Mathematical Tools for Reactor Physics

1 Delta function

Definition

$$\delta(\mathbf{r}-\mathbf{r}_0) = 0, \quad \text{if} \quad \mathbf{r} \neq \mathbf{r}_0$$

$$\int_V \delta(\mathbf{r}-\mathbf{r}_0)\, dV = \begin{cases} 1, & \text{if} \quad \mathbf{r}_0 \in V \\ 0, & \text{if} \quad \mathbf{r}_0 \notin V \end{cases}$$

Consequence

$$\int_V \delta(\mathbf{r}-\mathbf{r}_0) f(\mathbf{r})\, dV = f(\mathbf{r}_0) \int_V \delta(\mathbf{r}-\mathbf{r}_0)\, dV = \begin{cases} f(\mathbf{r}_0), & \text{if} \quad \mathbf{r}_0 \in V \\ 0, & \text{if} \quad \mathbf{r}_0 \notin V \end{cases}$$

δ in different coordinates

- Rectangular coordinates: $\delta(\mathbf{r}-\mathbf{r}_0) = \delta(x-x_0)\delta(y-y_0)\delta(z-z_0)$
- Cylinder coordinates: $\delta(\mathbf{r}-\mathbf{r}_0) = \frac{1}{r}\delta(r-r_0)\delta(\varphi-\varphi_0)\delta(z-z_0)$
- Spherical coordinates: $\delta(\mathbf{r}-\mathbf{r}_0) = \frac{1}{r^2 \sin\theta}\delta(r-r_0)\delta(\varphi-\varphi_0)\delta(\theta-\theta_0)$

Singular sources in terms of the delta function

- Point source at $\mathbf{r}_0$: $S(\mathbf{r}) = S\delta(\mathbf{r}-\mathbf{r}_0)$
- Plane symmetry
  - Plane source at $x = x_0$: $S(\mathbf{r}) = S\delta(x-x_0)$
- Cylinder symmetry
  - Thin, hollow cylindrical source of radius $\mathbf{r}_0$: $S(\mathbf{r}) = \frac{S}{2\pi r}\delta(r-r_0)$
  - Line source on z axis: $S(\mathbf{r}) = \frac{S}{2\pi r}\delta(r)$
- Spherical symmetry
  - Thin, hollow spherical source of radius $\mathbf{r}_0$: $S(\mathbf{r}) = \frac{S}{4\pi r^2}\delta(r-r_0)$
  - Point source at the origin: $S(\mathbf{r}) = \frac{S}{4\pi r^2}\delta(r)$

2 Bessel functions

J_n and Y_n are Bessel functions, I_n and K_n modifioied Bessel functions. J_n and Y_n have infinite number of zeros, I_n increases monotonically, and K_n decreases monotonically. At the origin and infinity the functions behave as follows:

$$J_0(0) = 1, \quad J_n(0) = 0 \quad (n \neq 0), \quad \lim_{x \to 0} Y_n(x) = -\infty$$

$$I_0(0) = 1, \quad I_n(0) = 0 \quad (n \neq 0), \quad \lim_{x \to 0} K_n(x) = \infty$$

$$\lim_{x \to \infty} I_n(x) = \infty, \quad \lim_{x \to \infty} K_n(x) = 0.$$

For derivatives:

$$\frac{d}{dx}\left(x^n K_n(x)\right) = -x^n K_{n-1}(x), \quad \text{for others} \quad \frac{d}{dx}\left(x^n Z_n(x)\right) = x^n Z_{n-1}(x)$$

$$I_0'(x) = I_1(x), \quad \text{for others} \quad Z_0'(x) = -Z_1(x)$$

$$I_n(x) K_{n-1}(x) + I_{n-1}(x) K_n(x) = \frac{1}{x}$$

3 Integral theorems

$$\oint_{\partial V} \mathbf{F} \cdot d\mathbf{S} = \int_V (\nabla \cdot \mathbf{F})\, dV \qquad \text{(Gauss theorem)}$$

$$\oint_{\partial S} \mathbf{F} \cdot d\mathbf{l} = \int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} \qquad \text{(Stokes theorem)}$$

4 Coordinate systems and differential operators

Rectangular coordinates $(x,\, y,\, z)$

$$\nabla u = \frac{\partial u}{\partial x}\mathbf{e}_x + \frac{\partial u}{\partial y}\mathbf{e}_y + \frac{\partial u}{\partial z}\mathbf{e}_z$$

$$\nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

$$\nabla \times \mathbf{F} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\mathbf{e}_x + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\mathbf{e}_y + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\mathbf{e}_z$$

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

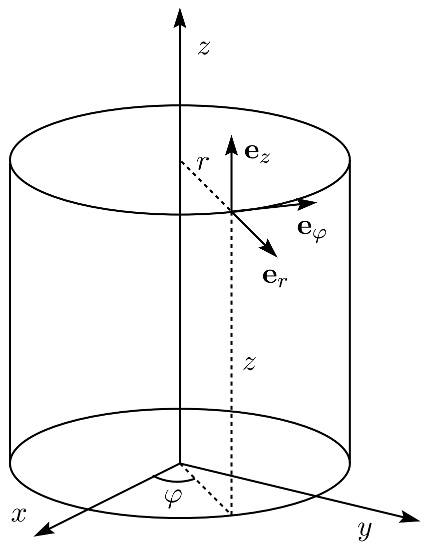
Cylindrical coordinates $(r,\ \varphi,\ z)$

