MS-C1421 Fourier analysis (Aalto University) Turunen / Vuojamo
10.4.2019, 13:00–16:00

Fill in the requested information to every answer sheet.

Calculators and literature forbidden.

This time you may assume the Fourier inverse formula known.
Remember Euler's formula $\mathrm{e}^{\mathrm{i}x} = \cos(x) + \mathrm{i}\sin(x)$.

About grading: Every exam problem will be graded from 0 to 6 points. Harmless small errors do not prevent from getting maximal points. You will get points if your answer contains at least some information (relevant definitions, pictures, calculations etc) — empty answer is surely worth zero. Remember to mention if you use some well-known properties of Fourier transforms.

1. Show that the Fourier integral transform preserves the inner product: in other words, show that

$$\langle \widehat{r}, \widehat{s} \rangle = \langle r, s \rangle$$

holds for "nice enough" non-periodic signals $r, s : \mathbb{R} \to \mathbb{C}$.
What does the conservation of energy mean here?

2. At time $t \in \mathbb{R}$ let

$$s(t) = \sin(3\pi t) + 7\cos(4\pi t).$$

(a) Find the Fourier coefficients of this 1-periodic signal $s : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$. That is, find the Fourier transform $r = \widehat{s} : \mathbb{Z} \to \mathbb{C}$.

(b) Find the energy of s. (Hint: use (a) and conservation of energy.)

(c) Find the Fourier transform $\widehat{r} = \widehat{\widehat{s}}$.

3. Find the discrete Fourier transform $r = \widehat{s}$ for the 4-periodic digital signal $s : \mathbb{Z}/4\mathbb{Z} \to \mathbb{C}$ when

$$s(0) = 6, \quad s(1) = 5, \quad s(2) = 4 \quad \text{and} \quad s(3) = 5.$$

Find also $\widehat{r} = \widehat{\widehat{s}}$. Write your answers real-valued.

4. Let $s(t) = \mathrm{e}^{-\pi t^2}$. Using information $\widehat{s} = s$, find

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}2\pi(t-u)\cdot\nu} \; s(u) \; k(t,u,\nu) \; \mathrm{d}u \; \mathrm{d}\nu,$$

where
(a) $k(t,u,\nu) = s(u)$,
(b) $k(t,u,\nu) = \widehat{s}(\nu)$.