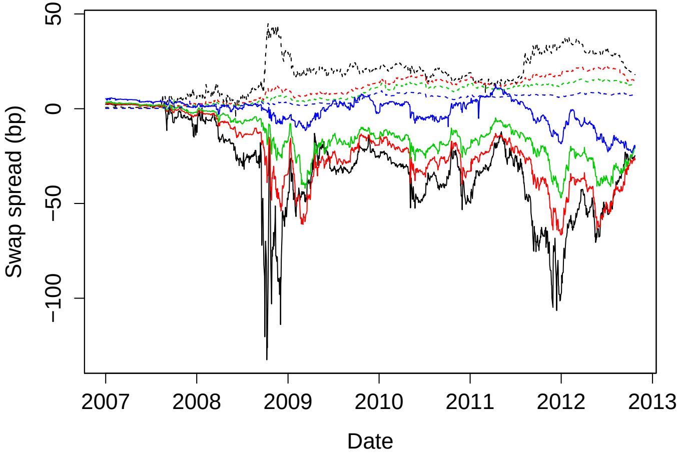
# Stationary stochastic processes and ARMA models

MS-C2128 Prediction and Time Series Analysis

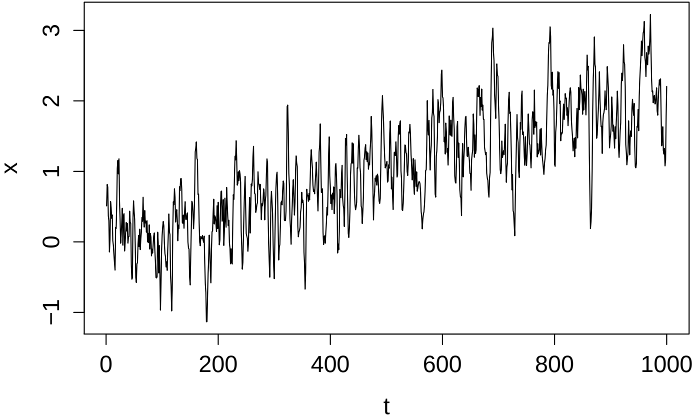
Fall term 2020

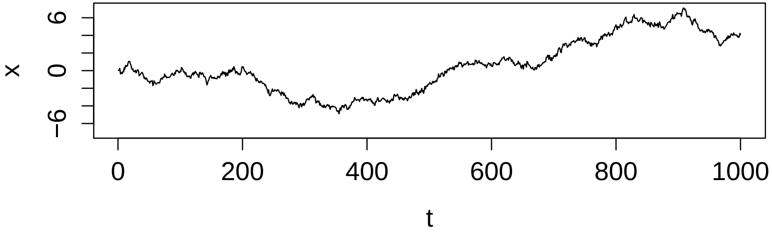
# Introduction

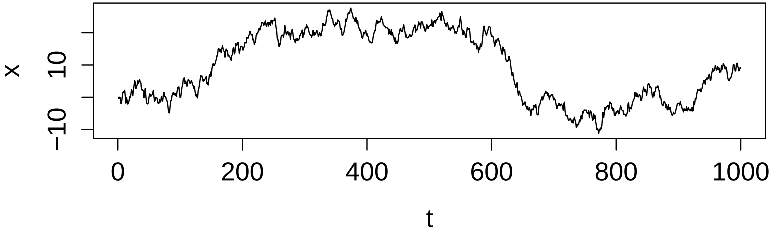
If we wish to deal with nasty, badly behaving, time series data...



...we should first be familiar with nicely behaving stochastic processes and their properties.




x
−6
0
6
0
200
400
600
800
1000
t


x
−10
10
0
200
400
600
800
1000
t

# Week 3: Stationary stochastic processes and ARMA models

1. Stationary stochastic processes
   1. Definition
   2. Autocorrelation function
   3. Partial autocorrelation function
   4. Lag and difference operators
   5. Difference stationarity
2. ARMA models
   1. Pure random process
   2. Different SARMA models
   3. Spectrum

# Content

# Stochastic processes

- A stochastic process $(x_t)_{t \in T}$ is a (time-)indexed collection of random variables defined on some common probability space. Each

  $$x_t, \quad t \in T$$

  is a random variable representing a value at time $t \in T$.
  - The joint distribution of the random variables $x_t$ defines fully the behaviour of the process $(x_t)_{t \in T}$.
- Here, we consider discrete time stochastic processes for which the index variable takes a discrete set of values. That is, we assume that $T \subset \mathbb{Z} := \{..., -2, -1, 0, 1, 2, ...\}$.
- We do not consider continuous time processes. (For example processes for which $T$ is the set of positive real numbers.)

