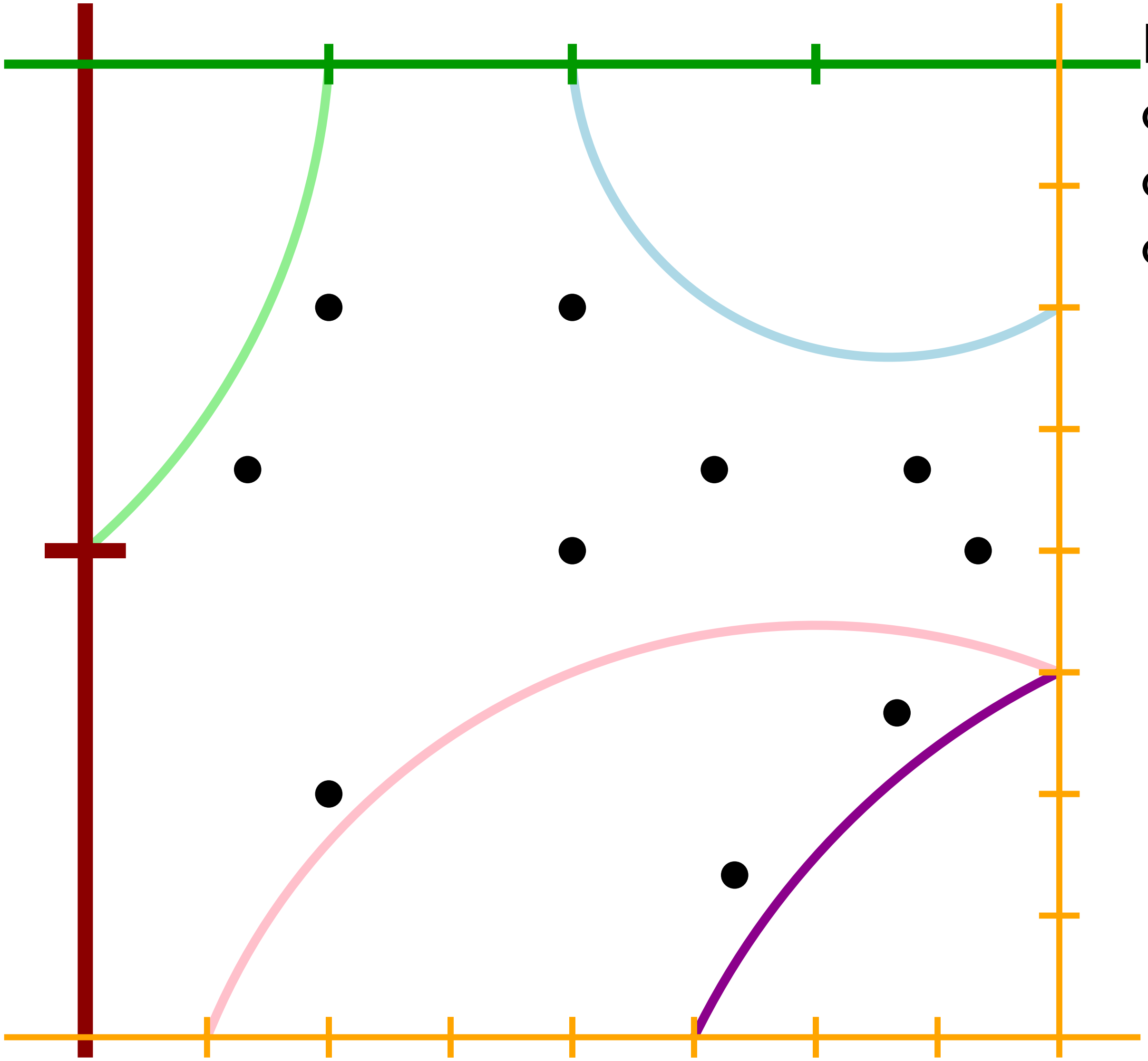
Each global well behaved tour induces the following in each square Q of the dissection:

- a path cover on the vertices in Q
- each portal of Q is visted 0,1 or 2 times by this path cover.
- a **crossing-free pairing** of the visited portals.



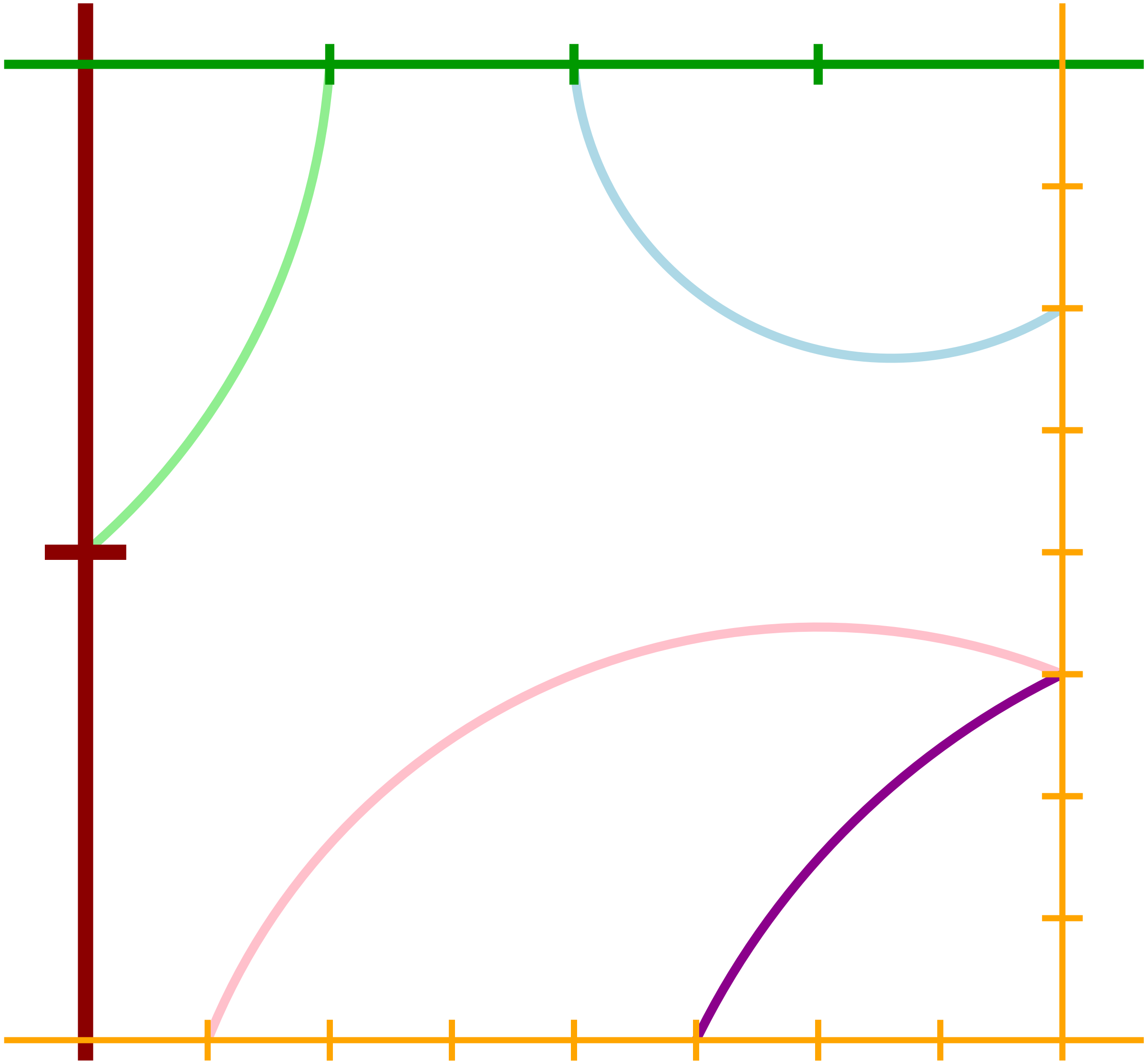
$$\text{max. } \underbrace{n^{O(1/\epsilon)}}_{\#\text{visit vectors}} \times \underbrace{2^{O(m)}}_{\#\text{real. pairings}} = n^{O(1/\epsilon)} \text{ total pairings}$$

