Aalto University
School of Electrical
Engineering

Lecture 9: Space-Vector Models

ELEC-E8405 Electric Drives (5 ECTS)

Marko Hinkkanen

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Learning Outcomes

After this lecture and exercises you will be able to:

- Include the number of pole pairs in the machine models
- Transform phase variables to a space vector (and vice versa)
- Transform space vectors to different coordinates
- Express the space-vector model of the synchronous machine in rotor coordinates
- Calculate steady-state operating points of the synchronous machine

Outline

Number of Pole Pairs

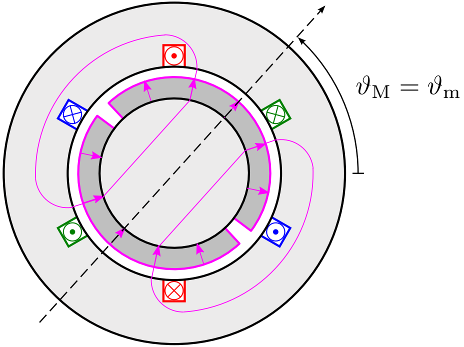
Space Vectors

Synchronous Machine Model in Stator Coordinates

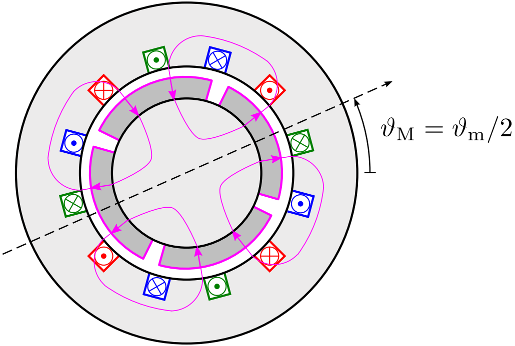
Coordinate Transformation

Synchronous Machine Model in Rotor Coordinates

Number of Pole Pairs p



2 poles ($p = 1$)



4 poles ($p = 2$)

Electrical angular speed $\omega_\mathrm{m} = p\,\omega_\mathrm{M}$ and electrical angle $\vartheta_\mathrm{m} = p\,\vartheta_\mathrm{M}$

Synchronous Rotor Speeds

- Stator (supply) frequency f (Hz)
- Electrical angular speed (rad/s)

$$\omega_\mathrm{m} = 2\pi f$$

- Rotor angular speed (rad/s)

$$\omega_\mathrm{M} = \frac{\omega_\mathrm{m}}{p}$$

- Rotor speed (r/min)

$$n = \frac{f}{p}\frac{60\ \text{s}}{\text{min}}$$

Speeds for $f = 50$ Hz

No of pole pairs p	Speed n (r/min)
1	3000
2	1500
3	1000
4	750
5	600
6	500

Note that in converter-fed motor drives, the rated supply frequency of the motor does not need to be 50 Hz.

1-Phase Machine

- Phase voltage

$$u_a = R i_a + \frac{d\psi_a}{dt}$$

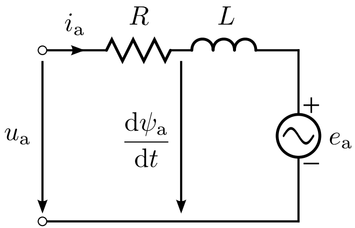
- Phase flux linkage

$$\psi_a = L i_a + \psi_{fa}$$

where $\psi_{fa} = \psi_f \cos(\vartheta_m)$

- Back-emf

$$e_a = \frac{d\psi_{fa}}{dt} = -\omega_m \psi_f \sin(\vartheta_m)$$



- Mechanical power

$$p_M = e_a i_a = T_M \omega_m / p$$

- Torque

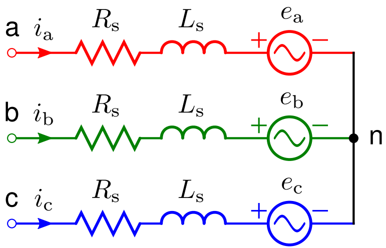
$$T_M = -p\, i_a \psi_f \sin(\vartheta_m)$$