# Time series as a stochastic process

- In time series analysis an observed time series is interpreted as a realization of some stochastic process.
  - In comparison, i.i.d. observations are interpreted as realizations of some random variable.
- In time series analysis, we wish to:
  (i) Find a suitable stochastic process that fits to the observed time series.
  (ii) Estimate the parameters of the corresponding stochastic process and conduct hypotheses testing.
  (iii) Construct predictions of the future behaviour of the time series.

# Expected value, variance and covariance: Definitions

The expected value of $x_t$, the variance of $x_t$ and the covariance of $x_t$ and $x_s$ are useful, if one wishes to describe characteristics of a stochastic process $(x_t)_{t \in T}$:

- The expected value of $x_t$ is defined as:

$$\mathrm{E}[x_t] = \mu_t, \quad t \in T$$

- The variance of $x_t$ is defined as:

$$\mathrm{var}(x_t) = \mathrm{E}[(x_t - \mu_t)^2] = \sigma_t^2, \quad t \in T$$

- The covariance of $x_t$ and $x_s$ is defined as:

$$\mathrm{cov}(x_t, x_s) = \mathrm{E}[(x_t - \mu_t)(x_s - \mu_s)] = \gamma_{ts}, \quad t, s \in T.$$

# Stationarity

Stochastic process $(x_t)_{t \in T}$ is called stationary (or weakly stationary) if:

(I) The expected value does not depend on time:

$$\mathrm{E}(x_t) = \mu, \quad \text{for all } t \in T$$

(II) The variance is finite and does not depend on time:

$$\mathrm{var}(x_t) = \sigma^2 < \infty, \quad \text{for all } t \in T$$

(III) The covariance of $x_t$ and $x_s$ does not depend on the time points $t$ and $s$. It only depends on the difference of $t$ and $s$:

$$\mathrm{cov}(x_t, x_s) = \gamma_{t-s}, \quad \text{for all } t, s \in T$$

- A process $(x_t)_{t \in T}$ is called strictly stationary if the joint distributions of $(x_{t1}, x_{t2}, ..., x_{tn})$ and $(x_{t1+h}, x_{t2+h}, ..., x_{tn+h})$ are the same for all $n, h, t1, t2, ..., tn$.

# Stationary stochastic processes

When you take a look at a realization of a stationary stochastic process you should NOT detect

1. Trend
2. Systematic changes in variance
3. Deterministic seasonality

# The importance of stationary processes in modeling time series data

Discussion

# Autocovariance: Definition

The $k$. autocovariance $\gamma_k$ of a stationary stochastic process $(x_t)_{t \in T}$ is defined as

$$\gamma_k := \gamma_{t-(t-k)} = \mathrm{cov}(x_t, x_{t-k}) = \mathrm{E}[(x_t - \mu)(x_{t-k} - \mu)], \quad t \in T, k \in \mathbb{Z}.$$

In particular

$$\gamma_0 = \mathrm{var}(x_t) = \sigma^2, \quad t \in T.$$

The autocovariance function of a stationary stochastic process $(x_t)_{t \in T}$ is a function of the autocovariances, $\gamma : \mathbb{Z} \to \mathbb{R}$,

$$\gamma(k) = \gamma_k \quad \text{for all } k \in \mathbb{Z}.$$

# Autocorrelation: Definition

The $k$. autocorrelation coefficient $\rho_k$ of a stationary stochastic process $(x_t)_{t \in T}$ is defined as:

$$\rho_k = \frac{\gamma_k}{\gamma_0}, \quad k \in \mathbb{Z}.$$

- The autocorrelation coefficient $\rho_k$ of $(x_t)_{t \in T}$ measures how strong the linear dependence of the variables $x_t$ and $x_{t-k}$ is.
  - (i) $\rho_0 = 1$
  - (ii) $\rho_{-k} = \rho_k$ for all $k \in \mathbb{Z}$
  - (iii) $|\rho_k| \leq 1$ for all $k \in \mathbb{Z}$.
- The autocorrelation function is the function $\rho : \mathbb{Z} \to [-1, 1]$,

$$\rho(k) = \rho_k, \quad \text{for all } k \in \mathbb{Z}.$$

# Partial autocorrelation: Definition

The $k$. partial autocorrelation coefficient $\alpha_k$ of a stationary stochastic process $(x_t)_{t \in T}$ is defined as:

$$\alpha_k = \text{cor}\left(x_t, x_{t-k} \mid x_{t-1}, ..., x_{t-k+1}\right) \quad , t \in T, \; k \in \mathbb{Z}$$

- Partial autocorrelation coefficient is the conditional correlation of $x_t$ and $x_{t-k}$ with respect to $x_{t-1}, ..., x_{t-k+1}$.
- Partial autocorrelation coefficient measures the correlation of $x_t$ and $x_{t-k}$, when the values $x_{t-1}, ..., x_{t-k+1}$ are known.
  - (i) $\alpha_0 = 1$
  - (ii) $\alpha_{-k} = \alpha_k$ for all $k \in \mathbb{Z}$
  - (iii) $|\alpha_k| \leq 1$ for all $k \in \mathbb{Z}$.

