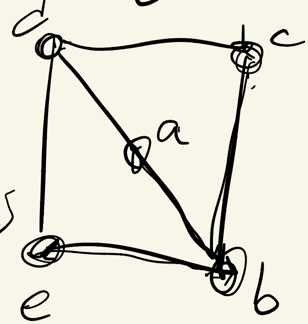
Graph theory

MS-E1050

$G = (V, E)$ edges

set of vertices/nodes



E set of unordered pairs of vertices

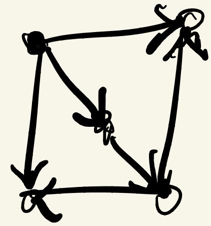
$$E = \{ \{ad\} \{ab\} \{bc\} \{cd\} \{de\} \{eb\} \}$$

$$= \{ad, ab, bc, cd, de, eb\}$$

<u>Simple</u> graph: no multiple edges, no loops, so all edges have <u>two</u> vertices

Directed graph: edges are ordered pairs

Multigraph:



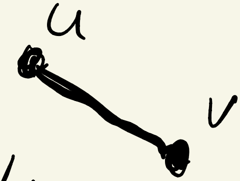
E multiset

loops allowed

In this course, unless otherwise stated, all graphs are simple, finite.

Def: If $\{u,v\} \overset{e}{=} \in E$,

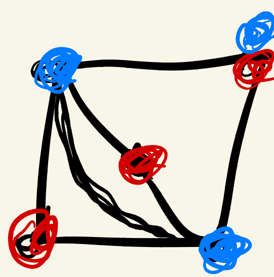
then:



- u and v are adjacent
- e is incident to u (and to v)
- v is a neighbour of u

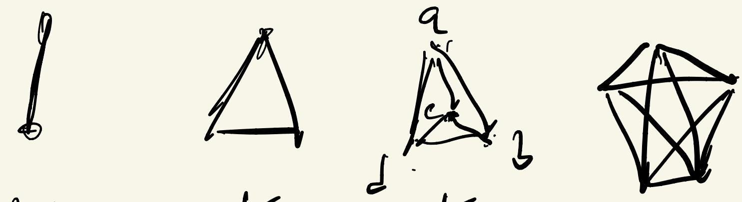
- An independent set in (or stable) $G=(V,E)$ is a set of nodes without edges between them.

- Clique is a set of nodes that are pairwise n-bours.



indep.

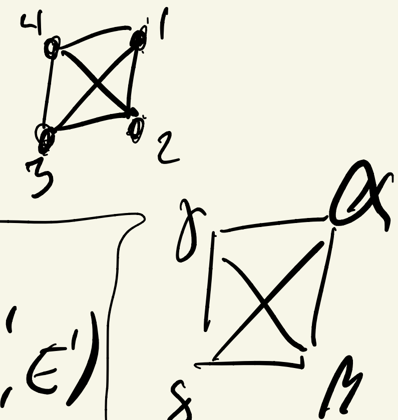
Clique on n nodes is denoted K_n (complete graph)



K_2 K_3 K_4 K_5

$\varphi : V \to V'$ is an <u>isomorphism</u> of graphs (V, E) and (V', E') if

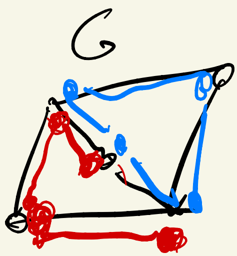
- φ is a bijection
- $\{u, v\} \in E \iff \{\varphi u, \varphi v\} \in E'$



G, H isomorphic if $\exists$ isomorphism between them.

("unlabelled" graph $\Longleftrightarrow$ isomorphism class of graphs)

H subgraph of G if
$= (V', E')$ $\quad = (V, E)$



$V' \subseteq V$

$E' \subseteq E$

H subgraph, not induced.

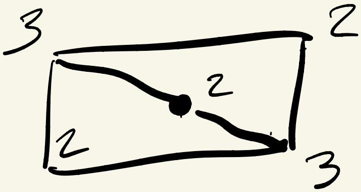
H induced subgraph of G
$= (V', E')$ $\quad = (V, E)$

if $V' \subseteq V$

$$E' = E \cap \binom{V'}{2}$$

2-elt subsets of V'

<u>Degree</u> of $v \in V$: number of edges incident to v

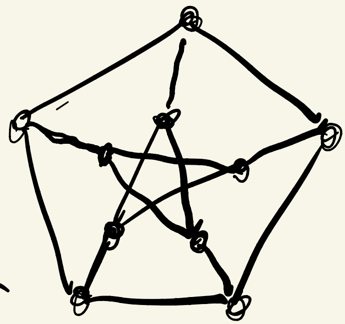


$d(v)$

$\delta(G) = 2 \quad \Delta(G) = 3$

G <u>Regular</u> if all vertices have same degree

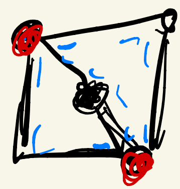
<u>Ex:</u> Petersen graph
3-regular



Minimal degree $\quad \delta(G) = \min_{v \in V} d(v)$

Maximal degree $\quad \Delta(G) = \max_{v \in V} d(v)$.