Synchronous Machine: Phase-Variable Model

$$u_a = R_s i_a + \frac{d\psi_a}{dt}$$

$$u_b = R_s i_b + \frac{d\psi_b}{dt}$$

$$u_c = R_s i_c + \frac{d\psi_c}{dt}$$



$$\psi_a = L_s i_a + \psi_f \cos(\vartheta_m)$$

$$\psi_b = L_s i_b + \psi_f \cos(\vartheta_m - 2\pi/3)$$

$$\psi_c = L_s i_c + \psi_f \cos(\vartheta_m - 4\pi/3)$$

$$e_a = -\omega_m \psi_f \sin(\vartheta_m)$$

$$e_b = -\omega_m \psi_f \sin(\vartheta_m - 2\pi/3)$$

$$e_c = -\omega_m \psi_f \sin(\vartheta_m - 4\pi/3)$$

$$T_M = -p\psi_f \left[i_a \sin(\vartheta_m) + i_b \sin(\vartheta_m - 2\pi/3) + i_c \sin(\vartheta_m - 4\pi/3)\right]$$

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Why Space Vectors?

1. Complex phasor models
   - Simple to use but limited to steady-state conditions
2. Phase-variable models
   - Valid both in transient and steady states
   - Too complicated
3. Space-vector models
   - Phase-variable models can be directly transformed to space-vector models
   - Compact representation, insightful physical interpretations
   - Commonly applied to analysis, modelling, and control of 3-phase systems

About Complex Numbers

- Complex number

$$\underline{z} = x + \mathrm{j}y$$

- Complex conjugate of $\underline{z}$

$$\underline{z}^* = x - \mathrm{j}y$$

- Magnitude of $\underline{z}$

$$z = |\underline{z}| = \sqrt{x^2 + y^2}$$

- Euler's formula

$$\mathrm{e}^{\mathrm{j}\vartheta} = \cos\vartheta + \mathrm{j}\sin\vartheta$$

- Rotating the position vector by 90°

$$\mathrm{j}\underline{z} = \mathrm{j}(x + \mathrm{j}y) = -y + \mathrm{j}x$$

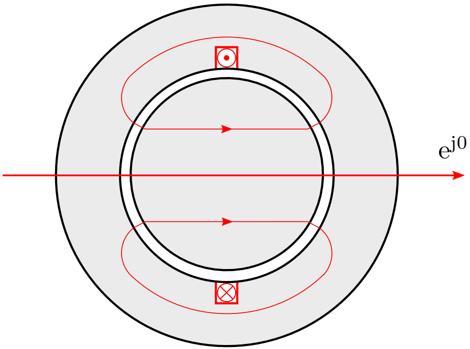
- Dot product

$$\begin{aligned}\mathrm{Re}\{\underline{z}_1\underline{z}_2^*\} &= \mathrm{Re}\{(x_1 + \mathrm{j}y_1)(x_2 - \mathrm{j}y_2)\} \\ &= x_1x_2 + y_1y_2\end{aligned}$$

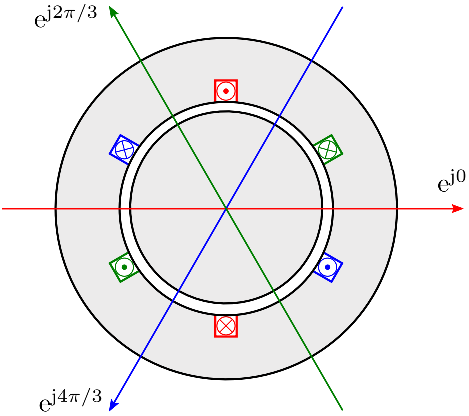
- Cross product

$$\begin{aligned}\mathrm{Im}\{\underline{z}_1\underline{z}_2^*\} &= \mathrm{Im}\{(x_1 + \mathrm{j}y_1)(x_2 - \mathrm{j}y_2\} \\ &= y_1x_2 - y_2x_1\end{aligned}$$

Magnetic Axes in the Complex Plane



Phase a



All 3 phases

Windings are sinusoidally distributed along the air gap

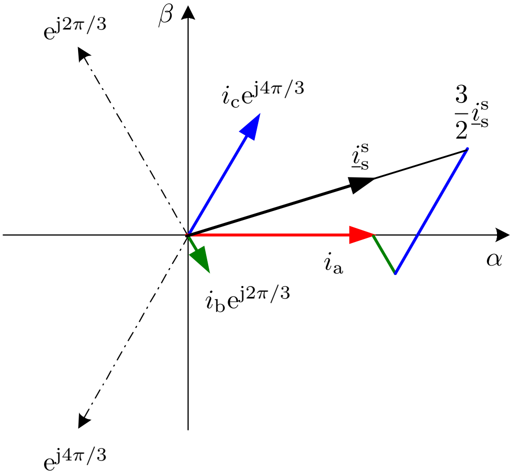
Space-Vector Transformation

- Space vector is a complex variable (signal)