The partial autocorrelation function is the function $\alpha : \mathbb{Z} \to [-1, 1]$,

$$\alpha(k) = \alpha_k, \quad \text{for all } k \in \mathbb{Z}.$$

# Autocorrelation and partial autocorrelation: Yule-Walker equations

$$\begin{bmatrix} 1 & \rho_1 & \rho_2 & \cdots & \rho_{k-1} \\ \rho_1 & 1 & \rho_1 & \cdots & \rho_{k-2} \\ \rho_2 & \rho_1 & 1 & \cdots & \rho_{k-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho_{k-1} & \rho_{k-2} & \rho_{k-3} & \cdots & 1 \end{bmatrix} \begin{bmatrix} \alpha_{k1} \\ \alpha_{k2} \\ \alpha_{k3} \\ \vdots \\ \alpha_{kk} \end{bmatrix} = \begin{bmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \\ \vdots \\ \rho_k \end{bmatrix},$$

where $\rho_k$ is the $k$. autocorrelation coefficient.
The $k$. partial autocorrelation coefficient $\alpha_k$ is obtained by solving $\alpha_{kk}$ from the equations above:

$$\alpha_k = \alpha_{kk}.$$

In particular

$$\alpha_2 = \alpha_{22} = \frac{\rho_2 - \rho_1^2}{1 - \rho_1^2}.$$

# Lag and difference: Definitions

Let $(x_t)_{t \in T}$ be a discrete time stochastic process.

- The lag operator $L$ is defined by:

$$Lx_t = x_{t-1}$$

- The difference operator $D$ is defined by:

$$Dx_t = x_t - x_{t-1}$$

**Remark**

The difference operator $D$ can be given in terms of the lag operator $L$

$$D = 1 - L,$$

as

$$(1 - L)x_t = x_t - Lx_t = x_t - x_{t-1} = Dx_t.$$

# Higher order lags and differences, Seasonal difference

- The p. lag $L^p$ is defined by:

$$L^p x_t = x_{t-p},$$

where $L^p = LL \cdots L$ ($p$ times): $L^p x_t = L^{p-1} L x_t = L^{p-1} x_{t-1}$.

- The p. difference $D^p$ is defined by:

$$D^p x_t = (1 - L)^p x_t,$$

where $D^p = DD \cdots D$ ($p$ times).

  - For the p. difference $D^p$ it holds that

$$D^p x_t = (1 - L)^p x_t = \sum_{i=0}^{p} (-1)^i \binom{p}{i} x_{t-i}.$$

- The seasonal difference $D_s$ is defined by:

$$D_s = 1 - L^s,$$

where $s$ is the length of the season (i.e. the period).

  - Now

$$D_s x_t = (1 - L^s) x_t = x_t - L^s x_t = x_t - x_{t-s}.$$

# Example: 2. difference

The second difference of $x_t$ can be calculated as follows:

- Approach 1:

$$\begin{aligned} D^2 x_t &= DDx_t = D(x_t - x_{t-1}) \\ &= Dx_t - Dx_{t-1} \\ &= x_t - x_{t-1} - (x_{t-1} - x_{t-2}) \\ &= x_t - 2x_{t-1} + x_{t-2} \end{aligned}$$

- Approach 2:

$$\begin{aligned} D^2 x_t &= (1 - L)^2 x_t = \left(1 - 2L + L^2\right) x_t \\ &= x_t - 2Lx_t + L^2 x_t \\ &= x_t - 2x_{t-1} + x_{t-2} \end{aligned}$$

# Difference stationarity

## Definition

Let $(x_t)_{t \in T}$ be a discrete time stochastic process.

- The process $(x_t)_{t \in T}$ is difference stationary of order $p$, if

$$D^q x_t \text{ is non-stationary for all } q = 0, 1, 2, ..., p-1,$$

  but $D^p x_t$ is stationary.

- The process $(x_t)_{t \in T}$ is difference stationary of order $p$ with respect to the season length $s$, if

$$D_s^q x_t \text{ is non-stationary for all } q = 0, 1, 2, ..., p-1,$$

  but $D_s^p x_t$ is stationary.

# Trend and seasonality

Differencing can be applied in order to remove a trend. Seasonal differencing can be applied in order to remove deterministic seasonality. Sometimes both are needed in order to obtain a stationary time series.

## Example

If the term (season legth) $s = 12$, we often apply the first difference (in order to remove the trend) and seasonal difference with period 12 (in order to remove seasonality). We then obtain the following series:

$$\begin{aligned} D_{12}Dx_t = DD_{12}x_t &= (1 - L)(1 - L^{12})x_t \\ &= \left(1 - L - L^{12} + L^{13}\right)x_t \\ &= x_t - x_{t-1} - \left(x_{t-12} - x_{t-13}\right). \end{aligned}$$

# Content

# ARMA processis

The family of ARMA processes is central in time series analysis.

