MS-C1650 Numeerinen analyysi, Exercise 4, Guidelines

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May 29, 2019

- These are not model solutions, but "getting started guidelines".
- You are allowed, and supposed, to use MATLAB (or Octave/Python/whatever), unless otherwise stated. It is probably a good idea to bring a laptop with you to the exercise session.
- Ask questions at exercise sessions, or at certain office hours (TBA)
- Errors and stupidities (in these notes): contact lauri.perkkio@aalto.fi

Exercise 1

In the first part, simply check that the funktion $y(t)?t^{3/2}$ solves the initual valua problem.

UPDATE The 23th May: Now check Peano existence theorem and Picard-Lindelöf theorem. In this case, $y'(t) = f(t, y) = \frac{3}{2}y^{1/3}$, so $f(y)$ is continuous but not Lipschitz continuous at $y = 0$. Beacause of this, the solution for this ODE exists/does not exist, and is unique/is not unique?

Second part: Denote $y_k = y(t_k)$ and $y'(t, y) = f(t, y)$ (In our case $f(t, y) = f(y) = y^{1/3}$) . Euler's method:

$$\begin{aligned} y_{k+1} &= y_k + hf(t_k, y_k) \\ &= y_k + hf(y_k) = ... \end{aligned}$$

Choose a time step $h > 0$ and make a table of the values of time (t_k,) the exact values solution $y(t_k)$ and the numerical solutions y_k (for $k = 0, 1, 2, ...$).

Exercise 2

Quadratures are the last week's integration rules of form

$$\int_{x_i}^{x_i+h} f(x)dx \approx h(A_o f(x_0) + ... + A_N f(x_N).$$

Let's do the part a) as an example.

Now $y'(t,y) = f = f(t)$, no dependence on y. $y_{k+1} = y(t_k) + y(t_k + h)$. This means that the solution is just an simple integral

$$\begin{aligned} y_{k+1} - y_k &= \int_{t_k}^{t_k+h} y'(t)dt \\ &= \int_{t_k}^{t_k+h} f(t)dt \\ &\approx \frac{h}{2}(f(t_k) + f(t_k+h)) \end{aligned}$$

So the Heun method corresponds to the quadrature with $A_0 = 1/2$, $A_1 = 1/2$, $x_0 = t_k$ and $x1 = t_k + h$. What is this particular quadrature rule called?

Exercise 3

We are dealing with a group of nonlinear ordinal differential equations

$$y' = f(y),$$

where

$$\begin{aligned} y &= [R, F]^T \\ f(y) &= [(2-F)R, (R-2)F]^T \\ y(0) &= [2, 1]^T. \end{aligned}$$

Let's solve the ODE numerically with the trapeziodal method

$$y_{k+1} = y_k + \frac{h}{2}\big[f(t_k, y_k) + f(t_{k+1}, y_{k+1})\big]. \tag{1}$$

Because we have y_{k+1} on the right side of the equation, the method is implicit and for every time step we have to solve a non linear equation group for y_{k+1} (the previous result y_k is already known). This equation can be solved for example by Newton's method.

We write the equation 1 in form

$$G(x) = x - y_k - \frac{h}{2}\big[f(y_k) + f(x)\big] = 0,$$

where we want to solve $x = y_{k+1}$.

We solve the root $x = [x_1, x_2]^T$ by Newton's method. Let's denote by x^k the result of k:th iteration. (Be careful: x_1 is the first component of the vector root x. However x^1 is itself a vector and the result of first iteration in Newton's method, which seeks to solve the root x.)

We can obtain the initual quess x^0 for example by Euler's method: $x^0 = y_k + hf(y_k)$. The formula of Newton's method for solving the equation group $G(x) = 0$ is:

$$x^{k+1} = x^k - J(G(x^k))^{-1}G(x^k),$$

where $J(F(x^k))$ is the Jacob matrix of G at x^k. Remember that

$$J = \begin{bmatrix} \partial G_1/\partial x_1 & \partial G_1/\partial x_2 \\ \partial G_2/\partial x_1 & \partial G_2/\partial x_2. \end{bmatrix}$$

Note that you have to present only the first step of the iteration in the exercise.

Exercise 4

A multistep method is convergent, if it is consistent and stable. Consistency means that the truncation error is $O(h^m)$, where $m \geq 1$.
Consistency: Lecture notes theorem 8.2.1 multistep method's truncation error (suom. katkaisuvirhe) is of order $p \geq 1$, if and only if

$$\sum_{l=0}^{m} a_l = 0 \qquad \text{and} \qquad \sum_{l=0}^{m} l^j a_l = j \sum_{l=0}^{m} l^{j-1} b_l$$

for all $j = 1, 2, ..., p$. The coefficients are from the definition of the multistep method

$$\sum_{l=0}^{m} a_l y_{k+l} = h \sum_{l=0}^{m} b_m f(t_{k+m}, y_{k+l}).$$

Zero-stableness: Denote by $\delta = \max_{i=0,1,...s}(|y_i - \tilde{y_i}|)$ the biggest error in initual values of y. The method is zero-stable, if there exist constant K and step size $h_0 > 0$ such that

$$|y_n - \tilde{y_n}| \leq K\delta$$

holds when $0 < h \leq h_0$ and $nh \leq T - t_0$. Here y_n, $\tilde{y_n}$ are the exact and numerical solution and y_0, $\tilde{y_0}$ are the corresponding initual values.

UPDATE Thu 23th May: Check root condition" for multistep methods; it is equivalent with zero-stability.

MATLAB

A long answer: find a book with "dynamical systems" and "numerical methods" in its topic.

A short answer:

For part a) you can choose an explicit method for example from the examples in the Lecture notes (Section 7). Analyze its stability and rate of convergence.

For part b) read carefully the hints for exercise 3 (this week). Form the correspondig equation group and follow the instruction. Note that in this exercise you are asked not to use the trapeziodal method, but implicit Euler's method

$$y_{k+1} = y_k + hf(t_{k+1}, y_{k+1}).$$

You can analyze the different behaviour of the methods with respect to the time step h for example by plotting the results obtained with different values of h.

CHALLENGE

"Challenge"-exercises are not discussed in the classroom, and they are not graded (as such). I.e., it is possible to get full points without returning these problems.