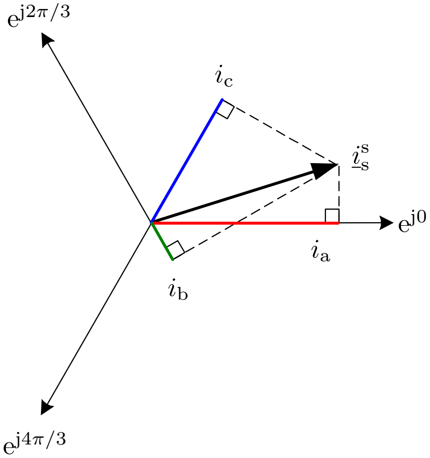
$$\boxed{\underline{i}_{\mathrm{s}}^{\mathrm{s}} = \frac{2}{3}\left(i_{\mathrm{a}} + i_{\mathrm{b}}\mathrm{e}^{\mathrm{j}2\pi/3} + i_{\mathrm{c}}\mathrm{e}^{\mathrm{j}4\pi/3}\right)}$$

  where i_{a}, i_{b}, and i_{c} are arbitrarily varying instantaneous phase variables
- Superscript s marks stator coordinates
- Same transformation applies for voltages and flux linkages
- Space vector does not include the zero-sequence component (not a problem since the stator winding is delta-connected or the star point is not connected)

Peak-value scaling of space vectors will be used in this course. Furthermore, we will use the subscript s to refer to the stator quantities, e.g., the stator current vector $\underline{i}_{\mathrm{s}}$ and the stator voltage vector $\underline{u}_{\mathrm{s}}$, since this is a very common convention in the literature.



$$\underline{i}_s^s = \frac{2}{3}\left(i_a + i_b e^{j2\pi/3} + i_c e^{j4\pi/3}\right)$$



$$i_a = \mathrm{Re}\left\{\underline{i}_s^s\right\}$$

$$i_b = \mathrm{Re}\left\{e^{-j2\pi/3}\underline{i}_s^s\right\}$$

$$i_c = \mathrm{Re}\left\{e^{-j4\pi/3}\underline{i}_s^s\right\}$$

Examples: Space Vectors Rotate in Steady State

- Positive sequence

$$i_\mathrm{a} = \sqrt{2}I_+\cos(\omega_\mathrm{m}t + \phi_+)$$
$$i_\mathrm{b} = \sqrt{2}I_+\cos(\omega_\mathrm{m}t - 2\pi/3 + \phi_+)$$
$$i_\mathrm{c} = \sqrt{2}I_+\cos(\omega_\mathrm{m}t - 4\pi/3 + \phi_+)$$

- Space vector

$$\underline{i}_\mathrm{s}^\mathrm{s} = \sqrt{2}I_+\,\mathrm{e}^{\mathrm{j}(\omega_\mathrm{m}t+\phi_+)}$$

- Negative sequence

$$i_\mathrm{a} = \sqrt{2}I_-\cos(\omega_\mathrm{m}t + \phi_-)$$
$$i_\mathrm{b} = \sqrt{2}I_-\cos(\omega_\mathrm{m}t - 4\pi/3 + \phi_-)$$
$$i_\mathrm{c} = \sqrt{2}I_-\cos(\omega_\mathrm{m}t - 2\pi/3 + \phi_-)$$

- Space vector

$$\underline{i}_\mathrm{s}^\mathrm{s} = \sqrt{2}I_-\,\mathrm{e}^{-\mathrm{j}(\omega_\mathrm{m}t+\phi_-)}$$

- Non-sinusoidal periodic waveform

$$\underline{i}_\mathrm{s}^\mathrm{s} = \sqrt{2}I_1\,\mathrm{e}^{\mathrm{j}(\omega_\mathrm{m}t+\phi_1)} + \sqrt{2}I_5\,\mathrm{e}^{-\mathrm{j}(5\omega_\mathrm{m}t+\phi_5)} + \sqrt{2}I_7\,\mathrm{e}^{\mathrm{j}(7\omega_\mathrm{m}t+\phi_7)}\ldots$$