- AR model = Autoregressive model
- MA model = Moving Average model
- ARMA model = Autoregressive Moving Average model
- SAR model = Seasonal AR model
- SMA model = Seasonal MA model
- SARMA model = Seasonal ARMA model
- ARIMA model = Integrated ARMA model
- SARIMA model = Integrated Seasonal ARMA model

# Pure stochastic process

Discrete time stochastic process $(\epsilon_t)_{t \in T}$ is a pure stochastic process, if

1. (i) $\mathrm{E}[\epsilon_t] = \mu$, $t \in T$
2. (ii) $\mathrm{var}(\epsilon_t) = \sigma^2$, $t \in T$
3. (iii) $\mathrm{cov}(\epsilon_t, \epsilon_s) = 0$, $t \neq s$

- If the expected value $\mu = 0$, then the pure stochastic process is called white noise and the following notation is used:

$$(\epsilon_t)_{t \in T} \sim WN(0, \sigma^2).$$

- If the random variables $\epsilon_t$ are independent and identically distributed, then the pure white noise process is called iid white noise and the following notation is used:

$$(\epsilon_t)_{t \in T} \sim IID(0, \sigma^2)$$

# AR($p$) model

An autoregressive process of order $p$ is given by:

$$x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} + ...\phi_p x_{t-p} + \epsilon_t, \quad (\epsilon_t)_{t \in T} \sim WN(0, \sigma^2).$$

- This process is called autoregressive, because $x_t$ depends on $x_{t-1}, x_{t-2}, ..., x_{t-p}$ and because it resembles multiple linear regression model

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + ... + \beta_p x_p + \epsilon$$

  where:

  - The response variable is $x_t$ and the explanatory variables are $x_{t-1}, x_{t-2}, ..., x_{t-p}$.
  - The regression coefficients are $\beta_0 = 0$ and $\beta_i = \phi_i$, $i = 1, ..., p$.
  - The residual is $\epsilon_t$.

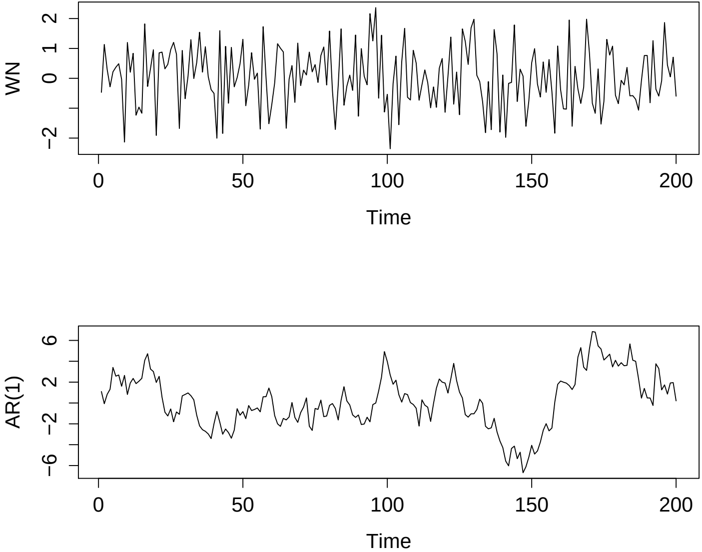
# AR($p$) model

**Example**

An AR(1) process is given by:

$$x_t = \phi_1 x_{t-1} + \epsilon_t, \quad (\epsilon_t)_{t \in T} \sim WN(0, \sigma^2)$$

# White noise vs AR(1)



# MA($q$) model

A moving average process of order $q$ is given by:

$$x_t = \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2} + \dots + \theta_q \epsilon_{t-q}, \quad (\epsilon_t)_{t \in T} \sim WN(0, \sigma^2)$$

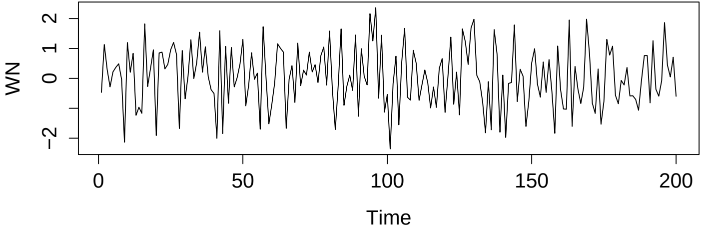
- The random variable $x_t$ is the weighted sum of the random variables $\epsilon_{t-q}, \dots, \epsilon_t$.