Dynamic Program (II)



For each square Q, and each crossing-free pairing P in Q, compute an optimal path cover respecting P.

Dynamic Program (III)



For a given square Q and pairing P:

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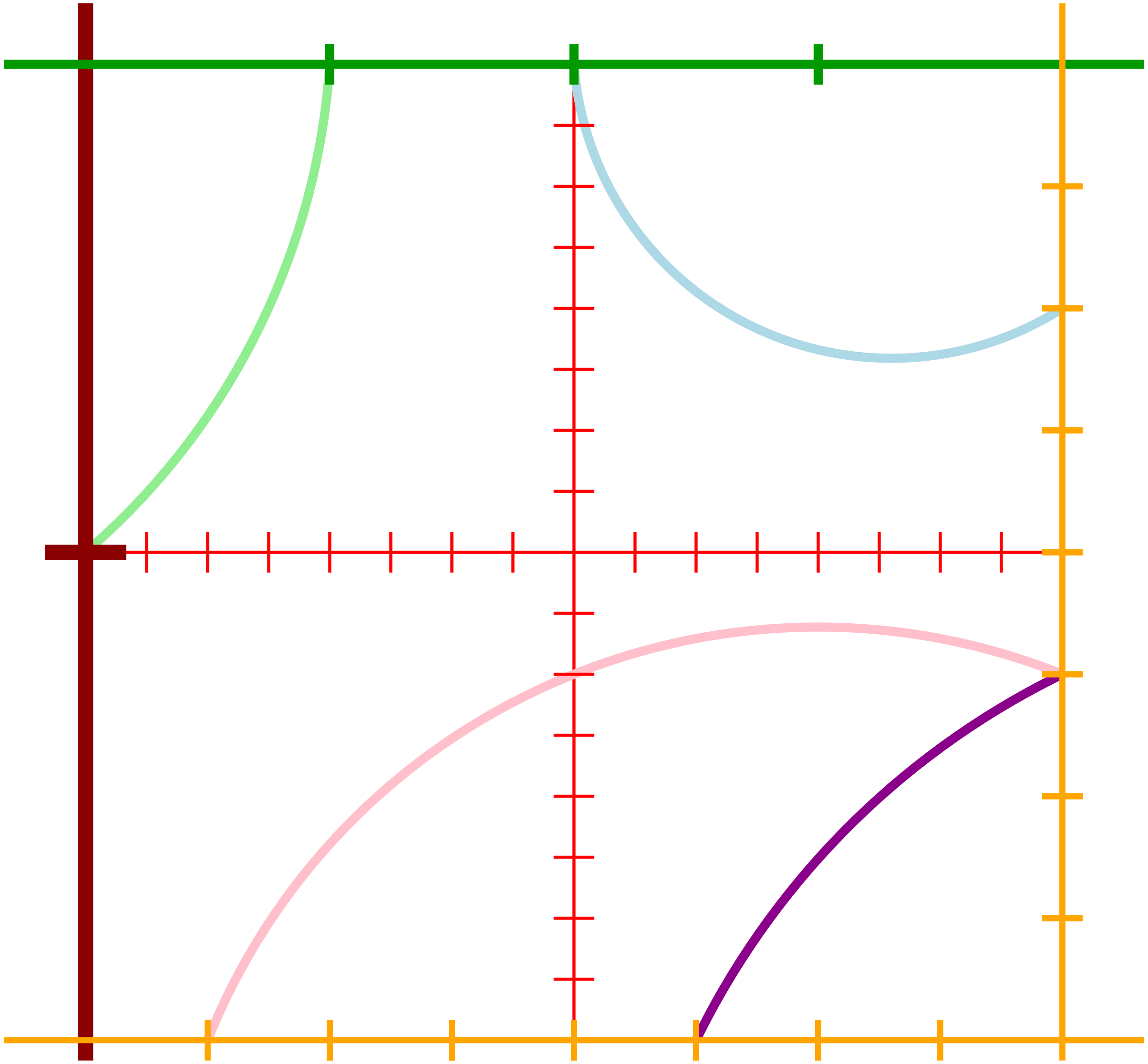

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- iterate over all $(n^{O(1/\epsilon)})^4 = n^{O(1/\epsilon)}$ crossing-free pairings of the child-squares