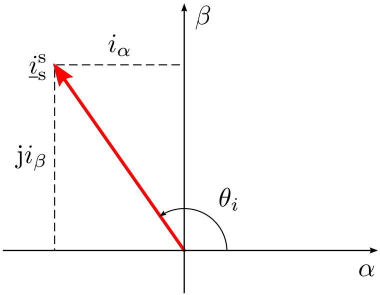
Representation in Component and Polar Forms

- Component form

$$\underline{i}_{\mathrm{s}}^{\mathrm{s}} = i_\alpha + \mathrm{j} i_\beta$$

- Polar form

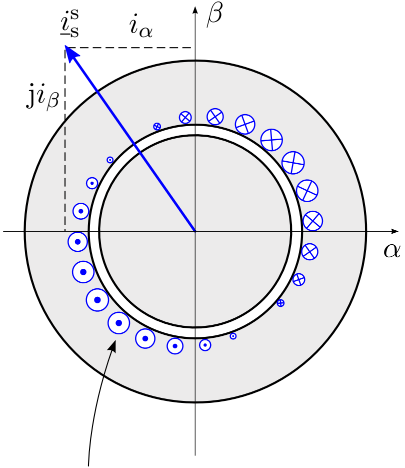
$$\begin{aligned}\underline{i}_{\mathrm{s}}^{\mathrm{s}} &= i_{\mathrm{s}} \mathrm{e}^{\mathrm{j}\theta_i} \\ &= \underbrace{i_{\mathrm{s}} \cos(\theta_i)}_{i_\alpha} + \mathrm{j} \underbrace{i_{\mathrm{s}} \sin(\theta_i)}_{i_\beta}\end{aligned}$$



- Generally, both the magnitude i_{s} and the angle θ_i may vary arbitrarily in time
- Positive sequence in steady state: $i_{\mathrm{s}} = \sqrt{2} I$ is constant and $\theta_i = \omega_{\mathrm{m}} t + \phi$

Physical Interpretation: Sinusoidal Distribution in Space

- 3-phase winding creates the current and the mmf, which are sinusoidally distributed along the air gap
- Space vector represents the instantaneous magnitude and angle of the sinusoidal distribution in space
- Magnitude and the angle can vary freely in time



Rotating current distribution produced by the 3-phase stator winding

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Space-Vector Model of the Synchronous Machine

- Stator voltage

$$\underline{u}_{\mathrm{s}}^{\mathrm{s}} = R_{\mathrm{s}}\underline{i}_{\mathrm{s}}^{\mathrm{s}} + \frac{\mathrm{d}\underline{\psi}_{\mathrm{s}}^{\mathrm{s}}}{\mathrm{d}t}$$

- Stator flux linkage

$$\underline{\psi}_{\mathrm{s}}^{\mathrm{s}} = L_{\mathrm{s}}\underline{i}_{\mathrm{s}}^{\mathrm{s}} + \psi_{\mathrm{f}}\mathrm{e}^{\mathrm{j}\vartheta_{\mathrm{m}}}$$

- Torque can be expressed in various forms
- Following form is convenient since it holds for other AC machines as well

$$T_{\mathrm{M}} = \frac{3p}{2}\,\mathrm{Im}\left\{\underline{i}_{\mathrm{s}}^{\mathrm{s}}\underline{\psi}_{\mathrm{s}}^{\mathrm{s}*}\right\}$$

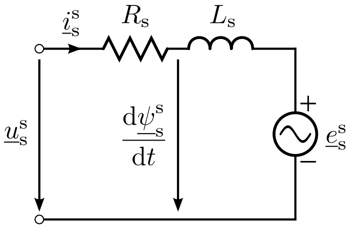
Derive these voltage and flux linkage equations starting from the phase-variable model and the definition of the space vector. Also show that the space-vector and phase-variable formulations for the torque are equal.

