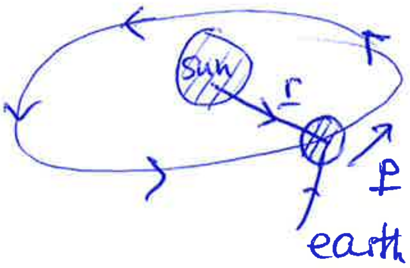
Angular momentum

In classical physics, the orbital angular momentum of a particle is given by

$$\underline{L} = \underbrace{\underline{r}}_{\text{position}} \times \underbrace{\underline{p}}_{\text{momentum}}$$



$$= \begin{vmatrix} \underline{x} & \underline{y} & \underline{z} \\ x & y & z \\ p_x & p_y & p_z \end{vmatrix}$$

$$= \underline{x}\, y p_z + \underline{y}\, z p_x + \underline{z}\, x p_y - \underline{x}\, z p_y - \underline{y}\, x p_z - \underline{z}\, y p_x$$

$$= \begin{pmatrix} y p_z - z p_y \\ z p_x - x p_z \\ x p_y - y p_x \end{pmatrix}$$

$\rightarrow L_x = y p_z - z p_y, \quad L_y = z p_x - x p_z, \quad L_z = x p_y - y p_x$

To obtain the corresponding quantum mechanical operators, we replace p_α by $-i\hbar \partial_\alpha$; i.e.

$$\hat{L}_x = \frac{\hbar}{i}\left(y \partial_z - z \partial_y\right)$$

$$\hat{L}_y = \frac{\hbar}{i}\left(z \partial_x - x \partial_z\right)$$

$$\hat{L}_z = \frac{\hbar}{i}\left(x \partial_y - y \partial_x\right)$$

We can start by working out the commutation relations of the operators, i.e. $[\hat{L}_x, \hat{L}_y]$ etc.

To evaluate the commutator, we apply it to a test function ψ:

$$[\hat{L}_x, \hat{L}_y]\psi = \left(\frac{\hbar}{i}\right)^2 \Big[(y\partial_z - z\partial_y)(z\partial_x - x\partial_z) - (z\partial_x - x\partial_z)(y\partial_z - z\partial_y) \Big] \psi$$

$$= \left(\frac{\hbar}{i}\right)^2 \Big[y\partial_z z\partial_x - y\partial_z x\partial_z - z\partial_y z\partial_x + z\partial_y x\partial_z - z\partial_x y\partial_z + z\partial_x z\partial_y + x\partial_z y\partial_z - x\partial_z z\partial_y \Big] \psi$$

$$= \left(\frac{\hbar}{i}\right)^2 \Big[y\partial_x\psi + yz\partial_z\partial_x\psi - yx\partial_z^2\psi - z^2\partial_y\partial_x\psi + zx\partial_y\partial_z\psi - zy\partial_x\partial_z\psi + z^2\partial_x\partial_y\psi + xy\partial_z^2\psi - x\partial_y\psi - xz\partial_z\partial_y\psi \Big]$$

$$= \left(\frac{\hbar}{i}\right)^2 \Big[y\partial_x - x\partial_y + \overbrace{(yz\partial_z\partial_x - zy\partial_x\partial_z)}^{0} + (xy\partial_z^2 - yx\partial_z^2) + (z^2\partial_x\partial_y - z^2\partial_y\partial_x) + (zx\partial_y\partial_z - xz\partial_z\partial_y) \Big] \psi$$

$$= \left(\frac{\hbar}{i}\right)^2 \big[y\partial_x - x\partial_y \big] \psi = \frac{\hbar}{i}(-\hat{L}_z)\psi$$

Thus, we find

$$[\hat{L}_x, \hat{L}_y] = i\hbar \hat{L}_z .$$

Similar arguments lead to the expressions

$$[\hat{L}_y, \hat{L}_z] = i\hbar \hat{L}_x \quad \& \quad [\hat{L}_z, \hat{L}_x] = i\hbar \hat{L}_y$$

