Aalto University
School of Science
Department of Computer Science
Combinatorics of Efficient Computations

Approximation Algorithms

Lecture 10: Scheduling Jobs on Parallel Machines

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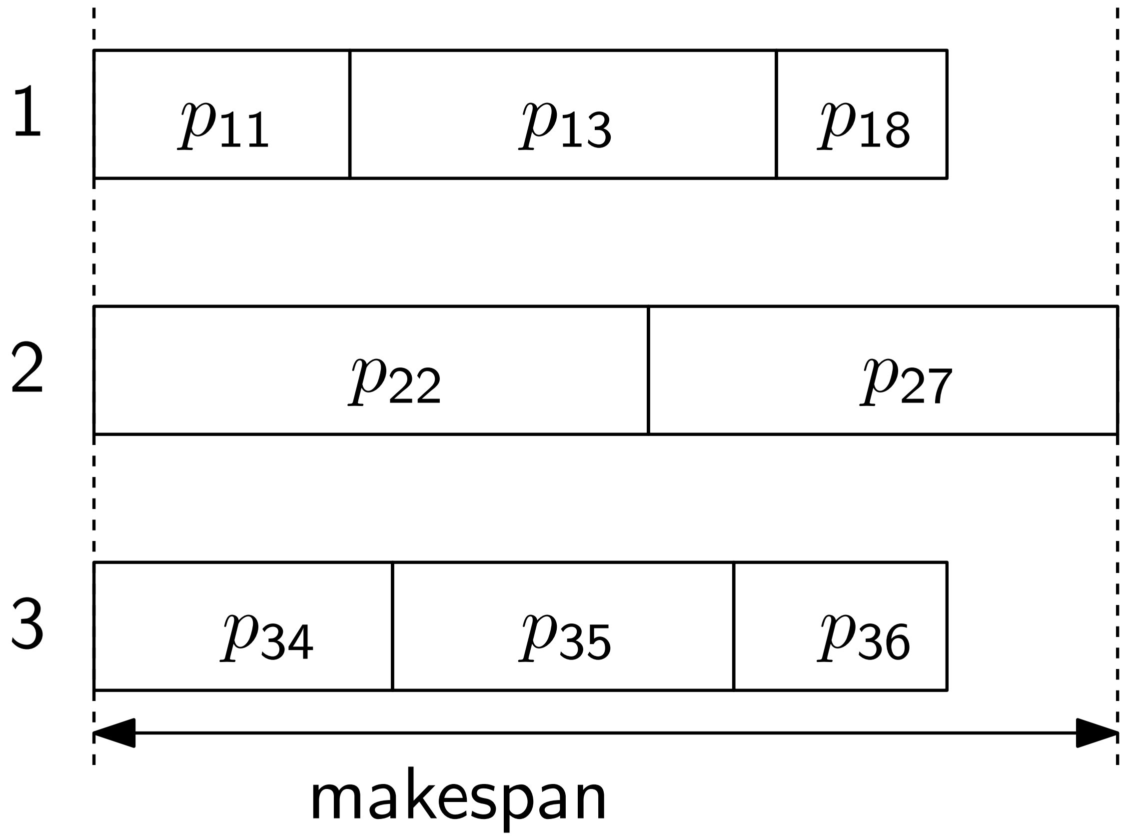
Scheduling on Parallel Machines

Given: A set J of **Jobs**, a set M of **machines** and for each $j \in J$ and $i \in M$ the **processing time** $p_{ij} \in \mathbb{N}^+$ of j on i.

Find: A **Schedule** $\sigma\colon J \to M$ of the jobs on the machines, which minimizes the total time to completion (**makespan**), i.e., minimizes the maximum time a machine is in use.



$J = \{1, 2, \ldots, 8\}$

$M = \{1, 2, 3\}$

A natural ILP

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\text{minimize} \quad & t \\
\text{s.t.} \quad & \sum_{i \in M} x_{ij} = 1, && j \in J \\
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Solution: A job with processing time m and m machines $\rightsquigarrow$ $\mathrm{OPT} = m$ and $\mathrm{OPT}_f = 1$

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Strengthen the ILP $\rightarrow$ implicit (non-linear) constraint:

If $p_{ij} > t$ then set $x_{ij} = 0$

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Define the "pruned" relaxation LP(T)

$$\sum_{i:\,(i,j)\in S_T} x_{ij} = 1, \qquad j \in J$$

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no objective function; just need to determine if a feasible solution exists.