Space-Vector Equivalent Circuit

- Stator voltage can be rewritten as

$$\underline{u}_{\mathrm{s}}^{\mathrm{s}} = R_{\mathrm{s}}\underline{i}_{\mathrm{s}}^{\mathrm{s}} + L_{\mathrm{s}}\frac{\mathrm{d}\underline{i}_{\mathrm{s}}^{\mathrm{s}}}{\mathrm{d}t} + \underline{e}_{\mathrm{s}}^{\mathrm{s}}$$

- Back-emf $\underline{e}_{\mathrm{s}}^{\mathrm{s}} = \mathrm{j}\omega_{\mathrm{m}}\psi_{\mathrm{f}}\mathrm{e}^{\mathrm{j}\vartheta_{\mathrm{m}}}$ is proportional to the speed



Torque

- Vectors in the polar form

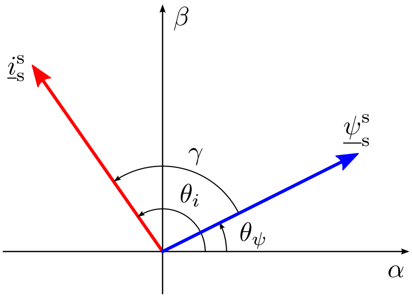
$$\underline{i}_s^s = i_s e^{j\theta_i} \qquad \underline{\psi}_s^s = \psi_s e^{j\theta_\psi}$$

- Instantaneous torque

$$T_M = \frac{3p}{2} \operatorname{Im}\left\{ \underline{i}_s^s \underline{\psi}_s^{s*} \right\} = \frac{3p}{2} i_s \psi_s \sin(\gamma)$$

where $\gamma = \theta_i - \theta_\psi$

- Nonzero γ is needed for torque production



Power

- Vectors in the component and polar forms

$$\underline{u}_\mathrm{s}^\mathrm{s} = u_\alpha + \mathrm{j}u_\beta = u_\mathrm{s}\mathrm{e}^{\mathrm{j}\theta_u} \qquad \underline{i}_\mathrm{s}^\mathrm{s} = i_\alpha + \mathrm{j}i_\beta = i_\mathrm{s}\mathrm{e}^{\mathrm{j}\theta_i}$$

- Instantaneous power fed to the stator

$$\begin{aligned} p_\mathrm{s} &= \frac{3}{2}\,\mathrm{Re}\left\{\underline{u}_\mathrm{s}^\mathrm{s}\underline{i}_\mathrm{s}^{\mathrm{s}*}\right\} \\ &= \frac{3}{2}\left(u_\alpha i_\alpha + u_\beta i_\beta\right) \\ &= \frac{3}{2}u_\mathrm{s} i_\mathrm{s}\cos(\varphi) \end{aligned}$$

where $\varphi = \theta_u - \theta_i$

The power calculated using the space vectors naturally agrees with the power $p_\mathrm{s} = u_\mathrm{a}i_\mathrm{a} + u_\mathrm{b}i_\mathrm{b} + u_\mathrm{c}i_\mathrm{c}$ calculated from the phase variables. Furthermore, in steady state, the rms-valued expression $P_\mathrm{s} = 3U_\mathrm{s}I_\mathrm{s}\cos(\varphi)$ is obtained, since $u_\mathrm{s} = \sqrt{2}U_\mathrm{s}$ and $i_\mathrm{s} = \sqrt{2}U_\mathrm{s}$ hold.

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Example: Stopping the Rotating Vector

- Positive-sequence space vector in stator coordinates

$$\underline{i}_\mathrm{s}^\mathrm{s} = \sqrt{2} I \, \mathrm{e}^{\mathrm{j}(\omega_\mathrm{m} t + \phi)}$$

- Rotating vector can be stopped by the transformation

$$\underline{i}_\mathrm{s} = \underline{i}_\mathrm{s}^\mathrm{s} \, \mathrm{e}^{-\mathrm{j}\omega_\mathrm{m} t} = \sqrt{2} I \, \mathrm{e}^{\mathrm{j}\phi}$$

- In other words, we observe the vector now in a coordinate system rotating at ω_m
- In rotating coordinates, the vector is denoted without a superscript and the components are marked with the subscripts d and q

$$\underline{i}_\mathrm{s} = i_\mathrm{d} + \mathrm{j} i_\mathrm{q}$$

Coordinate Transformation

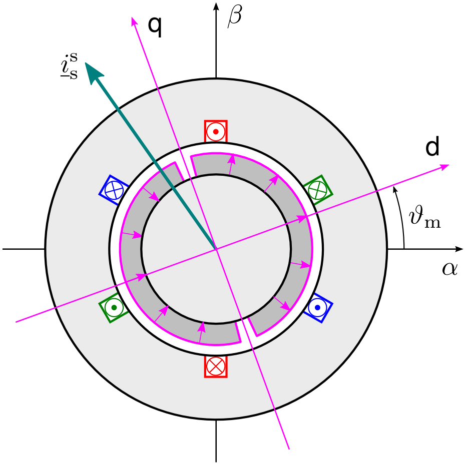
- Previous example assumed the rotor speed ω_m to be constant
- General dq transformation and its inverse are

$$\underline{i}_\mathrm{s} = \underline{i}_\mathrm{s}^\mathrm{s} \mathrm{e}^{-\mathrm{j}\vartheta_\mathrm{m}} \qquad \text{dq transformation}$$