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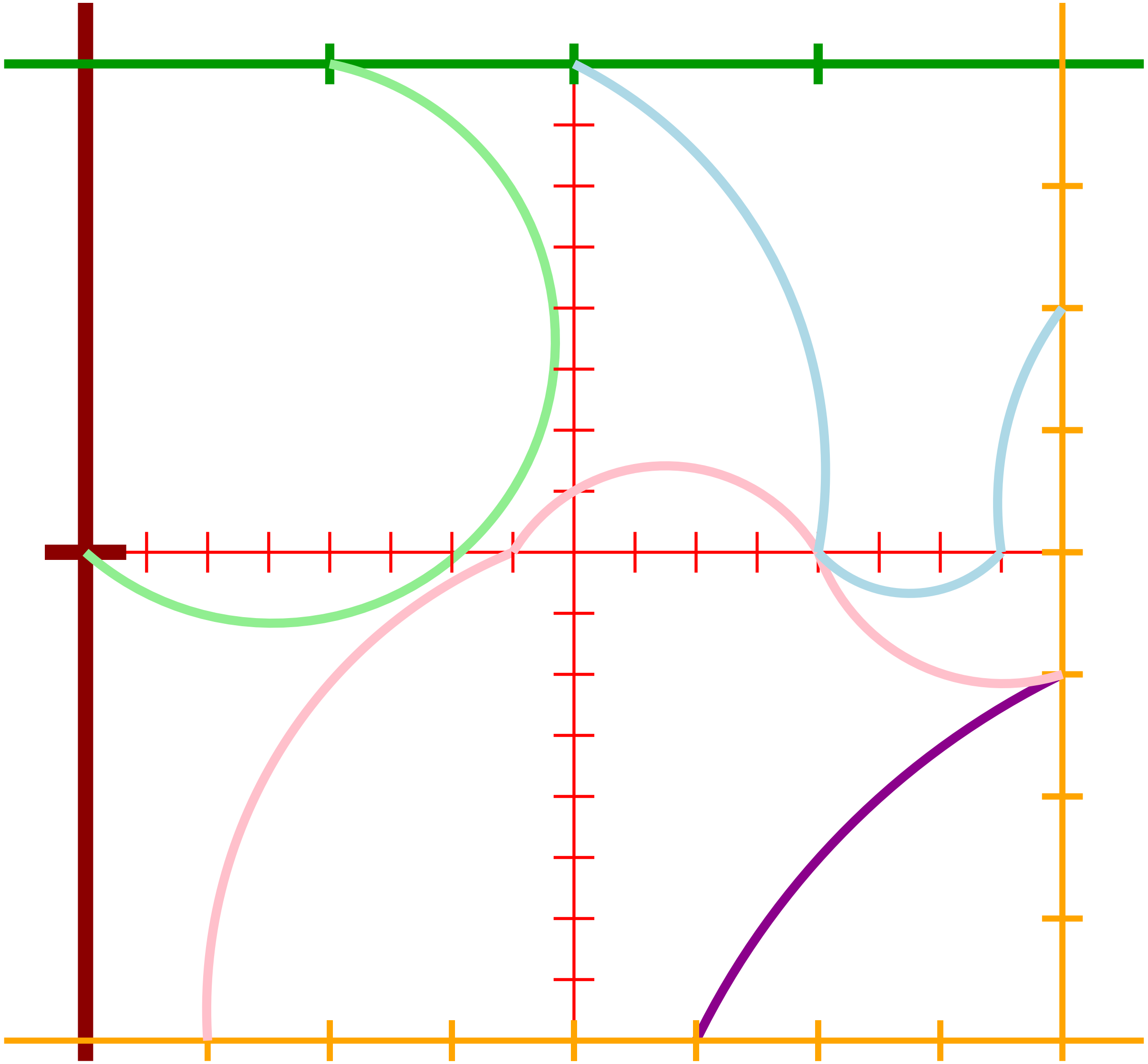

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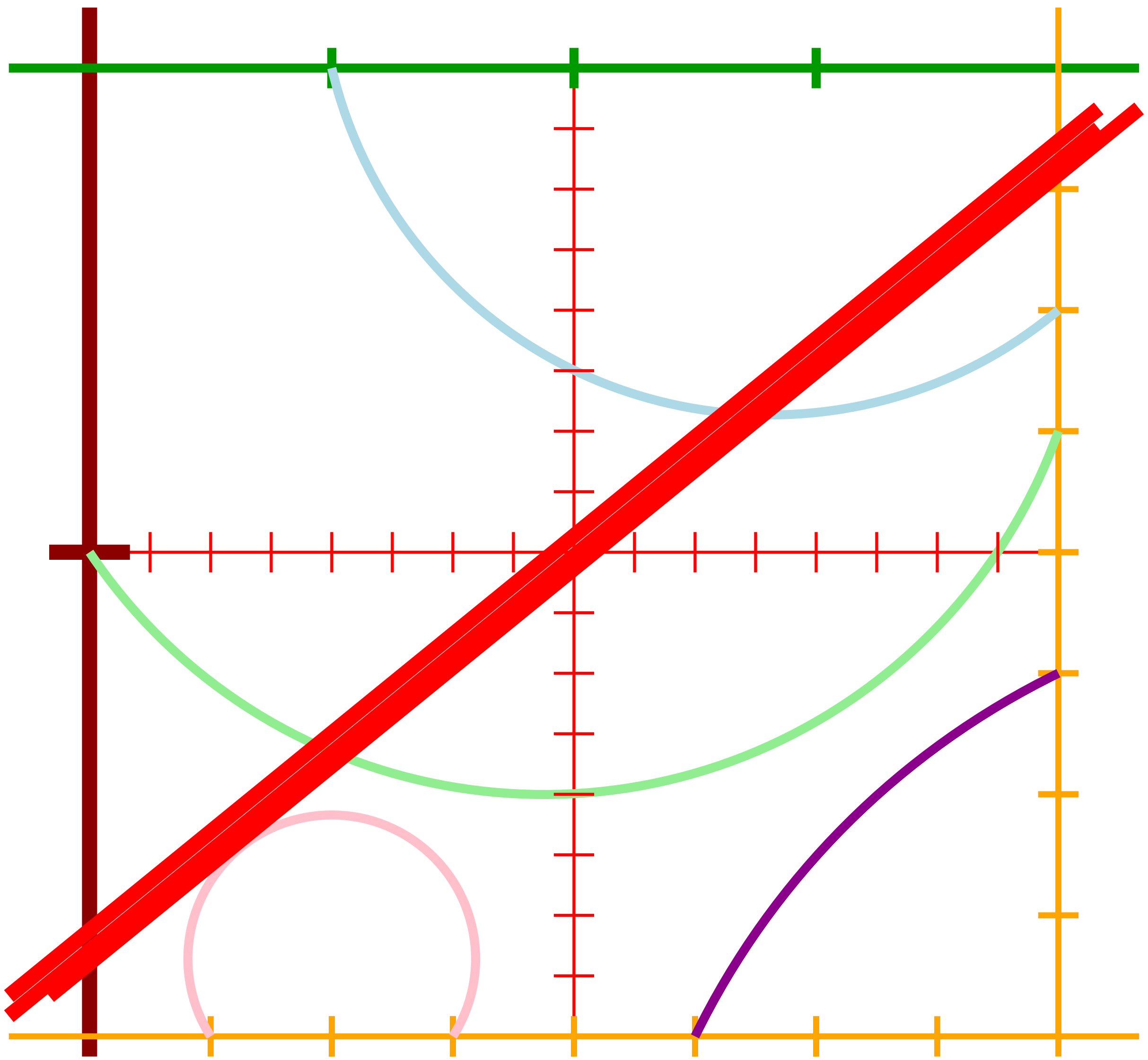

