Entanglement:

Let us again consider two electrons in a double-well potential, assuming that the spins are in the (singlet or triplet) state

$$|\Psi_{S/T}\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_L |\downarrow\rangle_R \pm |\downarrow\rangle_L |\uparrow\rangle_R\right),$$

where $|\theta = \uparrow,\downarrow\rangle_{\alpha=L,R}$ denotes the spin of the electron in the left/right potential well.

As an aside, we see that we can write the state as

$$|\Psi_{S/T}\rangle = \frac{1}{\sqrt{2}}\Big[\left(|L\rangle_1 |R\rangle_2 |\uparrow\rangle_1 |\downarrow\rangle_2 - |R\rangle_1 |L\rangle_2 |\downarrow\rangle_1 |\uparrow\rangle_2\right)$$
$$\pm \left(|L\rangle_1 |R\rangle_2 |\downarrow\rangle_1 |\uparrow\rangle_2 - |R\rangle_1 |L\rangle_2 |\uparrow\rangle_1 |\downarrow\rangle_2\right)\Big]$$

$$= \frac{1}{\sqrt{2}}\Big[|L\rangle_1 |R\rangle_2 \left(|\uparrow\rangle_1 |\downarrow\rangle_2 \pm |\downarrow\rangle_1 |\uparrow\rangle_2\right)$$
$$- |R\rangle_1 |L\rangle_2 \left(|\downarrow\rangle_1 |\uparrow\rangle_2 \pm |\uparrow\rangle_1 |\downarrow\rangle_2\right)\Big]$$

$$= \frac{1}{\sqrt{2}}\Big[|L\rangle_1 |R\rangle_2 \left(|\uparrow\rangle_1 |\downarrow\rangle_2 \pm |\downarrow\rangle_1 |\uparrow\rangle_2\right)$$
$$\mp |R\rangle_1 |L\rangle_2 \left(|\uparrow\rangle_1 |\downarrow\rangle_2 \pm |\downarrow\rangle_1 |\uparrow\rangle_2\right)\Big]$$

$$= \frac{1}{\sqrt{2}}\left(|L\rangle_1 |R\rangle_2 \mp |R\rangle_1 |L\rangle_2\right)\left(|\uparrow\rangle_1 |\downarrow\rangle_2 \pm |\downarrow\rangle_1 |\uparrow\rangle_2\right)$$

The state

$$|\Psi_{S/T}\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_L |\downarrow\rangle_R \pm |\downarrow\rangle_L |\uparrow\rangle_R \right)$$

is interesting because it cannot be written on the product form $|\theta\rangle_L |\theta'\rangle_R$.

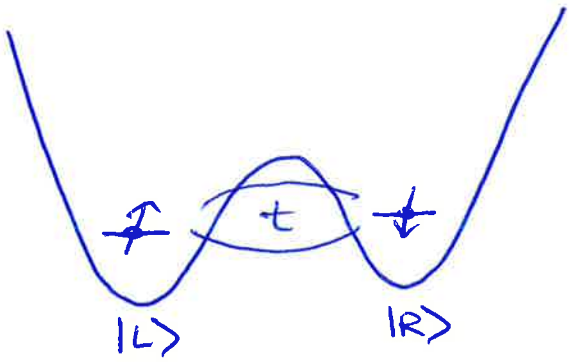
It is an <u>entangled</u> state. Before we discuss some of the peculiar features of entanglement, let us see how entanglement can be created. We begin by assuming that two electrons occupy two well-separated potential wells.

We measure the spins in each well and find, say,

$$|\psi\rangle = |\uparrow\rangle_L |\downarrow\rangle_R$$

(If any other direction is found, we can use electron spin resonance to flip the spins and prepare the spin state above)

Next, we bring the two electrons together, so that they can tunnel between the wells.



Now the spins interact due to the exchange coupling

$$\hat{\mathcal{H}} = -J\, \hat{\underline{S}}_1 \cdot \hat{\underline{S}}_2 \quad \text{with } J = \frac{4t^2}{U}$$

The time-evolution of the spins reads (omitting overall phase factors in several places)

$$|\psi(t)\rangle = e^{-i\hat{\mathcal{H}}t/\hbar}\, |\uparrow\rangle_L |\downarrow\rangle_R$$

$$= e^{-i\hat{\mathcal{H}}t/\hbar} \frac{1}{\sqrt{2}} \Big[\frac{1}{\sqrt{2}} \big(|\uparrow\rangle_L |\downarrow\rangle_R + |\downarrow\rangle_L |\uparrow\rangle_R \big) + \frac{1}{\sqrt{2}} \big(|\uparrow\rangle_L |\downarrow\rangle_R - |\downarrow\rangle_L |\uparrow\rangle_R \big) \Big]$$

$$= e^{-i\hat{\mathcal{H}}t/\hbar} \frac{1}{\sqrt{2}} \big[|\text{triplet}\rangle + |\text{singlet}\rangle \big]$$

$$= \frac{1}{\sqrt{2}} \big[|\text{triplet}\rangle + e^{+iJt/\hbar} |\text{singlet}\rangle \big]$$