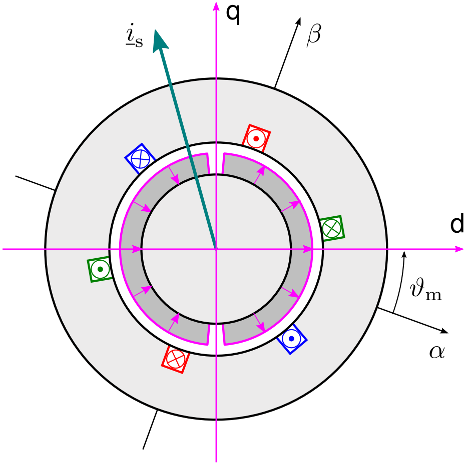
$$\underline{i}_\mathrm{s}^\mathrm{s} = \underline{i}_\mathrm{s} \mathrm{e}^{\mathrm{j}\vartheta_\mathrm{m}} \qquad \alpha\beta \text{ transformation}$$

where the rotor angle is

$$\vartheta_\mathrm{m} = \int \omega_\mathrm{m} \mathrm{d}t$$

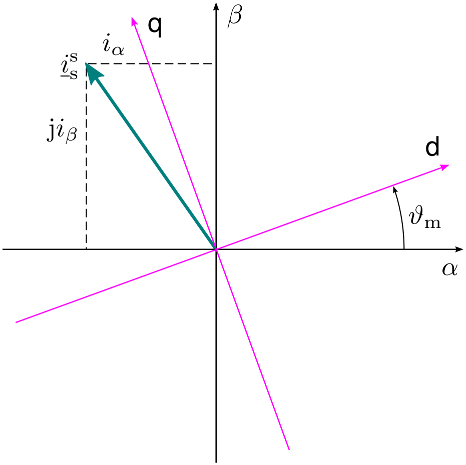


Stator coordinates ($\alpha\beta$)

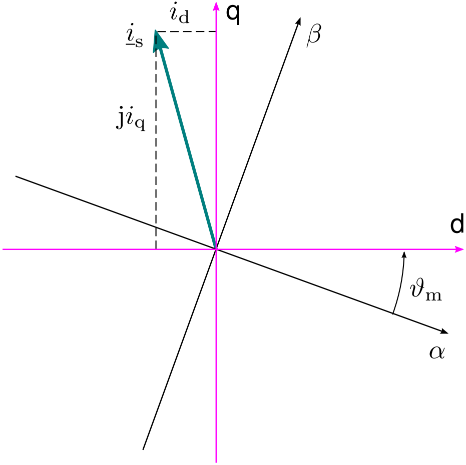


Rotor coordinates (dq)

$$\underline{i}_s = \underline{i}_s^s \, e^{-j\vartheta_m}$$



Stator coordinates ($\alpha\beta$)



Rotor coordinates (dq)

$$\underline{i}_\mathrm{s} = \underline{i}_\mathrm{s}^\mathrm{s}\,\mathrm{e}^{-\mathrm{j}\vartheta_\mathrm{m}}$$

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- Substitute $\underline{\psi}_\mathrm{s}^\mathrm{s} = \underline{\psi}_\mathrm{s}\mathrm{e}^{\mathrm{j}\vartheta_\mathrm{m}}$, $\underline{u}_\mathrm{s}^\mathrm{s} = \underline{u}_\mathrm{s}\mathrm{e}^{\mathrm{j}\vartheta_\mathrm{m}}$, and $\underline{i}_\mathrm{s}^\mathrm{s} = \underline{i}_\mathrm{s}\mathrm{e}^{\mathrm{j}\vartheta_\mathrm{m}}$