While the individual components of the angular momentum do not commute, the situation is different for the square of the angular momentum:

$$\hat{L}^2 \equiv \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$$

To see this, we evaluate the commutator

$$[\hat{L}^2, \hat{L}_x] = \underbrace{[\hat{L}_x^2, \hat{L}_x]}_{0} + [\hat{L}_y^2, \hat{L}_x] + [\hat{L}_z^2, \hat{L}_x]$$

$$= \hat{L}_y^2 \hat{L}_x - \hat{L}_x \hat{L}_y^2 + \hat{L}_z^2 \hat{L}_x - \hat{L}_x \hat{L}_z^2$$

$$= \hat{L}_y \hat{L}_y \hat{L}_x - \hat{L}_y \hat{L}_x \hat{L}_y + \hat{L}_y \hat{L}_x \hat{L}_y - \hat{L}_x \hat{L}_y \hat{L}_y$$

$$+ \hat{L}_z \hat{L}_z \hat{L}_x - \hat{L}_z \hat{L}_x \hat{L}_z + \hat{L}_z \hat{L}_x \hat{L}_z - \hat{L}_x \hat{L}_z \hat{L}_z$$

$$= \hat{L}_y [\hat{L}_y, \hat{L}_x] + [\hat{L}_y, \hat{L}_x] \hat{L}_y + \hat{L}_z [\hat{L}_z, \hat{L}_x] + [\hat{L}_z, \hat{L}_x] \hat{L}_z$$

$$= \hat{L}_y(-i\hbar\hat{L}_z) + (-i\hbar\hat{L}_z)\hat{L}_y + \hat{L}_z(i\hbar\hat{L}_y) + (i\hbar\hat{L}_y)\hat{L}_z$$

$$= 0$$

In a similar way, one finds

$$[\hat{L}^2, \hat{L}_x] = [\hat{L}^2, \hat{L}_y] = [\hat{L}^2, \hat{L}_z] = 0$$

Since $\hat{L}^2$ and $\hat{L}_\alpha$ commute we can hope to find a common set of eigenvectors.

To see this, assume that $\hat{L}^2$ has non-degenerate eigenvalues of the form

$$\hat{L}^2|\psi_n\rangle = \lambda_n|\psi_n\rangle$$

[The degenerate case is discussed in Zettili p. 177]

Now, we have

$$\underbrace{\langle\psi_m|\underbrace{[\hat{L}^2, \hat{L}_\alpha]}_{0}|\psi_n\rangle}_{0\,=} = \langle\psi_m|\hat{L}^2\hat{L}_\alpha|\psi_n\rangle - \langle\psi_m|\hat{L}_\alpha\hat{L}^2|\psi_n\rangle$$

$$= \lambda_m\langle\psi_m|\hat{L}_\alpha|\psi_n\rangle - \lambda_n\langle\psi_m|\hat{L}_\alpha|\psi_n\rangle$$

$$= (\lambda_m - \lambda_n)\langle\psi_m|\hat{L}_\alpha|\psi_n\rangle$$

$\Rightarrow \langle\psi_m|\hat{L}_\alpha|\psi_n\rangle \propto \delta_{mn}$

$\Rightarrow \hat{L}_\alpha$ is diagonal in the basis $\{|\psi_n\rangle\}$ ✓

To find the eigenvalues, we introduce ladder operators (cf. the harmonic oscillator),

$$\hat{L}_{\pm} \equiv \hat{L}_x \pm i \hat{L}_y$$

The commutator with $\hat{L}_z$ reads

$$\begin{aligned}[\hat{L}_z, \hat{L}_{\pm}] &= [\hat{L}_z, \hat{L}_x] \pm i [\hat{L}_z, \hat{L}_y] \\ &= i\hbar \hat{L}_y \pm i (-i\hbar \hat{L}_x) \\ &= \pm \hbar (\hat{L}_x \pm i \hat{L}_y) \\ &= \pm \hbar \hat{L}_{\pm}\end{aligned}$$

and we also have $[\hat{L}^2, \hat{L}_{\pm}] = 0$.