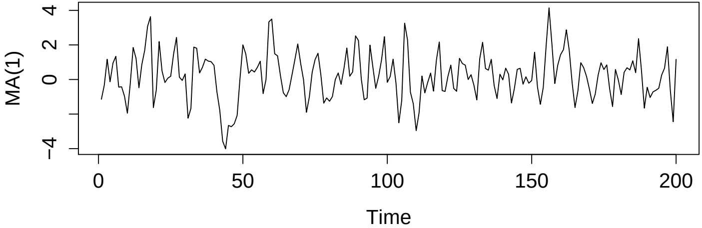
**Example**

A MA(1) process is given by:

$$x_t = \epsilon_t + \theta_1 \epsilon_{t-1}, \quad (\epsilon_t)_{t \in T} \sim WN(0, \sigma^2)$$

# White noise vs MA(1)



# ARMA($p$, $q$) model

An autoregressive moving average process with an AR part of order $p$ and a MA part of order $q$ is given by:

$$x_t - \phi_1 x_{t-1} - \phi_2 x_{t-2} - \ldots - \phi_p x_{t-p} = \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2} + \ldots + \theta_q \epsilon_{t-q},$$

where $(\epsilon_t)_{t \in T} \sim WN(0, \sigma^2)$.

- $x_t$ depends on both, the random variables $x_{t-1}, .., x_{t-p}$ and the random variables $\epsilon_{t-1}, ..., \epsilon_{t-q}$.

**Example**

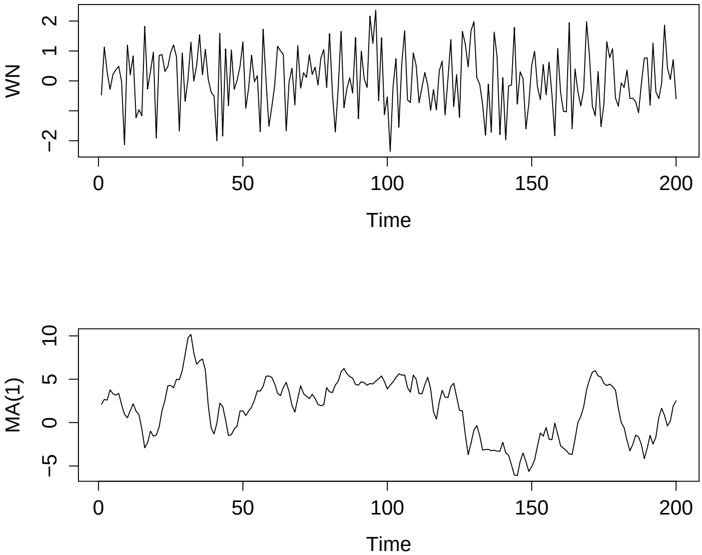
An ARMA(1,1) process is given by:

$$x_t - \phi_1 x_{t-1} = \epsilon_t + \theta_1 \epsilon_{t-1}$$

or equivalently

$$x_t = \phi_1 x_{t-1} + \theta_1 \epsilon_{t-1} + \epsilon_t$$

# White noise vs ARMA(1,1)



# SAR$(P)_s$ model and SMA$(Q)_s$ model

- A seasonal AR process of order $P$, with period $s$ is given by:

$$x_t = \Phi_1 x_{t-s} + \Phi_2 x_{t-2s} + \ldots + \Phi_P x_{t-Ps} + \epsilon_t, \quad (\epsilon_t)_{t \in T} \sim WN(0, \sigma^2).$$

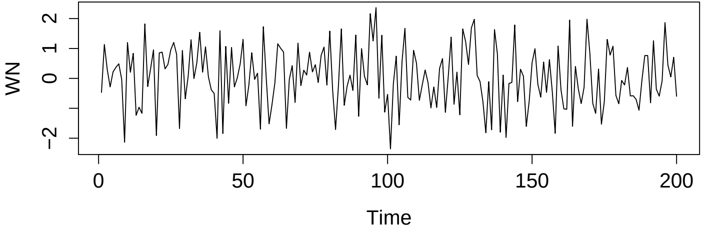
- A seasonal MA process of order $Q$, with period $s$ is given by:

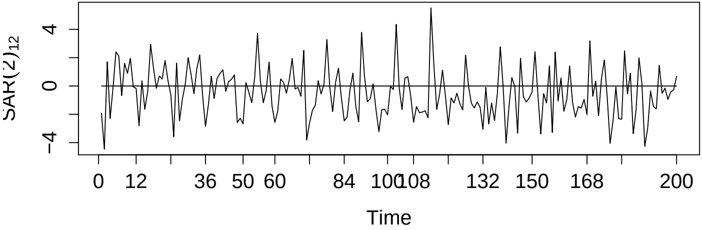
$$x_t = \epsilon_t + \Theta_1 \epsilon_{t-s} + \Theta_2 \epsilon_{t-2s} + \ldots + \Theta_Q \epsilon_{t-Qs}, \quad (\epsilon_t)_{t \in T} \sim WN(0, \sigma^2).$$