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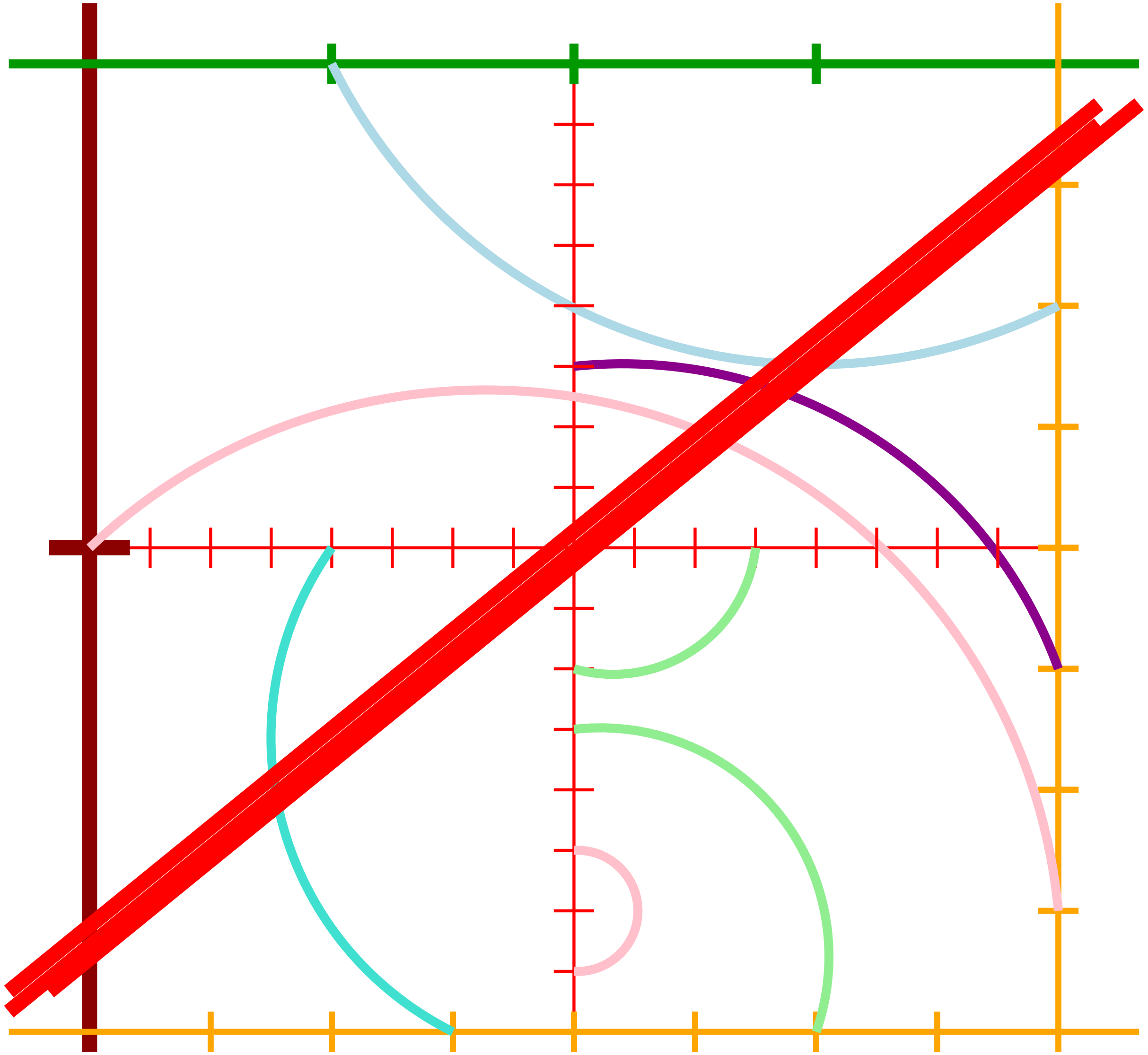

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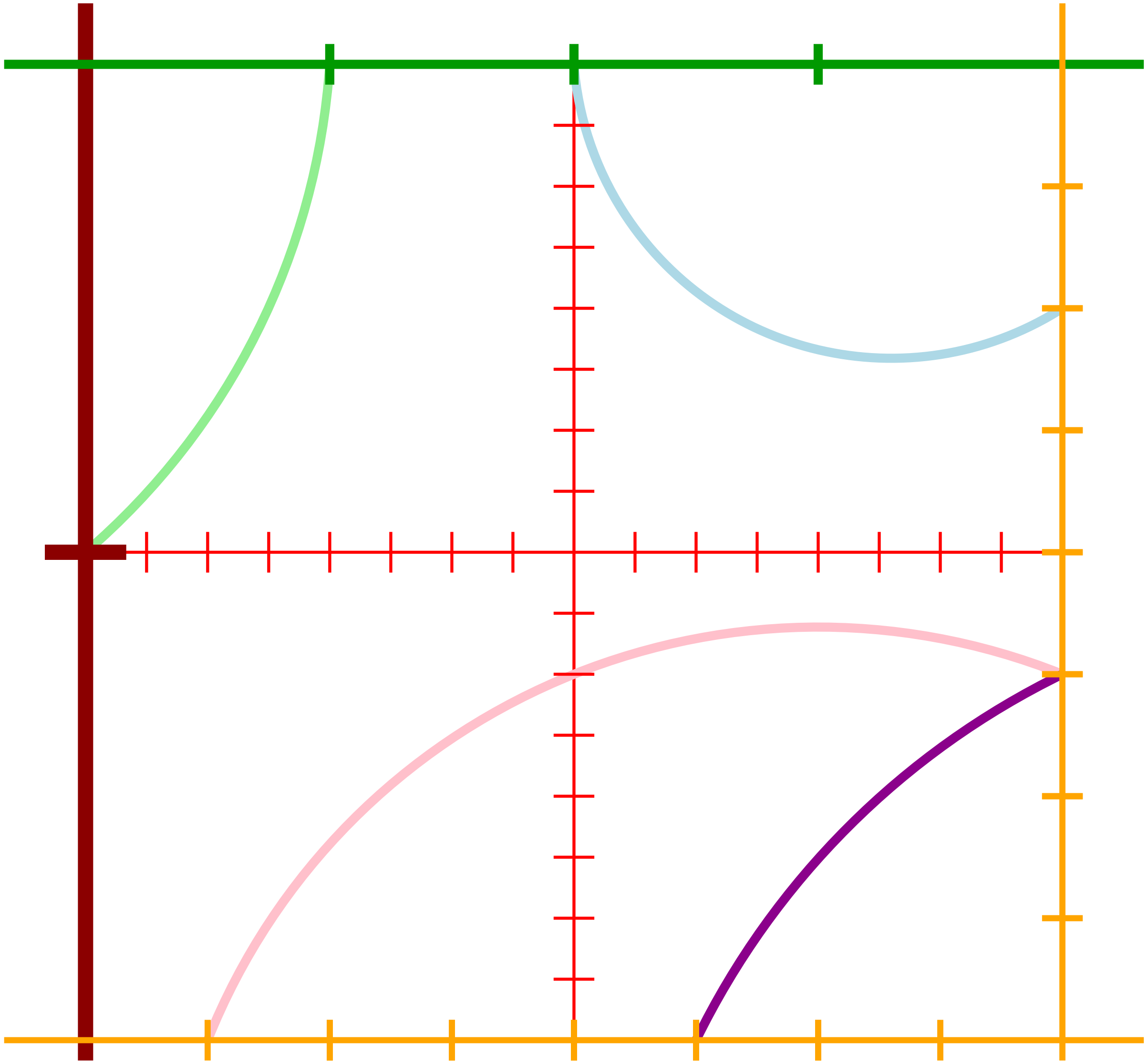