Let us know consider an eigenstate ψ_n of $\hat{L}^2$.
We easily see that

$$\hat{L}^2 (L_{\pm}\psi) = L_{\pm} (\hat{L}^2 \psi) = L_{\pm} \lambda_n \psi = \lambda_n L_{\pm} \psi$$

so that $L_{\pm}\psi$ is also an eigenstate of $\hat{L}^2$.
with eigenvalue λ_n.

We also see that

$$\begin{aligned}\hat{L}_z (L_{\pm}\psi) &= \hat{L}_z (L_{\pm}\psi) + L_{\pm} L_z \psi - L_{\pm} L_z \psi \\ &= [\hat{L}_z, \hat{L}_{\pm}] \psi + \hat{L}_{\pm} \hat{L}_z \psi\end{aligned}$$

$$= \pm\hbar \hat{L}_\pm \psi + L_\pm \underbrace{L_z \psi}_{= \mu_n \psi}$$

$$= (\mu_n \pm \hbar) \hat{L}_\pm \psi$$

$\Rightarrow$ $L_\pm \psi$ is an eigenvector of $\hat{L}_z$ with eigenvalue $\mu_n \pm \hbar$

$\Rightarrow$ $L_\pm$ lowers (−) / raises (+) the eigenvalue by $\hbar$

Based on physical arguments, the z-component of the angular momentum cannot become arbitrarily large (positive or negative); it cannot exceed the total angular momentum. Thus, at some point, we must reach the largest value of the angular momentum, such that

$$\hat{L}_+ \psi_{max} = 0$$

We write the corresponding eigenvalue as

$$\hat{L}_z \psi_{max} = \hbar l \psi_{max},$$

where l is known as the azimuthal quantum number.

We recall that $\hat{L}^2 \psi_{max} = \lambda_n \psi_{max}$

We also have that

$$\hat{L}_\pm \hat{L}_\mp = (\hat{L}_x \pm i\hat{L}_y)(\hat{L}_x \mp i \hat{L}_y)$$

$$= \hat{L}_x^2 + \hat{L}_y^2 \mp i(\hat{L}_x\hat{L}_y - \hat{L}_y\hat{L}_x)$$

$$= \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2 - \hat{L}_z^2 \mp i(i\hbar\hat{L}_z)$$

$$= \hat{L}^2 - \hat{L}_z^2 \mp i(i\hbar\hat{L}_z)$$

$$\Rightarrow \hat{L}^2 = \hat{L}_\pm \hat{L}_\mp + \hat{L}_z^2 \mp \hbar \hat{L}_z$$

It then follows that

$$\hat{L}^2 \psi_{max} = (\hat{L}_-\hat{L}_+ + \hat{L}_z^2 + \hbar\hat{L}_z)\psi_{max}$$

$$= (0 + \hbar^2\ell^2 + \hbar^2\ell)\psi_{max}$$

$$= \underbrace{\hbar^2\ell(\ell+1)}_{\lambda_n}\psi_{max}$$

$$\Rightarrow \quad \lambda_n = \hbar^2\ell(\ell+1)$$

For the lowest eigenvalue, we similarly have

$$\hat{L}_-\psi_{min} = 0 \;, \quad \hat{L}_z\psi_{min} = \hbar\tilde{\ell}\psi_{min}$$

and $\quad \hat{L}^2\psi_{min} = \lambda_n\psi_{min}$

In this case, we use that

$$\hat{L}^2 \psi_{min} = (\hat{L}_+\hat{L}_- + \hat{L}_z^2 - \hbar \hat{L}_z)\psi_{min}$$
$$= (0 + (\hbar\tilde{\ell})^2 - \hbar\hbar\tilde{\ell})\psi_{min}$$
$$= \hbar^2\tilde{\ell}(\tilde{\ell}-1)\psi_{min} \Rightarrow \lambda_n = \hbar^2\tilde{\ell}(\tilde{\ell}-1)$$