**Example**

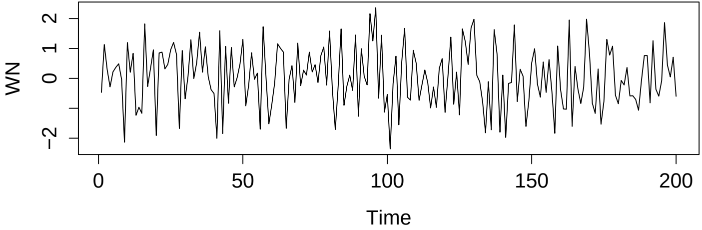
- A SAR$(2)_{12}$ process is given by:
  $x_t = \Phi_1 x_{t-12} + \Phi_2 x_{t-24} + \epsilon_t$
- A SMA$(1)_6$ process is given by: $x_t = \epsilon_t + \Theta_1 \epsilon_{t-6}$

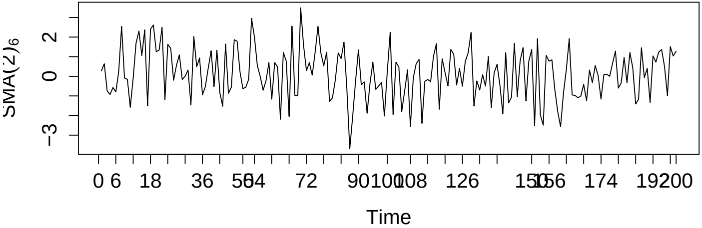
# White noise vs SAR(2)$_{12}$



# White noise vs SMA(1)$_6$



# SARMA$(P, Q)_s$ model

A seasonal ARMA process with period $s$, an AR part of order $P$ and a MA part of order $Q$ is given by:

$$x_t - \Phi_1 x_{t-s} - \dots - \Phi_P x_{t-Ps} = \epsilon_t + \Theta_1 \epsilon_{t-s} + \dots + \Theta_Q \epsilon_{t-Qs},$$

where $(\epsilon_t)_{t \in T} \sim WN(0, \sigma^2)$.

**Example**

A SARMA$(2,1)_4$ process is given by:

$$x_t - \Phi_1 x_{t-4} - \Phi_2 x_{t-8} = \epsilon_t + \Theta_1 \epsilon_{t-4}$$

or equivalently

$$x_t = \Phi_1 x_{t-4} + \Phi_2 x_{t-8} + \Theta_1 \epsilon_{t-4} + \epsilon_t$$

# Lag polynomials

We next consider cases with an AR part, a seasonal AR part, an MA part and a seasonal MA part. We start by getting familiar with lag polynomials.

# Lag polynomials: Definition

Lag polynomial of order $r$ is given by:

$$\delta_r(L) = 1 + \delta_1 L + \delta_2 L^2 + ... + \delta_r L^r.$$

- It now follows from the linearity of the operator $L$, that

$$\begin{aligned}\delta_r(L)x_t &= \left(1 + \delta_1 L + \delta_2 L^2 + ... + \delta_r L^r\right)x_t \\ &= x_t + \delta_1 L x_t + \delta_2 L^2 x_t + ... + \delta_r L^r x_t \\ &= x_t + \delta_1 x_{t-1} + \delta_2 x_{t-2} + ... + \delta_r x_{t-r}.\end{aligned}$$

**Example**

If $\phi(L) := 1 - \phi_1 L$ and $\Phi(L) := 1 - \Phi_1 L^{12}$, then we have that

$$\begin{aligned}\phi(L)\Phi(L)x_t &= \left(1 - \phi_1 L\right)\left(1 - \Phi_1 L^{12}\right)x_t \\ &= \left(1 - \phi_1 L - \Phi_1 L^{12} + \phi_1\Phi_1 L^{13}\right)x_t \\ &= x_t - \phi_1 x_{t-1} - \Phi_1 x_{t-12} + \phi_1\Phi_1 x_{t-13}.\end{aligned}$$

# SARMA$(p, q)(P, Q)_s$ model

A multiplicative seasonal ARMA process with period $s$, a pure AR part of order $p$, a pure MA part of order $q$, a seasonal AR part of order $P$, and a seasonal MA part of order $Q$ is given by:

$$\Phi_P^s(L)\phi_p(L)x_t = \Theta_Q^s(L)\theta_q(L)\epsilon_t, \quad (\epsilon_t)_{t\in T} \sim WN(0, \sigma^2),$$

where $\phi_p$, $\theta_q$, $\Phi_P^s$ and $\Theta_Q^s$ are the following lag polynomials

$$\begin{aligned}
\phi_p(L) &= 1 - \phi_1 L - \phi_2 L^2 - ... - \phi_p L^p \\
\theta_q(L) &= 1 + \theta_1 L + \theta_2 L^2 + ... + \theta_q L^q \\
\Phi_P^s(L) &= 1 - \Phi_1 L^s - \Phi_2 L^{2s} - ... - \Phi_P L^{Ps} \\
\Theta_Q^s(L) &= 1 + \Theta_1 L^s + \Theta_2 L^{2s} + ... + \Theta_Q L^{Qs}
\end{aligned}$$

(Here, it is customary to assume that the polynomials $\Phi_P^s(L)\phi_p(L)$ and $\Theta_Q^s(L)\theta_q(L)$ do not share roots.)