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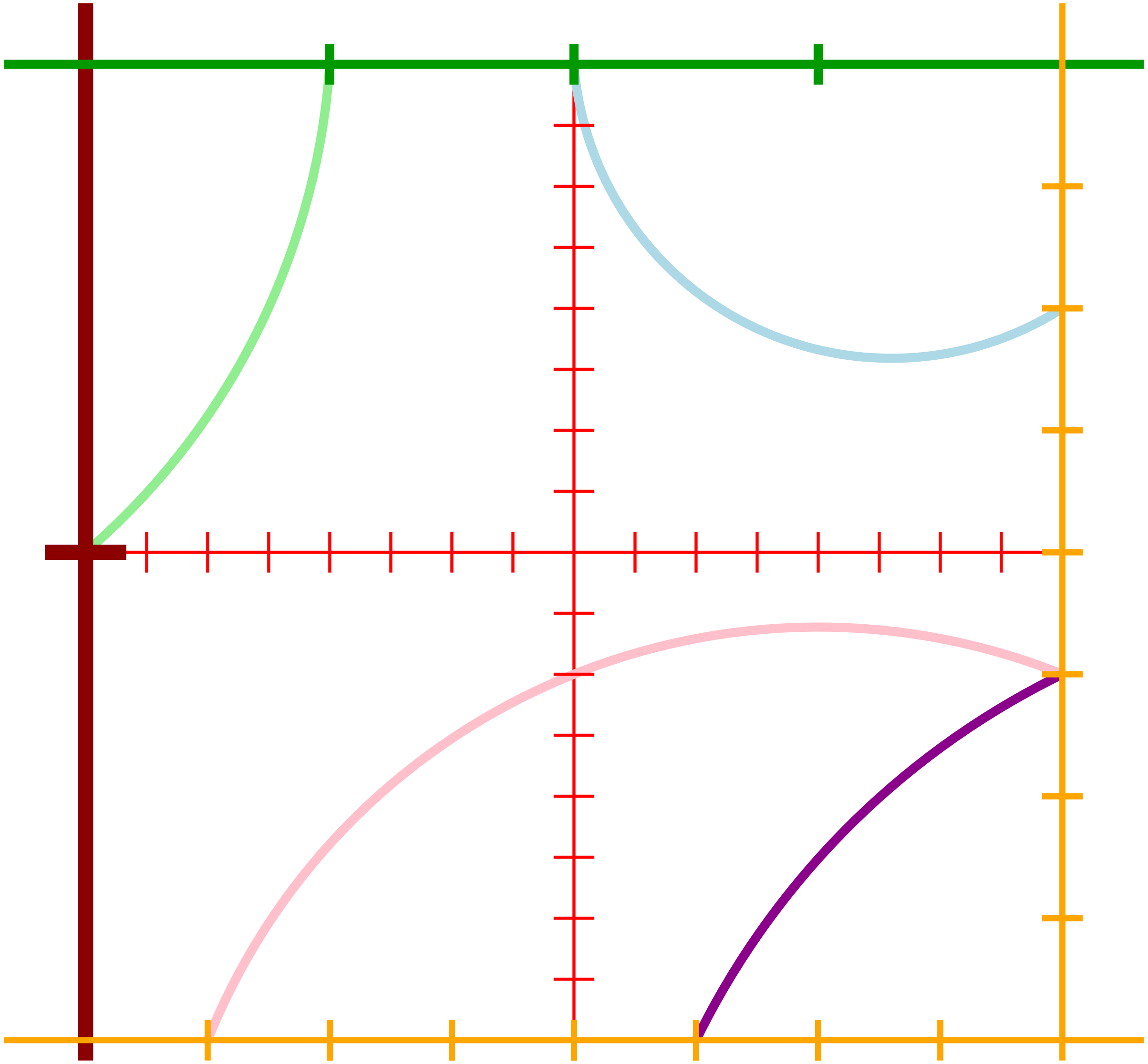

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□

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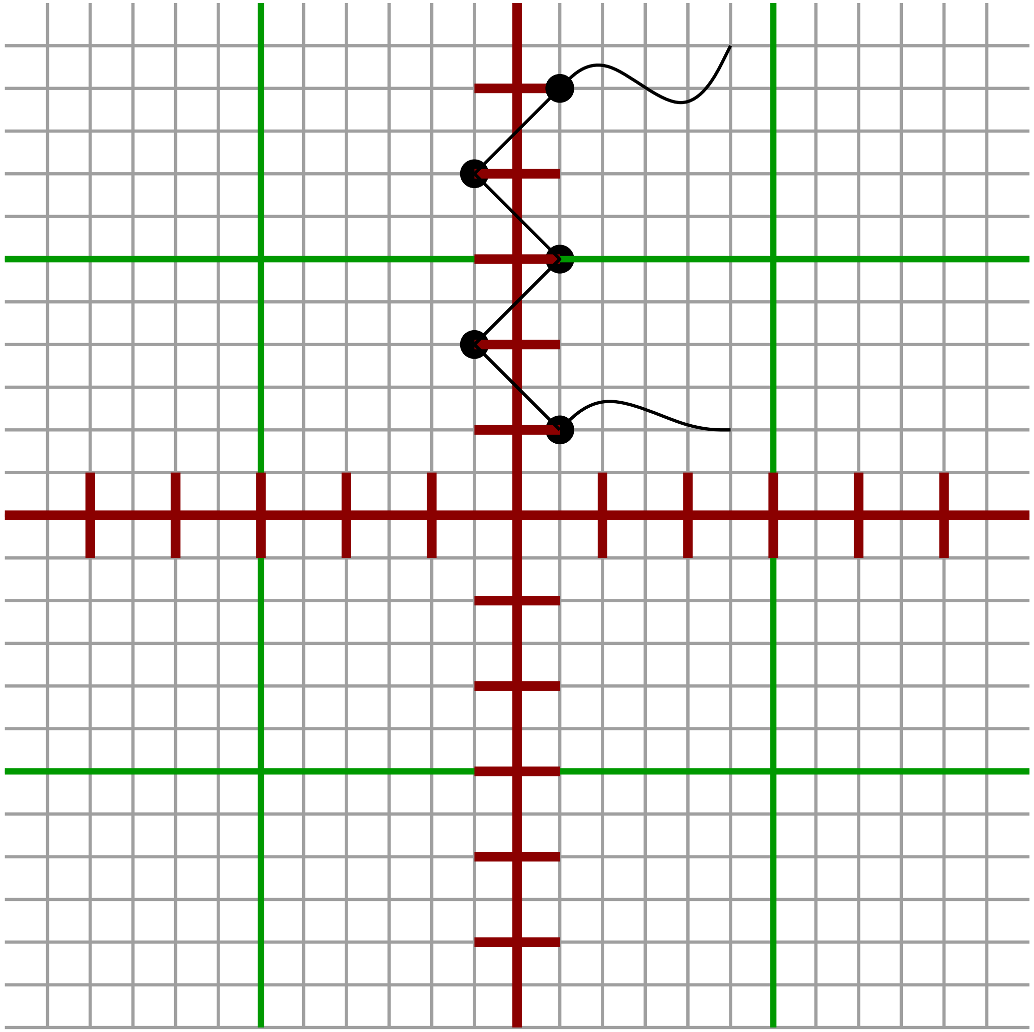

