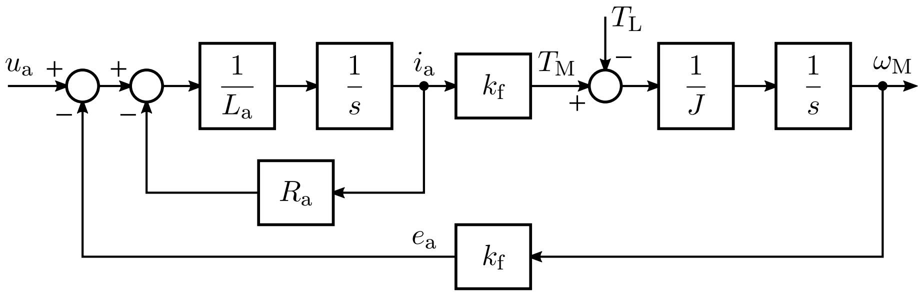
Problem 1: Transfer functions of a DC motor

The block diagram of a DC motor is shown in the figure.

(a) Derive the transfer functions

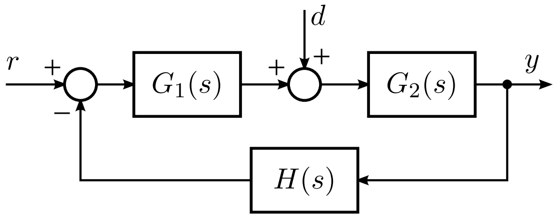
$$G_{\omega u}(s) = \frac{\omega_\mathrm{M}(s)}{u_\mathrm{a}(s)} \quad \text{and} \quad G_{\omega T}(s) = \frac{\omega_\mathrm{M}(s)}{T_\mathrm{L}(s)}$$

(b) Replace the electric dynamics of the machine with the DC gain and formulate the transfer functions $G_{\omega u}(s)$ and $G_{\omega T}(s)$.



Solution

(a) Consider a closed-loop system shown in the figure.

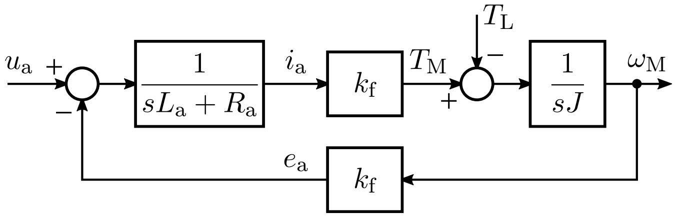


The following equations hold for the closed-loop transfer functions:

$$\frac{y(s)}{r(s)} = \frac{G_1(s)G_2(s)}{1 + G_1(s)G_2(s)H(s)} \tag{1}$$

$$\frac{y(s)}{d(s)} = \frac{G_2(s)}{1 + G_1(s)G_2(s)H(s)} \tag{2}$$

It is relatively easy to derive these equations if one has forgotten them. Using (1), the block diagram given in the problem is first transformed to the following form:



Using (1), we can write the transfer function from the voltage to the speed as

$$G_{\omega u}(s) = \frac{\omega_\mathrm{M}(s)}{u_\mathrm{a}(s)} = \frac{\frac{1}{sL_\mathrm{a}+R_\mathrm{a}} k_\mathrm{f} \frac{1}{sJ}}{1 + \frac{1}{sL_\mathrm{a}+R_\mathrm{a}} k_\mathrm{f} \frac{1}{sJ} k_\mathrm{f}} = \frac{\frac{k_\mathrm{f}}{JL_\mathrm{a}}}{s^2 + s\frac{R_\mathrm{a}}{L_\mathrm{a}} + \frac{k_\mathrm{f}^2}{JL_\mathrm{a}}}$$

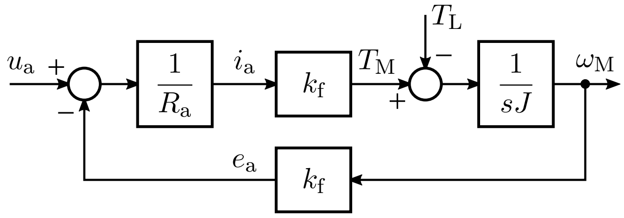
Using (2), the transfer function from the load torque to the speed becomes

$$G_{\omega T}(s) = \frac{\omega_\mathrm{M}(s)}{T_\mathrm{L}(s)} = -\frac{\frac{1}{sJ}}{1 + \frac{1}{sL_\mathrm{a}+R_\mathrm{a}} k_\mathrm{f} \frac{1}{sJ} k_\mathrm{f}} = -\frac{\frac{1}{J}\left(s + \frac{R_\mathrm{a}}{L_\mathrm{a}}\right)}{s^2 + s\frac{R_\mathrm{a}}{L_\mathrm{a}} + \frac{k_\mathrm{f}^2}{JL_\mathrm{a}}}$$

(b) The electric dynamics of the machine

$$Y_\mathrm{a}(s) = \frac{1}{sL_\mathrm{a} + R_\mathrm{a}}$$

is replaced with the DC gain by substituting $s = 0$. The corresponding block is shown in the following figure:



The transfer functions can be derived from this block diagram in a fashion similar to Part (a) of the problem. The same result is obtained by multiplying the numerator and denominator of the derived transfer functions by L_a and then substituting $L_\mathrm{a} = 0$:

$$G_{\omega u}(s) = \frac{\omega_\mathrm{M}(s)}{u_\mathrm{a}(s)} = \frac{\frac{k_\mathrm{f}}{JR_\mathrm{a}}}{s + \frac{k_\mathrm{f}^2}{JR_\mathrm{a}}}$$

$$G_{\omega T}(s) = \frac{\omega_\mathrm{M}(s)}{T_\mathrm{L}(s)} = -\frac{1/J}{s + \frac{k_\mathrm{f}^2}{JR_\mathrm{a}}}$$

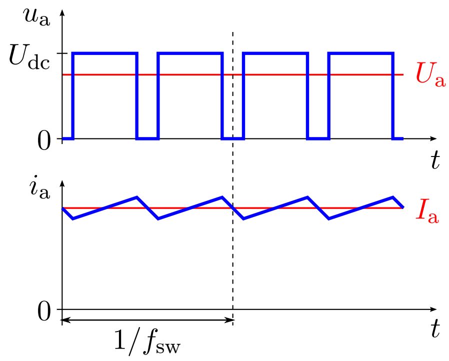
The time constant of the first-order system is

$$\tau = \frac{JR_\mathrm{a}}{k_\mathrm{f}^2}$$

which depends strongly on the flux factor k_f of the motor (and increases when the flux factor is decreased).

Problem 2: Current ripple

The parameters of a DC motor are: $R_a = 1\ \Omega$, $L_a = 10$ mH, and $k_f = 4$ Vs. The average steady-state current taken by the motor is $I_a = 100$ A and the rotor speed is 560 r/min. The motor is supplied from a four-quadrant DC-DC converter, where the unipolar PWM is applied. The DC-bus voltage is $U_{dc} = 450$ V and the switching (carrier) frequency is $f_{sw} = 4$ kHz. Calculate the peak-to-peak current ripple.



Solution

The electrical dynamics of the DC motor are governed by

$$L_a \frac{di_a(t)}{dt} = u_a(t) - R_a i_a(t) - e_a(t) \tag{3}$$

The rotor angular speed is

$$\omega_M = 2\pi \cdot \frac{560\ \text{r/min}}{60\ \text{s/min}} = 58.6\ \text{rad/s}$$

and the back-emf is $E_a = k_f\omega_M = 4\ \text{Vs} \cdot 58.6\ \text{rad/s} = 234.6$ V. Based on (3), the average armature voltage is

$$U_a = R_a I_a + E_a = 1\ \Omega \cdot 100\ \text{A} + 234.6\ \text{V} = 334.6\ \text{V}$$

in steady state.

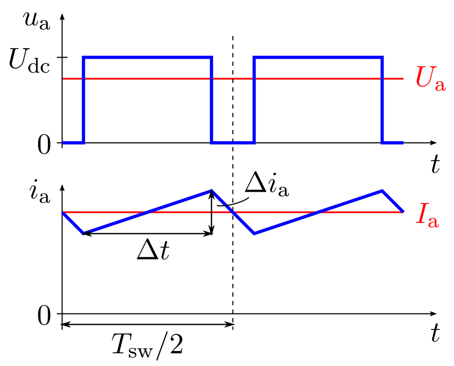
The figure below illustrates the waveforms of the armature voltage u_a and the armature current i_a. The average voltage during the switching period T_{sw} is

$$\overline{u}_a = \frac{1}{T_{sw}} \int_0^{T_{sw}} u_a(t)dt = \frac{2\Delta t}{T_{sw}} U_{dc} \tag{4}$$

where Δt is the duration of the positive voltage pulse (see the figure). The duration is

$$\Delta t = \frac{U_a}{U_{dc}} \frac{T_{sw}}{2} = \frac{U_a}{U_{dc}} \frac{1}{2f_{sw}}$$

since $U_a = \overline{u}_a$ in steady state.



In the time scale of switching periods, the dynamics in (3) can be approximated as

$$L_a \frac{di_a(t)}{dt} = u_a(t) - R_a I_a - E_a = u_a(t) - U_a \tag{5}$$

The change Δi_a in the current during the positive voltage pulse $u_a(t) = U_{dc}$ is

$$\begin{aligned} \Delta i_a &= \frac{U_{dc} - U_a}{L_a} \Delta t \\ &= \frac{U_{dc} - U_a}{L_a} \frac{U_a}{U_{dc}} \frac{1}{2 f_{sw}} \\ &= \frac{450 \text{ V} - 334.6 \text{ V}}{10 \text{ mH}} \cdot \frac{334.6 \text{ V}}{450 \text{ V}} \frac{1}{2 \cdot 4 \text{ kHz}} = 1.1 \text{ A} \end{aligned}$$

This peak-to-peak current ripple is roughly 1% of the average current.

Remark 1: Naturally, the same result would be obtained, if the zero-voltage condition $u_a(t) = 0$ were used:

$$\Delta i_a = \frac{U_a}{L_a} \left(\frac{T_{sw}}{2} - \Delta t \right)$$

Remark 2: The switching period is $T_{sw} = 1/f_{sw} = 1/(4 \text{ kHz}) = 250$ μs and the electrical time constant of the armature winding is $\tau_a = L_a/R_a = 10$ ms. Since τ_a is much longer (40 times) than T_{sw}, the approximation (5) holds well.