We then see that if $Jt/\hbar = \pi \rightarrow t^* = \frac{\pi\hbar}{J}$,

we get $|\psi(t^*)\rangle = \frac{1}{\sqrt{2}} \big[|\text{triplet}\rangle - |\text{singlet}\rangle \big]$

$$= \frac{1}{2} \big[|\downarrow\rangle_L |\uparrow\rangle_R + |\downarrow\rangle_L |\uparrow\rangle_R \big] = |\downarrow\rangle_L |\uparrow\rangle_R$$

In this case, we have <u>swapped</u> the state of the spins

Instead, if we consider $t^*/2 = \frac{\pi}{2}\frac{\hbar}{J}$, we get

$$|\psi(t^*/2)\rangle = \frac{1}{\sqrt{2}}\Big[\,|\text{triplet}\rangle + i\,|\text{singlet}\rangle\,\Big]$$

$$= \frac{1}{\sqrt{2}}\Big[\frac{1}{\sqrt{2}}\big(|\uparrow\rangle_L|\downarrow\rangle_R + |\downarrow\rangle_L|\uparrow\rangle_R\big) + \frac{i}{\sqrt{2}}\big(|\uparrow\rangle_L|\downarrow\rangle_R - |\downarrow\rangle_L|\uparrow\rangle_R\big)\Big]$$

$$= \frac{1}{2}\Big[(1+i)\,|\uparrow\rangle_L|\downarrow\rangle_R + (1-i)\,|\downarrow\rangle_L|\uparrow\rangle_R\Big]$$

$$= \frac{1+i}{2}\Big[\,|\uparrow\rangle_L|\downarrow\rangle_R + \frac{1-i}{1+i}\,|\downarrow\rangle_L|\uparrow\rangle_R\Big]$$

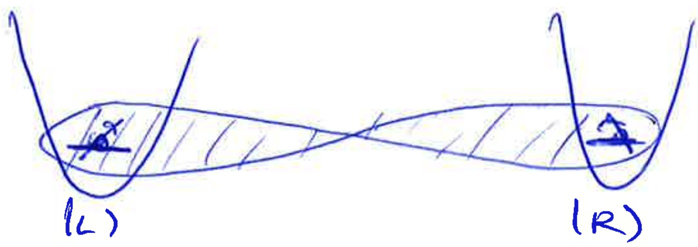
overall phase factor $\times\frac{1}{\sqrt{2}}$ which can be omitted

Now we use that $\frac{1-i}{1+i} = \frac{(1-i)^2}{(1+i)(1-i)} = \frac{1-1-2i}{1+1} = -i$,

so we find

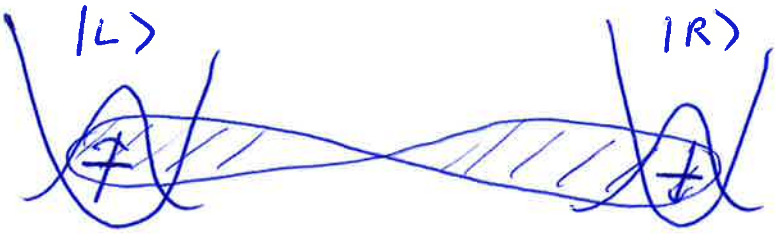
$$|\psi(t^*/2)\rangle = \frac{1}{\sqrt{2}}\Big[\,|\uparrow\rangle_L|\downarrow\rangle_R - i\,|\downarrow\rangle_L|\uparrow\rangle_R\Big]$$

which is an entangled spin state. Finally, by moving the two electrons apart, we have created long-distance entanglement



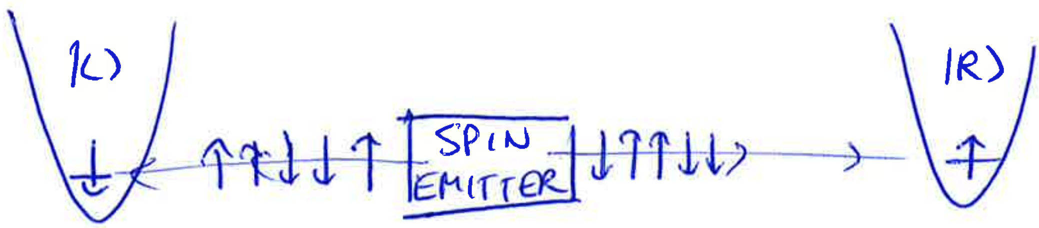
Bell inequality:

Let us now consider some of the remarkable and counterintuitive properties of entangled states, in particular the singlet state.