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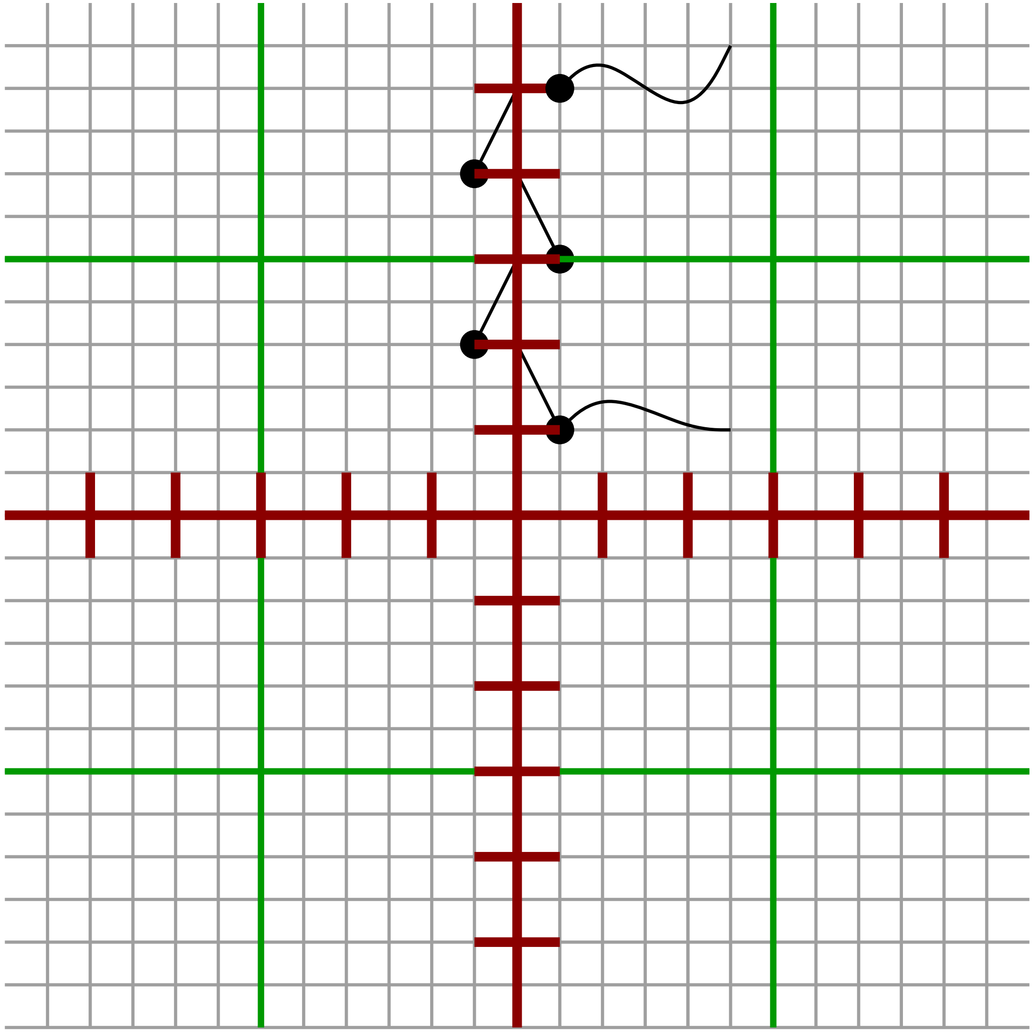

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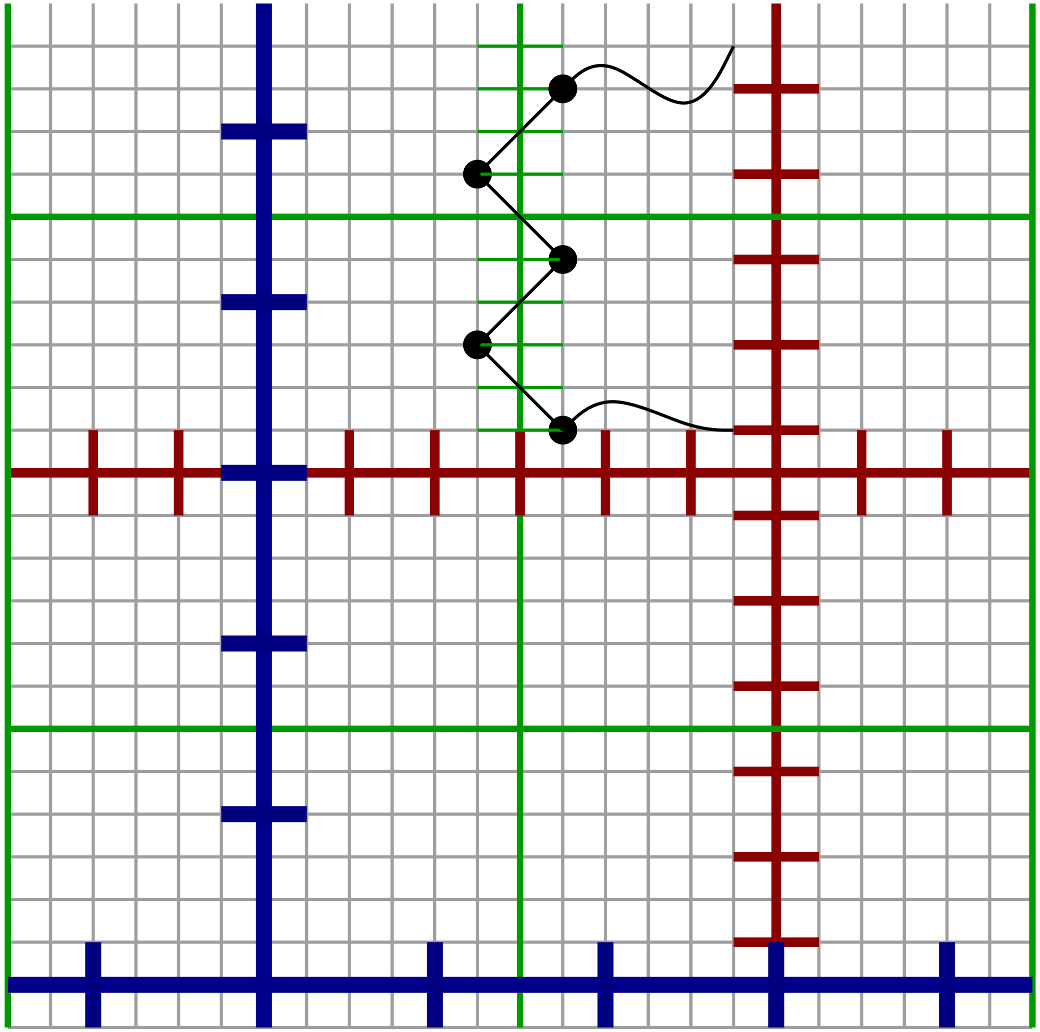