We can now conclude that

$$\lambda_n = \hbar^2 \ell(\ell+1) = \hbar^2\tilde{\ell}(\tilde{\ell}-1)$$
$$\Rightarrow \tilde{\ell} = \ell+1 \text{ or } \tilde{\ell} = -\ell \Rightarrow \tilde{\ell} = -\ell \text{ since } \tilde{\ell} < \ell$$

Thus, the eigenvalues of $\hat{L}_z$ are $m\hbar$, where m goes from $-\ell$ to $+\ell$ in N integer steps, i.e.

$\ell = -\ell + N \Rightarrow \ell = N/2$, which must be an integer or a half-integer. Hence, we find that the eigenvalues are

$$\hat{L}^2 \psi_{m,\ell} = \hbar^2 \ell(\ell+1)\psi_{m,\ell}, \quad \hat{L}_z \psi_{m,\ell} = \hbar m \psi_{m,\ell},$$

where $\ell = 0, 1/2, 1, 3/2, \ldots$ & $m = -\ell, -\ell+1, \ldots, \ell-1, \ell$

For a given value of ℓ, there are $2\ell+1$ values of m.

The eigenfunctions $\psi_{m,\ell}$ are the spherical harmonics $Y_{\ell,m}(\theta,\varphi)$, see Zettili, sec. 5.7.

We shall now argue that ℓ must be an integer for orbital angular momentum. (More generally, it can also be a half-integer!)

To see that m and ℓ must be integers for orbital angular momentum, we consider again $\hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x$ with $[\hat{x},\hat{p}_x] = [\hat{y},\hat{p}_y] = i\hbar$.

We now define the operators

$$\hat{q}_1 = \frac{1}{\sqrt{2}}\left(\hat{x} + \hat{p}_y/\hbar\right)$$

$$\hat{q}_2 = \frac{1}{\sqrt{2}}\left(\hat{x} - \hat{p}_y/\hbar\right)$$

$$\hat{p}_1 = \frac{1}{\sqrt{2}}\left(p_x/\hbar - y\right)$$

$$\hat{p}_2 = \frac{1}{\sqrt{2}}\left(p_x/\hbar + y\right)$$

With the commutators $[\hat{q}_1,\hat{q}_2] = [\hat{p}_1,\hat{p}_2] = 0$, $[\hat{q}_1,\hat{p}_2] = [\hat{q}_2,\hat{p}_1] = 0$, and $[\hat{q}_1,\hat{p}_1] = [\hat{q}_2,\hat{p}_2] = i$

We then have

$$\hat{x} = \frac{1}{\sqrt{2}}\left(\hat{q}_1 + \hat{q}_2\right); \quad \frac{\hat{p}_y}{\hbar} = \frac{1}{\sqrt{2}}\left(\hat{q}_1 - \hat{q}_2\right)$$

$$\hat{y} = \frac{1}{\sqrt{2}}\left(\hat{p}_2 - \hat{p}_1\right); \quad \frac{\hat{p}_x}{\hbar} = \frac{1}{\sqrt{2}}\left(\hat{p}_2 + \hat{p}_1\right)$$