$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow_z\rangle_L\, |\downarrow_z\rangle_R - |\downarrow_z\rangle_L\, |\uparrow_z\rangle_R\right)$$

If we measure the left spin, we will always find that the right spin points in the opposite direction. In principle, this is not very surprising. Just imagine a "classical" device that sends out spins to the left and to the right that always point in opposite directions:



In this case, we would indeed find that the left and right spins always point in opposite directions. In particular, the state of the spins was already determined before the measurement, and there was no "spooky action at a distance" in the sense that the measurement in one location immediately affected a measurement in a different location. This is the essence of "local realism" as advocated by Einstein, Podolsky, and Rosen (EPR).

Thus, according to EPR, the measurement outcomes could be explained by having a large ensemble of spins with 50% being $|\uparrow_z\rangle_L |\downarrow_z\rangle_R$ and 50% being $|\downarrow_z\rangle_L |\uparrow_z\rangle_R$.

Now, as we have seen in the exercises, the singlet state is always of the form

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow_{\underline{n}}\rangle_L |\downarrow_{\underline{n}}\rangle_R - |\downarrow_{\underline{n}}\rangle_L |\uparrow_{\underline{n}}\rangle_R \right)$$

no matter what direction $\underline{n}$ we choose for our spin basis. For example, with $\underline{n} = \underline{x}$, We have

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow_x\rangle_L |\downarrow_x\rangle_R - |\downarrow_x\rangle_L |\uparrow_x\rangle_R \right),$$

thus; if we measure along the x-direction, we find $|\uparrow_x\rangle_L |\downarrow_x\rangle_R$ half of the time, and $|\downarrow_x\rangle_L |\uparrow_x\rangle_R$ the other half of the time.

Now, imagine that we measure along the z-direction half of the time and along the x-direction the other half.

According to local realism, we could explain the measurement outcomes by having a large ensemble of particles with the following properties:

25% of pairs with the left particle having $\uparrow_z$ and $\uparrow_x$ and the right particle having $\downarrow_z$ and $\downarrow_x$

25% with $\uparrow_z$ and $\downarrow_x$ (left) and $\downarrow_z$ and $\uparrow_x$ (right)

25% —"— $\downarrow_z$ and $\uparrow_x$ (left) and $\uparrow_z$ and $\downarrow_x$ (right)

25% —"— $\downarrow_z$ and $\downarrow_x$ (left) and $\uparrow_z$ and $\uparrow_x$ (right)

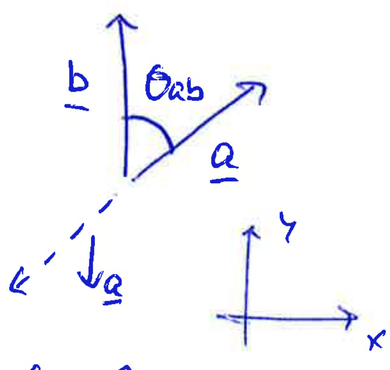
From this distribution, we would also find that we would measure $\uparrow_z$ for the left particle <u>and</u> $\uparrow_x$ for the right particle 25% of the time. However, this is also consistent with quantum mechanics. Indeed, if we calculate the probability that the left particle is in the state $|\uparrow_{\underline{a}}\rangle$ and the right one is in the state $|\uparrow_{\underline{b}}\rangle$, we find

$$P(\underline{a},\underline{b}) = \left| {}_R\langle \uparrow_{\underline{b}} | \, {}_L\langle \uparrow_{\underline{a}} | \, |\psi\rangle_{LR} \right|^2$$

$$= \left| {}_R\langle \uparrow_{\underline{b}} | \, {}_L\langle \uparrow_{\underline{a}} | \frac{1}{\sqrt{2}} \left(|\uparrow_{\underline{a}}\rangle_L |\downarrow_{\underline{a}}\rangle_R - |\downarrow_{\underline{a}}\rangle_L |\uparrow_{\underline{a}}\rangle_R \right) \right|^2$$

$$= \frac{1}{2} \left| {}_R\langle \uparrow_{\underline{b}} | \downarrow_{\underline{a}} \rangle_R \right|^2$$