$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \\ z = z \end{cases}$$

$$\begin{cases} r = \sqrt{x^2+y^2} \\ \varphi = \arctan\frac{y}{x} \\ z = z \end{cases}$$

$$\begin{cases} \mathbf{e}_x = \mathbf{e}_r\cos\varphi - \mathbf{e}_\varphi\sin\varphi \\ \mathbf{e}_y = \mathbf{e}_r\sin\varphi + \mathbf{e}_\varphi\cos\varphi \\ \mathbf{e}_z = \mathbf{e}_z \end{cases}$$

$$\begin{cases} \mathbf{e}_r = \mathbf{e}_x\cos\varphi + \mathbf{e}_y\sin\varphi \\ \mathbf{e}_\varphi = -\mathbf{e}_x\sin\varphi + \mathbf{e}_y\cos\varphi \\ \mathbf{e}_z = \mathbf{e}_z \end{cases}$$



$$\nabla u = \frac{\partial u}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial u}{\partial\varphi}\mathbf{e}_\varphi + \frac{\partial u}{\partial z}\mathbf{e}_z$$

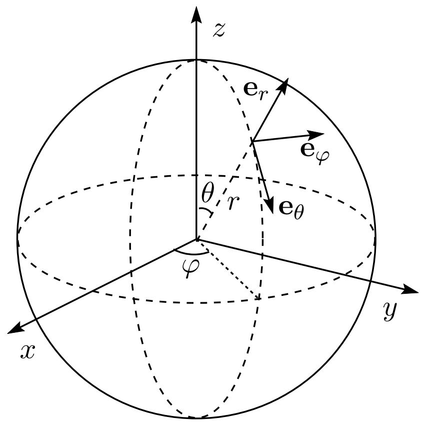
$$\nabla\cdot\mathbf{F} = \frac{1}{r}\frac{\partial(rF_r)}{\partial r} + \frac{1}{r}\frac{\partial F_\varphi}{\partial\varphi} + \frac{\partial F_z}{\partial z}$$

$$\nabla\times\mathbf{F} = \left(\frac{1}{r}\frac{\partial F_z}{\partial\varphi} - \frac{\partial F_\varphi}{\partial z}\right)\mathbf{e}_r + \left(\frac{\partial F_r}{\partial z} - \frac{\partial F_z}{\partial r}\right)\mathbf{e}_\varphi + \frac{1}{r}\left(\frac{\partial(rF_\varphi)}{\partial r} - \frac{\partial F_r}{\partial\varphi}\right)\mathbf{e}_z$$

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial\varphi^2} + \frac{\partial^2 u}{\partial z^2}$$

Spherical coordinates (r, θ, φ)

$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases}$$



$$\begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \theta = \arccos \frac{z}{\sqrt{x^2+y^2+z^2}} \\ \varphi = \arctan \frac{y}{x} \end{cases}$$

$$\begin{cases} \mathbf{e}_x = \mathbf{e}_r \sin\theta\cos\varphi + \mathbf{e}_\theta \cos\theta\cos\varphi - \mathbf{e}_\varphi \sin\varphi \\ \mathbf{e}_y = \mathbf{e}_r \sin\theta\sin\varphi + \mathbf{e}_\theta \cos\theta\sin\varphi + \mathbf{e}_\varphi \cos\varphi \\ \mathbf{e}_z = \mathbf{e}_r \cos\theta - \mathbf{e}_\theta \sin\theta \end{cases}$$

$$\begin{cases} \mathbf{e}_r = \mathbf{e}_x \sin\theta\cos\varphi + \mathbf{e}_y \sin\theta\sin\varphi + \mathbf{e}_z \cos\theta \\ \mathbf{e}_\theta = \mathbf{e}_x \cos\theta\cos\varphi + \mathbf{e}_y \cos\theta\sin\varphi - \mathbf{e}_z \sin\theta \\ \mathbf{e}_\varphi = -\mathbf{e}_x \sin\varphi + \mathbf{e}_y \cos\varphi \end{cases}$$

$$\nabla u = \frac{\partial u}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial u}{\partial \theta}\mathbf{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial u}{\partial \varphi}\mathbf{e}_\varphi$$

$$\nabla \cdot \mathbf{F} = \frac{1}{r^2}\frac{\partial (r^2 F_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial (F_\theta \sin\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial F_\varphi}{\partial \varphi}$$

$$\begin{aligned} \nabla \times \mathbf{F} = {} & \frac{1}{r\sin\theta}\left(\frac{\partial (F_\varphi \sin\theta)}{\partial \theta} - \frac{\partial F_\theta}{\partial \varphi}\right)\mathbf{e}_r + \frac{1}{r\sin\theta}\left(\frac{\partial F_r}{\partial \varphi} - \sin\theta \frac{\partial (r F_\varphi)}{\partial r}\right)\mathbf{e}_\theta \\ & + \frac{1}{r}\left(\frac{\partial (r F_\theta)}{\partial r} - \frac{\partial F_r}{\partial \theta}\right)\mathbf{e}_\varphi \end{aligned}$$

$$\nabla^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial u}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial \varphi^2}$$

References

L. Råde and B. Westergren, BETA – Mathematics Handbook for Science and Engineering, 3rd ed. Studentlitteratur, Lund, 1995.