# SARMA$(p, q)(P, Q)_s$ model

Consider a SARMA$(p, q)(P, Q)_s$ process

$$\Phi_P^s(L)\phi_p(L)x_t = \Theta_Q^s(L)\theta_q(L)\epsilon_t, \quad (\epsilon_t)_{t\in T} \sim WN(0, \sigma^2),$$

Now

- The AR part is of order $p$; The corresponding parameters are: $\phi_1, \phi_2, ..., \phi_p$
- The seasonal AR part is of order $P$; The corresponding parameters are: $\Phi_1, \Phi_2, ..., \Phi_P$
- The MA part is of order $q$; The corresponding parameters are: $\theta_1, \theta_2, ..., \theta_q$
- The seasonal MA part is of order $Q$; The corresponding parameters are: $\Theta_1, \Theta_2, ..., \Theta_Q$

# SARMA$(p, q)(P, Q)_s$ model

Note that the SARMA$(p, q)(P, Q)_s$ models

$$\Phi_P^s(L)\phi_p(L)x_t = \Theta_Q^s(L)\theta_q(L)\epsilon_t, \quad (\epsilon_t)_{t\in T} \sim WN(0, \sigma^2)$$

cover all the following processes:

- AR$(p)$
- MA$(q)$
- ARMA$(p, q)$
- SAR$(P)_s$
- SMA$(Q)_s$
- SARMA$(P, Q)_s$

# Roots of the lag polynomials

Based on the fundamental theorem of algebra, the lag polynomials of order $r$

$$\delta_r(L) = 1 + \delta_1 L + \delta_2 L^2 + ... + \delta_r L^r$$

have $r$ roots(, that may or may not be complex valued).

**Example**

Let $\phi(L) = 1 - L + \frac{1}{2}L^2$. The the roots of the polynomial $\phi(L)$

$$L_1 = 1 + i \quad \text{and} \quad L_2 = 1 - i$$

lie outside of the unit circle:

$$\|L_1\|^2 = \|L_2\|^2 = 2.$$

In what follows, when we consider different SARMA$(p, q)(P, Q)_s$ models, we assume that $\mathrm{E}[x_{t-v}\epsilon_t] = 0$ for all $v \geq 1$. Moreover, we assume that the corresponding polynomials $\Phi_P^s(L)\phi_p(L)$ and $\Theta_Q^s(L)\theta_q(L)$ do not share roots.

# SARMA$(p, q)(P, Q)_s$ model: Stationarity

SARMA$(p, q)(P, Q)_s$ process $x_t$ is stationary, if and only if the roots of the lag polynomials of the AR part

$$\phi_p(L) = 1 - \phi_1 L - \phi_2 L^2 - ... - \phi_p L^p$$
$$\Phi_P^s(L) = 1 - \Phi_1 L^s - \Phi_2 L^{2s} - ... - \Phi_P L^{Ps}$$

lie outside of the unit circle.

**Fact**

A SARMA process can not be analyzed using auto- and partial autocorrelation functions unless it is stationary.

# SARMA$(p,q)(P,Q)_s$ model: Stationarity

A SARMA$(p,q)(P,Q)_s$ process $x_t$ is stationary if and only if it has an MA($\infty$) representation

$$x_t = \Psi(L)\epsilon_t, \quad (\epsilon_t)_{t\in T} \sim WN(0,\sigma^2),$$

where

$$\Psi(L) = \phi^{-1}(L)\Phi^{-1}(L)\theta(L)\Theta(L) = \sum_{i=0}^{\infty} \psi_i L^i, \quad (\psi_0 = 1),$$

and where the series

$$\sum_{i=0}^{\infty} \psi_i$$

converges absolutely.

# SARMA$(p,q)(P,Q)_s$ model: Invertibility

A SARMA$(p,q)(P,Q)_s$ process is called invertible, if it has an AR($\infty$) representation

$$\Pi(L)x_t = \epsilon_t, \quad (\epsilon_t)_{t\in T} \sim WN(0,\sigma^2),$$

where

$$\Pi(L) = \theta^{-1}(L)\Theta^{-1}(L)\phi(L)\Phi(L) = \sum_{i=0}^{\infty} \pi_i L^i, \quad (\pi_0 = 1)$$

and where the series

$$\sum_{i=0}^{\infty} \pi_i$$

converges absolutely.