Prop The number of nodes of odd degree is even.



2 odd

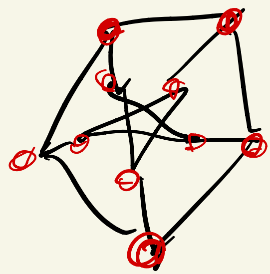
Proof

$$2\cdot|E| = \#\{\text{edge-vertex incidences}\}$$

$$2\cdot 6 =$$

$$= \sum_{v\in V} d(v)$$

$$= 2+2+2+3+3$$



10 odd

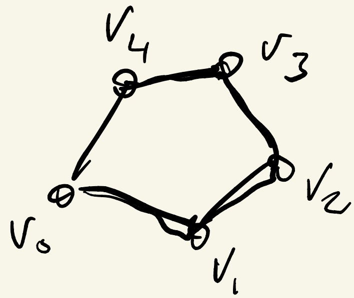
So the number of odd terms in the sum is even. ∎

PATHS & CYCLES

A **Path** of length n is a sequence $v_0 \dots v_n$ of **distinct** vertices s.t. $\{v_i v_{i+1}\} \in E$ for all $i = 0 \dots n-1$

"Path" = "simple walk"

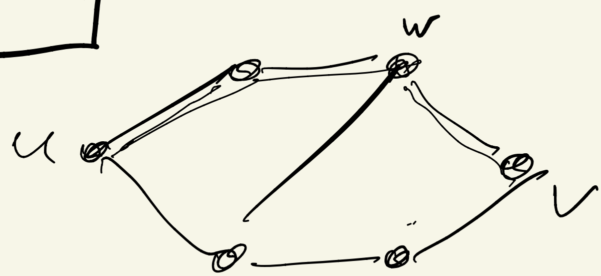
Cycle of length $n+1$ is a path $v_0 \dots v_n$ s.t. $\{v_n v_0\} \in E$



Distances $d_G(u,v)$ is the length of a shortest path from u to v.

if no path $u-v$, then $d(u,v) = \infty$

$d(u,v)=3$

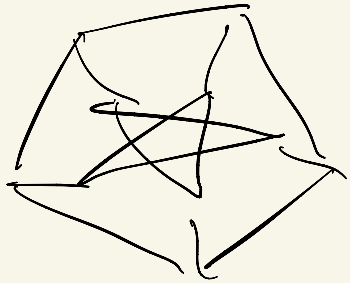


$$d(u,v) + d(v,w) \geq d(u,w)$$
$$d(u,v) \geq 0$$
$$d(u,v) = 0 \iff u = v$$

Diameter of $G=(V,E)$

is $\max_{u,v \in V} d(u,v)$

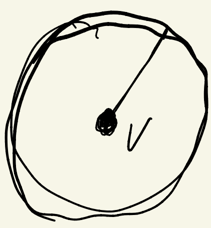
Def: **Girth** $g(G)$ is the length of the shortest cycle in G.



$g(P) = 5$

Def: Radius of $G = (V, E)$

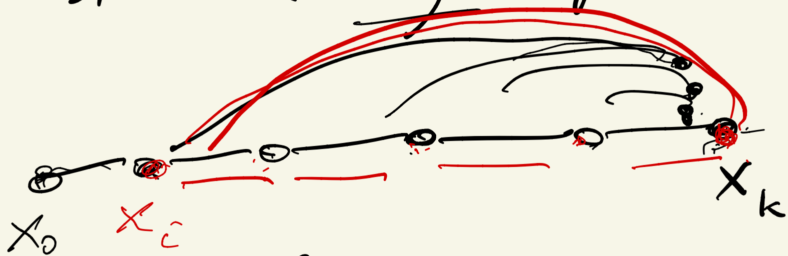
$$r(G) = \min_{v \in V} \left(\max_{u \in V} d(u, v) \right)$$



the $v \in V$ that minimizes this is **central** vertex.

Prop Every graph with $\delta \geq 2$ contains a path of length δ and a cycle of length $\geq \delta + 1$.