To calculate the overlap, we place the x-y coordinate system in the $\underline{a}$-$\underline{b}$ plane and write



$$|\downarrow_{\underline{a}}\rangle = \hat{U}_{\underline{z}}(\pi - \theta_{ab}) |\uparrow_{\underline{b}}\rangle = e^{-i\left(\frac{\pi-\theta_{ab}}{2}\right)\hat{\sigma}_z} |\uparrow_{\underline{b}}\rangle$$

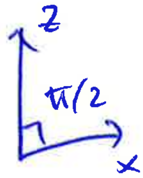
$$= \left[\hat{\mathbb{1}} \cos\left(\frac{\pi-\theta_{ab}}{2}\right) - i \hat{\sigma}_z \sin\left(\frac{\pi-\theta_{ab}}{2}\right) \right] |\uparrow_{\underline{b}}\rangle$$

Since $|\uparrow_{\underline{b}}\rangle$ has no component in the z-direction, we have $\langle \uparrow_{\underline{b}} | \hat{O}_z | \uparrow_{\underline{b}} \rangle = 0$, and thus

$$P(\underline{a},\underline{b}) = \frac{1}{2}\cos^2\left(\frac{\pi - \theta_{ab}}{2}\right)$$

$$= \frac{1}{2}\sin^2(\theta_{ab}/2)$$

From this result, we get



$$P(\underline{z},\underline{x}) = \frac{1}{2}\sin^2\left(\frac{\pi/2}{2}\right) = \frac{1}{2}\left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{4} \checkmark,$$

which indeed implies 25%

So also in the case of two measurement directions, we can explain the outcomes based on local realism.

Bell realized that "local realism" runs into trouble, if we consider three different measurement directions $\underline{a}, \underline{b}, \underline{c}$.
In this case, local realism would require N pairs of electron with the following properties:

# of pairs	Left particle	Right particle
N_1	$(\uparrow_a, \uparrow_b, \uparrow_c)$	$(\downarrow_a, \downarrow_b, \downarrow_c)$
N_2	$(\uparrow_a, \uparrow_b, \downarrow_c)$	$(\downarrow_a, \downarrow_b, \uparrow_c)$
N_3	$(\uparrow_a, \downarrow_b, \uparrow_c)$	$(\downarrow_a, \uparrow_b, \downarrow_c)$
N_4	$(\uparrow_a, \downarrow_b, \downarrow_c)$	$(\downarrow_a, \uparrow_b, \uparrow_c)$
N_5	$(\downarrow_a, \uparrow_b, \uparrow_c)$	$(\uparrow_a, \downarrow_b, \downarrow_c)$
N_6	$(\downarrow_a, \uparrow_b, \downarrow_c)$	$(\uparrow_a, \downarrow_b, \uparrow_c)$
N_7	$(\downarrow_a, \downarrow_b, \uparrow_c)$	$(\uparrow_a, \uparrow_b, \downarrow_c)$
N_8	$(\downarrow_a, \downarrow_b, \downarrow_c)$	$(\uparrow_a, \uparrow_b, \uparrow_c)$

(Note that the spin of the left particle is always opposite to the right particle as required for a singlet)

The total number of pairs is $N = N_1 + N_2 + \dots + N_8$

Based on this table, we find

$$P(\uparrow_a, \uparrow_b) = \frac{N_3 + N_4}{N}$$

$$P(\uparrow_a, \uparrow_c) = \frac{N_2 + N_4}{N}$$

$$P(\uparrow_c, \uparrow_b) = \frac{N_3 + N_7}{N}$$

Next, we use the obvious inequality

$$N_3 + N_4 \leq N_3 + N_4 + N_2 + N_7 = (N_2 + N_4) + (N_3 + N_7)$$

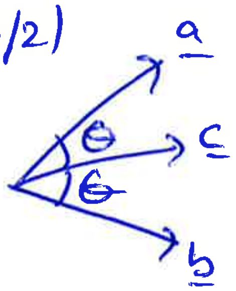
$$\Rightarrow \quad \boxed{P(\uparrow_a, \uparrow_b) \leq P(\uparrow_a, \uparrow_c) + P(\uparrow_c, \uparrow_b)} \quad \text{Bell's inequality!}$$

this is the prediction based on "local realism".

However, QM allows us to violate Bell's inequality.