# SARMA$(p, q)(P, Q)_s$ model: Invertibility

A SARMA$(p, q)(P, Q)_s$ process is invertible, if and only if the roots of the lag polynomials of the MA part

$$\theta_q(L) = 1 + \theta_1 L + \theta_2 L^2 + ... + \theta_q L^q$$
$$\Theta_Q^s(L) = 1 + \Theta_1 L^s + \Theta_2 L^{2s} + ... + \Theta_Q L^{Qs}$$

lie outside of the unit circle.

**Fact**

The autocorrelation function of a SARMA process does not define the MA and the seasonal MA parts of the process uniquely unless the process is invertible.

# Stationarity and invertibility

## Example

1. An AR($p$) process:

$$x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} + ... \phi_p x_{t-p} + \epsilon_t, \quad (\epsilon_t)_{t \in T} \sim WN(0, \sigma^2).$$

   - Is stationary iff the roots of the lag polynomial (of the AR part) lie outside of the unit circle.
   - Is always invertible.

2. A MA($q$) process

$$x_t = \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2} + ... + \theta_q \epsilon_{t-q}, \quad (\epsilon_t)_{t \in T} \sim WN(0, \sigma^2).$$

   - Is always stationary.
   - Is invertible iff the roots of the lag polynomial (of the MA part) lie outside of the unit circle.

# Spectrum of a stationary process

- If the analysis of a time series is based on correlation functions, we say that the analysis takes place in the time domain.
- The analysis of a stationary time series can also be conducted in the frequency domain.
  - In the frequency domain, the analysis of a time series is based on the so called spectral density function $f(\lambda)$ of the process.
  - The analysis conducted in the frequency domain is especially useful in revealing cyclic components of the process.
- The autocovariance function $\gamma_k$ and the spectral density function $f(\lambda)$ of a stationary process have exactly the same information.

The spectral density function $f(\lambda)$ (also called the power spectral function or spectrum) of a stationary process $(x_t)_{t\in T}$ is given by

$$f(\lambda) = \frac{1}{2\pi}\left(\gamma_0 + 2\sum_{k=1}^{\infty}\gamma_k \cos(\lambda k)\right), \quad \lambda \in [0, \pi],$$

where $\gamma_k$ is the $k$. autocovariance of $(x_t)_{t\in T}$.

- $\lambda$: (angular) frequency
- $2\pi/\lambda$: period
- $\lambda/2\pi$: the number of cycles per time unit

**Fact**

$$\gamma_k = \int_{-\pi}^{\pi} f(\lambda)\cos(\lambda k)d\lambda = 2\int_{0}^{\pi} f(\lambda)\cos(\lambda k)d\lambda,$$

for all $k = 0, 1, 2, \ldots$ In particular $\mathrm{var}(x_t) = \gamma_0 = 2\int_0^{\pi} f(\lambda)d\lambda$

# Spectrum of a stationary process: Aliasing

$$f(\lambda) = \frac{1}{2\pi}\left(\gamma_0 + 2\sum_{k=1}^{\infty}\gamma_k \cos(\lambda k)\right), \quad \lambda \in [0, \pi],$$

- We see that the frequencies $\lambda$, $-\lambda$, $\lambda$ and $\lambda \pm 2s\pi$, $s = 1, 2, ...$ have the same values.
- This phenomena is called aliasing.
- One can examine the spectral density function only on the interval $[0, \pi]$.

# Spectrum and the cyclic components of a stationary process

Consider a stationary process that has a cyclic component with period $s$. Then the corresponding spectral density function obtains its maximal values at $\lambda_s = 2\pi/s$, the basic frequency, and also at harmonic frequencies

$$k\lambda_s, \quad k = 1, 2, ..., \lfloor s/2 \rfloor,$$

where $\lfloor s/2 \rfloor = \max\{m \in \mathbb{Z} \mid m \leq s/2\}$.

**Example**

If $s = 4$, then $\lambda_4 = \pi/2$ and there is only one harmonic frequency $\pi$. If $s = 12$, then $\lambda_{12} = \pi/6$ and the harmonic frequencies are $2\pi/6$, $3\pi/6$, $4\pi/6$, $5\pi/6$ and $\pi$.

References:


1. Brockwell, P., Davis, R. (2009): Time Series – Theory and Methods, Springer
2. Hamilton, J. (1994): Time Series Analysis, Princeton University Press

# Next week:

1. Characteristics of the ARMA models
   1. Statistical properties of the stationary ARMA models
   2. ARIMA and SARIMA models
2. Fitting an ARMA model
   1. Estimation
   2. Box-Jenkins method
   3. Decomposition of time series