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$$x \mapsto (x + a) \text{ mod } L$$
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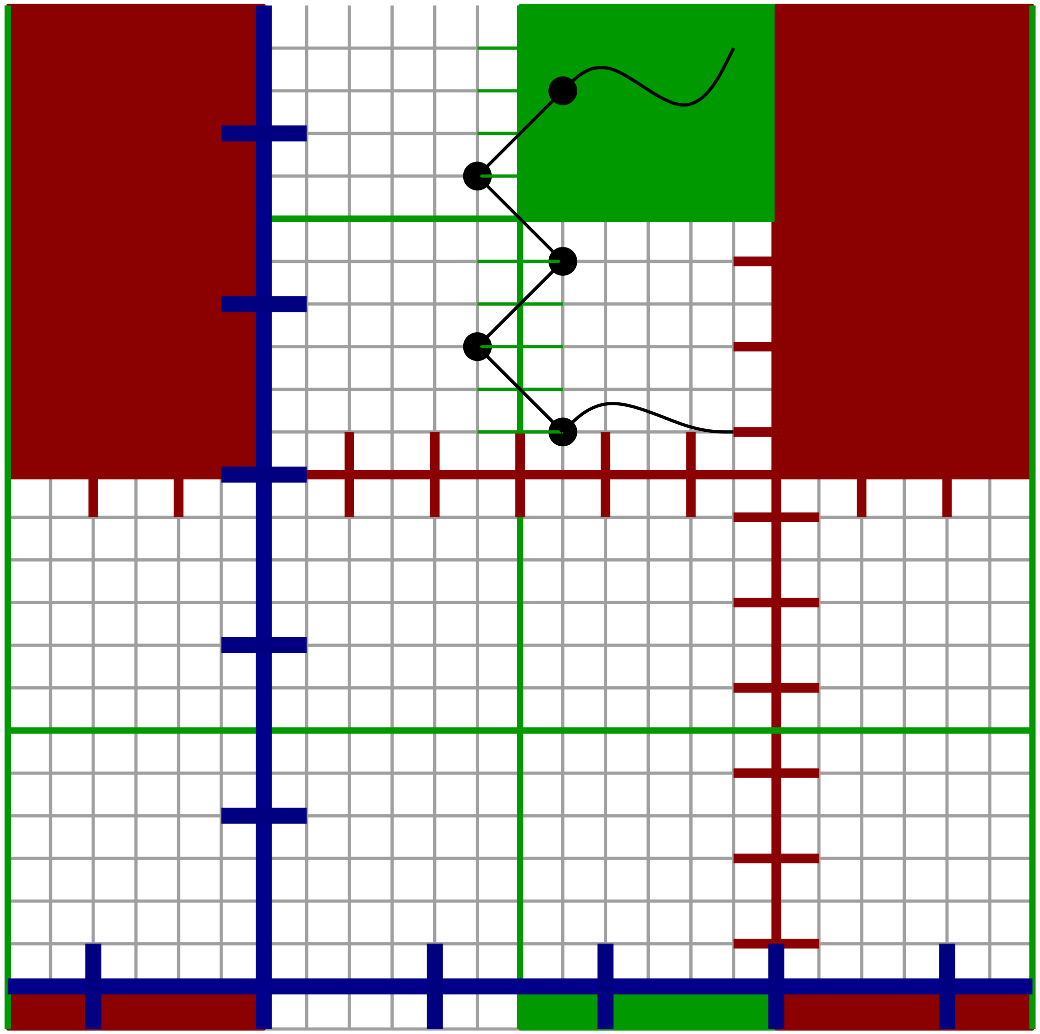

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- Squares in the dissection tree are "wrapped around"
- dynamic program must be modified accordingly.

Shifted Dissections (II)

Lem. Let π be an optimal tour and $N(\pi)$ be the number of crossings with the lines of the $L \times L$-grid. Then we have $N(\pi) \leq \sqrt{2} \cdot \mathsf{OPT}$

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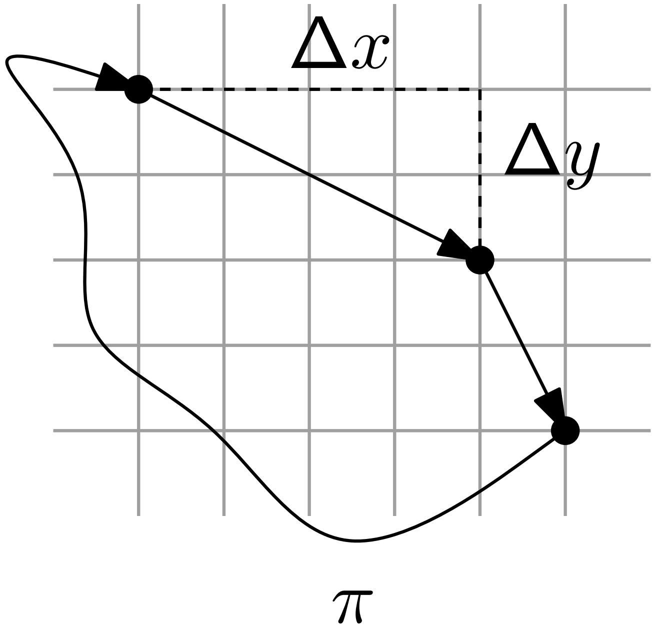
Lem. Let π be an optimal tour and $N(\pi)$ be the number of crossings with the lines of the $L \times L$-grid. Then we have $N(\pi) \leq \sqrt{2} \cdot \mathrm{OPT}$

Proof.

- Consider a tour as an ordered cyclic sequence.