and

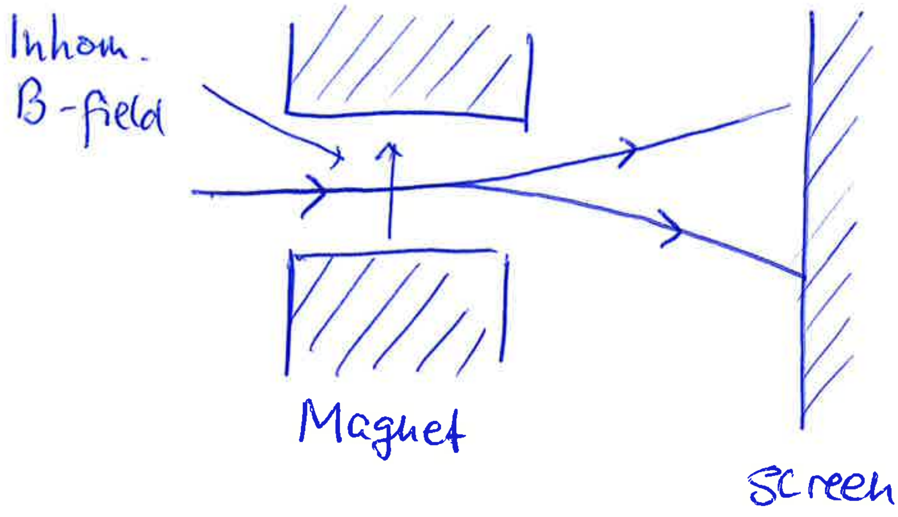
$$\frac{\hat{L}_z}{\hbar} = \frac{1}{2}\left(\hat{q}_1 + \hat{q}_2\right)\left(\hat{q}_1 - \hat{q}_2\right) - \frac{1}{2}\left(\hat{p}_2 - \hat{p}_1\right)\left(\hat{p}_2 + \hat{p}_1\right)$$

$$= \frac{1}{2}\left(\hat{q}_1^2 + \hat{p}_1^2\right) - \frac{1}{2}\left(\hat{q}_2^2 + \hat{p}_2^2\right)$$

These are two harmonic oscillators with $M = 1$ and $\omega_0 = 1$

$$\Rightarrow \quad \frac{E_z}{\hbar} = \left(n_1 + \frac{1}{2}\right) - \left(n_2 + \frac{1}{2}\right) = n_1 - n_2 = \underline{\text{integer!}}$$

The Stern-Gerlach experiment and spin



In 1922, Stern and Gerlach observed that a beam of silver atoms were split into two when passing through a non-uniform magnetic field. The atoms were in the groundstate with vanishing orbital angular momentum, so no splitting was epected.

(A non-uniform magnetic field will exhibit a force on a magnetic dipole; $\underline{F} = \underline{\nabla}(\underline{\mu} \cdot \underline{B})$)

In the case of silver, only the outermost electron is unpaired, and it is the angular momentum of this electron that causes the beam to split into two. This observation led Goudsmit and Uhlenbeck to postulate that the electron has an additional intrinsic angular momentum called spin

The magnetic dipole moment of the spin is

$$\bar{\mu}_s = -g_s \frac{e}{2m_e c} \bar{S},$$

where $g_s \simeq 2$ is called the Landé factor, and

$$\bar{S} = \begin{pmatrix} \hat{S}_x \\ \hat{S}_y \\ \hat{S}_z \end{pmatrix} = \frac{\hbar}{2} \begin{pmatrix} \hat{\sigma}_x \\ \hat{\sigma}_y \\ \hat{\sigma}_z \end{pmatrix}$$

in terms of the Pauli matrices $\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z$.

Notice how the spin operators fulfill the algebra of angular momentum, namely

$$[\hat{S}_x, \hat{S}_y] = i\hbar \hat{S}_z$$

$$[\hat{S}_y, \hat{S}_z] = i\hbar \hat{S}_x$$

$$[\hat{S}_z, \hat{S}_x] = i\hbar \hat{S}_y$$

Moreover, bosons (e.g. photons, phonons, magnons) have integer spin, $s = 0, 1, 2, \ldots$, while fermions (e.g. electrons, protons, neutrons) have half-integer spin, $s = 1/2, 3/2, 5/2, \ldots$, in particular $s = 1/2$ for electrons.