$$\underline{u}_\mathrm{s}\mathrm{e}^{\mathrm{j}\vartheta_\mathrm{m}} = R_\mathrm{s}\underline{i}_\mathrm{s}\mathrm{e}^{\mathrm{j}\vartheta_\mathrm{m}} + \frac{\mathrm{d}}{\mathrm{d}t}\left(\underline{\psi}_\mathrm{s}\mathrm{e}^{\mathrm{j}\vartheta_\mathrm{m}}\right) \quad \Rightarrow \quad \boxed{\underline{u}_\mathrm{s} = R_\mathrm{s}\underline{i}_\mathrm{s} + \frac{\mathrm{d}\underline{\psi}_\mathrm{s}}{\mathrm{d}t} + \mathrm{j}\omega_\mathrm{m}\underline{\psi}_\mathrm{s}}$$

$$\underline{\psi}_\mathrm{s}\mathrm{e}^{\mathrm{j}\vartheta_\mathrm{m}} = L_\mathrm{s}\underline{i}_\mathrm{s}\mathrm{e}^{\mathrm{j}\vartheta_\mathrm{m}} + \psi_\mathrm{f}\mathrm{e}^{\mathrm{j}\vartheta_\mathrm{m}} \quad \Rightarrow \quad \boxed{\underline{\psi}_\mathrm{s} = L_\mathrm{s}\underline{i}_\mathrm{s} + \psi_\mathrm{f}}$$

- Torque is proportional to i_q

$$\boxed{T_\mathrm{M} = \frac{3p}{2}\,\mathrm{Im}\left\{\underline{i}_\mathrm{s}\underline{\psi}_\mathrm{s}^*\right\} = \frac{3p}{2}\psi_\mathrm{f} i_\mathrm{q}}$$

while i_d does not contribute to the torque

Power Balance

- Stator voltage can be rewritten as

$$\underline{u}_\mathrm{s} = R_\mathrm{s}\underline{i}_\mathrm{s} + L_\mathrm{s}\frac{\mathrm{d}\underline{i}_\mathrm{s}}{\mathrm{d}t} + \mathrm{j}\omega_\mathrm{m}L_\mathrm{s}\underline{i}_\mathrm{s} + \underline{e}_\mathrm{s}$$

  where $\underline{e}_\mathrm{s} = \mathrm{j}\omega_\mathrm{m}\psi_\mathrm{f}$ is the back-emf
- Power balance is obtained from the stator voltage equation

$$p_\mathrm{s} = \frac{3}{2}\mathrm{Re}\left\{\underline{u}_\mathrm{s}\underline{i}_\mathrm{s}^*\right\} = \underbrace{\frac{3}{2}R_\mathrm{s}|\underline{i}_\mathrm{s}|^2}_{\text{Losses}} + \underbrace{\frac{3}{2}\frac{L_\mathrm{s}}{2}\frac{\mathrm{d}|\underline{i}_\mathrm{s}|^2}{\mathrm{d}t}}_{\text{Rate of change of energy in } L_\mathrm{s}} + \underbrace{T_\mathrm{M}\frac{\omega_\mathrm{m}}{p}}_{\text{Mechanical power}}$$

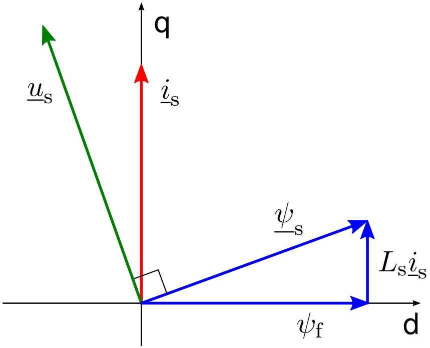
- Middle term is zero in steady state

Vector Diagram

- In steady state, $\mathrm{d}/\mathrm{d}t = 0$ holds in rotor coordinates
- Stator voltage

$$\begin{aligned} \underline{u}_\mathrm{s} &= R_\mathrm{s}\underline{i}_\mathrm{s} + \mathrm{j}\omega_\mathrm{m}\underline{\psi}_\mathrm{s} \\ &= R_\mathrm{s}\underline{i}_\mathrm{s} + \mathrm{j}\omega_\mathrm{m}(L_\mathrm{s}\underline{i}_\mathrm{s} + \psi_\mathrm{f}) \end{aligned}$$

- Steady-state operating points can be illustrated by means of vector diagrams



Assumption: $R_\mathrm{s} \approx 0$