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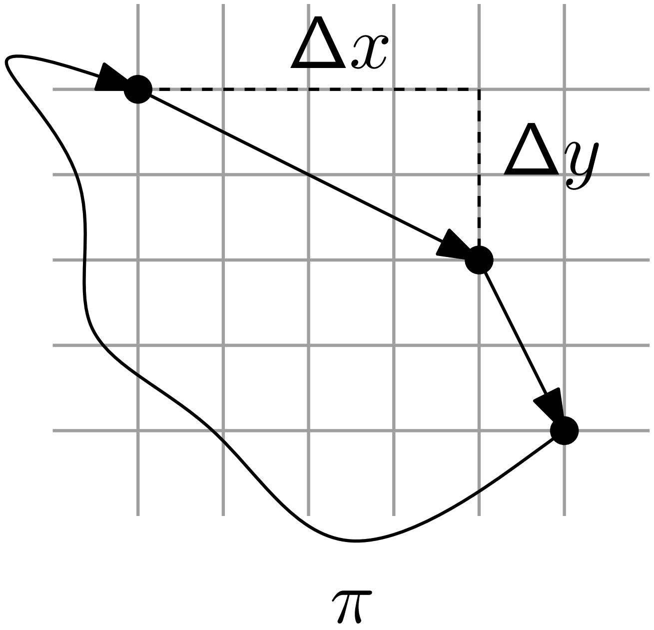

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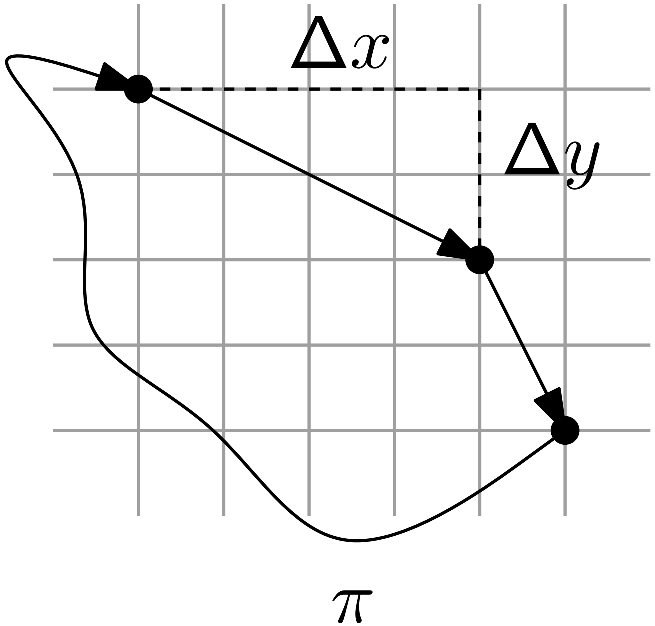

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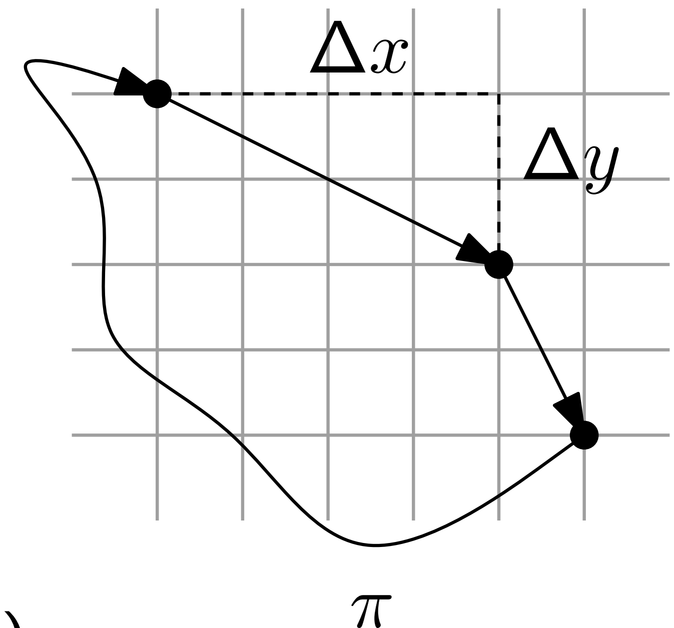

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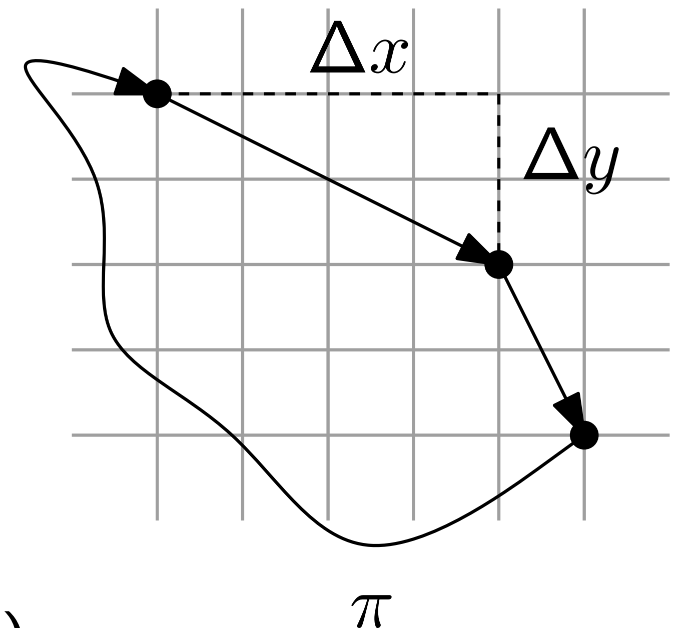

□

Shifted Dissections (III)

Thm. Let $a, b \in [0, L-1]$ be chosen independently and uniformaly at random. Then the expected cost of an optimal well behaved tour with respect to the (a, b)-shifted dissection is at most $(1 + \sqrt{2}\epsilon)$OPT.

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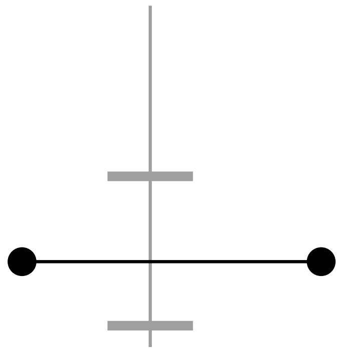

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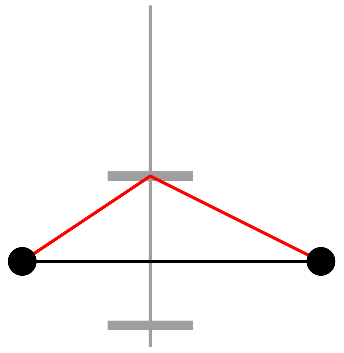

detour per intersection $\leq$ inter-portal distance.

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- Thus, the expected increase in tour length due to this intersection is at most:

$$\sum_{i=0}^{k} \frac{L}{2^i m} \frac{2^i}{L} \leq \frac{k}{m} \leq \epsilon$$

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$\square$

Approximation Scheme.

Thm. There is a PTAS for euclidean TSP, i.e., for each $\epsilon > 0$ one can find a $(1+\epsilon)$-approximation in $n^{O(1/\epsilon)}$ time.

Proof.

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Proof. Try all L^2 different (a, b)-shifted dissections.
By the previous thm., one of these is good enough. $\square$