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Use binary search to find the smallest T so that LP(T) has a solution and let T^* be this value of T.

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Lem. 2

Any extreme-point solution to LP(T) must set at least $n - m$ jobs integrally.

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Why is this useful ....?

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Thm. This algorithm is a 2-approximation.
(assuming we have the F-perfect matching)

Pseudo-Trees and -Forests

A connected graph with vertex set V is called a **Pseudo-Tree**, when it has at most $|V|$ edges.

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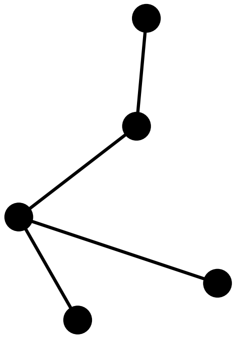

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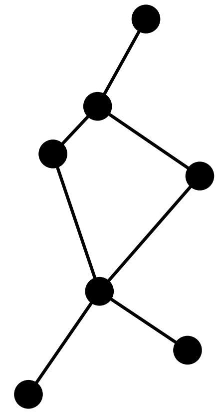

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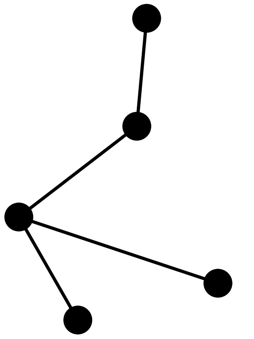

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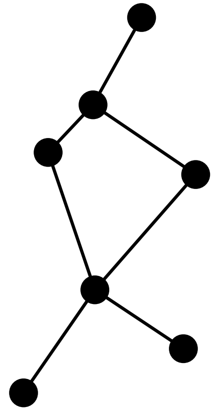

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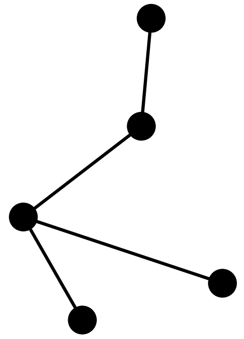

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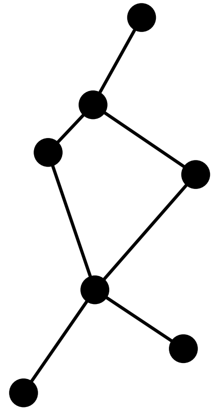

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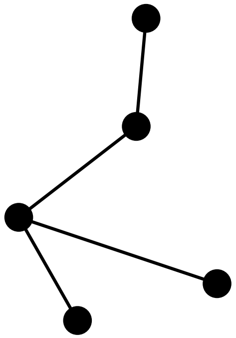

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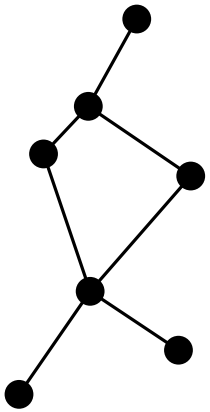

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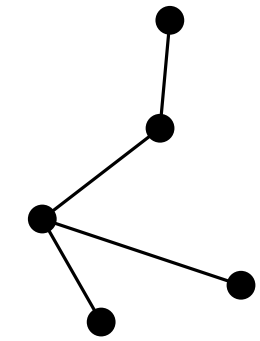

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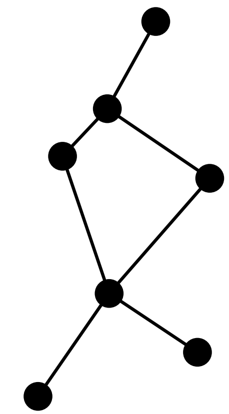

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Algorithm:

- LP(T) has no feasible solutions for any $T < m$.
- extreme-pt. solution: assign $1/m$ of j_1 and $m - 1$ other jobs to each machine. $\rightsquigarrow$ $2m - 1$ makespan.

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