Quantum mechanically, we have

$$P(\underline{a}, \underline{b}) \; (= P(\uparrow_a, \uparrow_b)) = \frac{1}{2} \sin^2(\theta_{ab}/2)$$



Thus, if we choose the directions as

so that $\theta_{ab} = 2\theta$ and $\theta_{ac} = \theta_{cb} = \theta$,

we get

$$\frac{1}{2} \sin^2\theta \leq \sin^2(\theta/2).$$

However, this inequality is violated for any $\theta \leq \frac{\pi}{2}$

For instance, with $\theta \ll 1$, we easily get

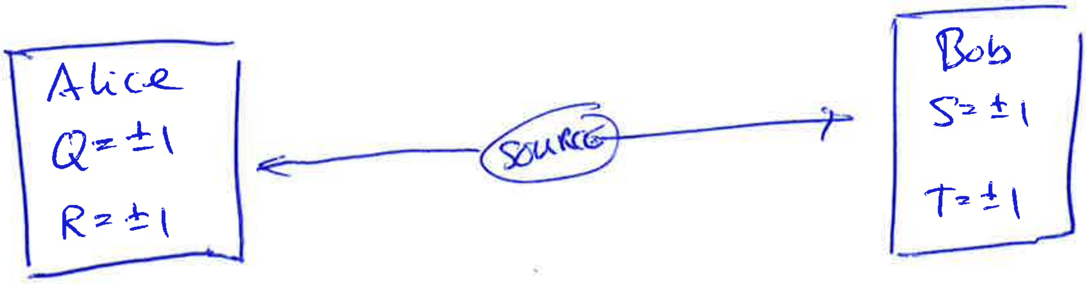
$$\frac{1}{2} \sin^2\theta \simeq \frac{1}{2}\theta^2 \quad \text{and} \quad \sin^2(\theta/2) \simeq \frac{1}{4}\theta^2,$$

and clearly $\frac{1}{2}\theta^2 \geq \frac{1}{4}\theta^2$!

$\Rightarrow$ local realism cannot be true!

CSHS:

Bell's original inequality from 1964 turns out to be difficult to realize in experiment, since it requires highly accurate control of the measurement directions. In 1969, Clauser, Horne, Shimony, and Holt (CSHS) devised another inequality to test local realism. To this end, let us consider the setup below.



In this experiment, Alice and Bob each receive a particle and can choose either to measure $Q=\pm 1$ or $R=\pm 1$ for Alice, and $S=\pm 1$ or $T=\pm 1$ for Bob. To test local realism, we consider the correlations between these measurements. In particular, it will be useful to consider the quantity

$$QS + RS + RT - QT = (Q+R)S + (R-Q)T$$

Now, since $Q = \pm 1$ and $R = \pm 1$, we must have either $(Q+R)S = \pm 2$ and $(R-Q)T = 0$

or $(Q+R)S = 0$ and $(R-Q)T = \pm 2$,

since $S = \pm 1$ and $T = \pm 1$. Thus, we conclude that

$$QS + RS + RT - QT = \pm 2$$

We now make two assumptions:

(Realism): The particles have definite values of Q, R, T, S before the measurements.

(Locality): Alice's measurements do not influence Bob's measurements and vice versa.

Thus, the measurement outcomes are determined by the classical probability distribution $P(Q,R,S,T)$ that a given pair of particles have the values Q, R, S, T. Using this probability distribution, we can evaluate the average of $QS+RS+RT-QT$:

$$\langle QS + RS + RT - QT \rangle = \sum_{Q,R,S,T} (QS + RS + RT - QT) P(Q,R,S,T)$$

$$= \sum_{Q,R,S,T} QS\, P(Q,R,S,T) + \ldots = \langle QS \rangle + \langle RS \rangle + \langle RT \rangle - \langle QT \rangle$$

However, we also have

$$\langle QS+RS+RT-QT\rangle = \sum_{Q,R,S,T} \underbrace{(QS+RS+RT-QT)}_{=\pm 2\,\leq +2} P(Q,R,S,T)$$

$$\leq 2 \underbrace{\sum_{Q,R,S,T} P(Q,R,S,T)}_{=1} = 2$$

We thereby obtain the CHS inequality

$$\boxed{\langle QS\rangle + \langle RS\rangle + \langle RT\rangle - \langle QT\rangle \leq 2}$$

If this inequality is violated, local realism cannot be true. Importantly, the left-hand side can be measured and has been found to exceed 2!

In the exercises, you will see that if you use

$Q = \hat{\sigma}_z$, $R = \hat{\sigma}_x$, $S = -(\hat{\sigma}_x + \hat{\sigma}_z)/\sqrt{2}$, and $T = (\hat{\sigma}_z - \hat{\sigma}_x)/\sqrt{2}$

for a singlet state, you find

$$\langle QS\rangle + \langle RS\rangle + \langle RT\rangle - \langle QT\rangle = 2\sqrt{2} > 2 \;!$$