Proof: Consider a longest path



All n'bours of x_k are in the path, because otherwise we could extend the path.

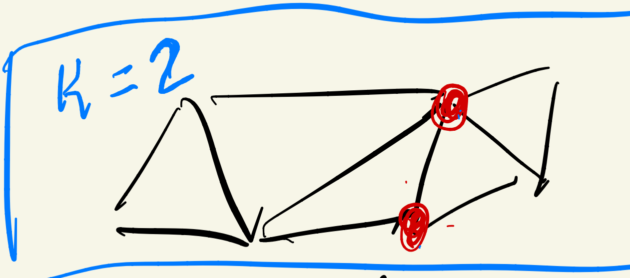
So $\delta \leq d(x_k) \leq k$

Let x_i be the first n'bour of x_k in the path.

Then $x_i \ldots x_k x_i$ cycle with $\geq \delta + 1$ nodes. $\blacksquare$

CONNECTIVITY

Def: G is connected, if there is a path between any pair of nodes.

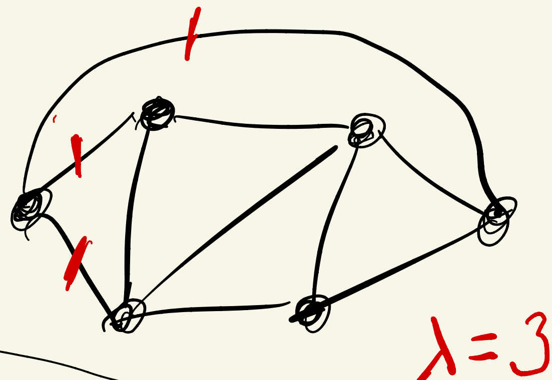


Def: $G=(V,E)$ is k-connected if $|V| \geq k$ and for all subsets $U \subseteq V$ w. $|U| < k$, $\underbrace{G[V-U]}_{\text{induced subgraph on vertices } V-U}$ is connected.

Def: Largest k s.t. G is k-connected is the <u>connectivity</u> $K(G)$

Def $G = (V,E)$ is k-edge-connected if $|E| \geq k$ and for all $F \subseteq E$, $|F| < k$, $(V, E-F)$ is connected

$\lambda(G)$ (edge connectivity) is largest k s.t. G is k-edge connected.



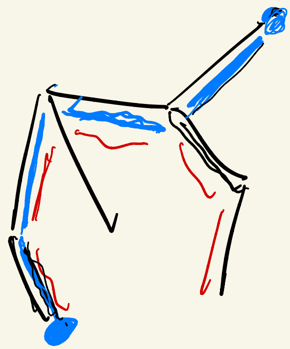
Next time

Prop:

$$\kappa(G) \leq \lambda(G) \leq \delta(G)$$

Tree is a connected graph with no cycles

maximal path endpoints v have $d(v)=1$.



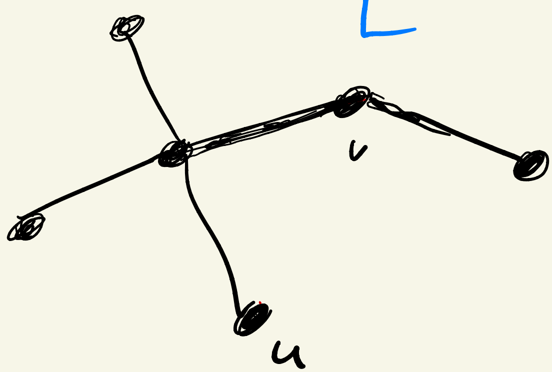
Forest is a disjoint union of trees.

i.e. a graph w no cycles.

So every tree has at least two **leaves** (ie degree 1 nodes)

Thm 1.5.1. TFAE:

1) $T = (V,E)$ is a tree
2) For any $u, v \in V$, there is a unique path $u - v$
3) T is maximally acyclic on V [If $E \subsetneq F$, then (V,F) has cycle]
4) T is minimally connected on V. [If $F \subsetneq E$, then (V,F) is disconn.]

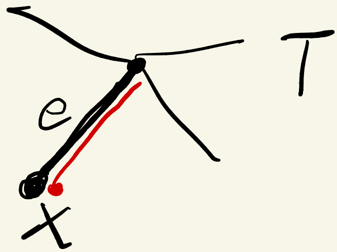


Claim: A tree with $|V| = n$ has $|E| = n-1$

Pf: By induction on n.

(base case $n=1$)

ind. step:
Add leaf x.



$T \setminus \{x\}$ is tree with $n-1$ nodes $n-2$ edges

Indeed, if $G=(V,E)$ is connected, then

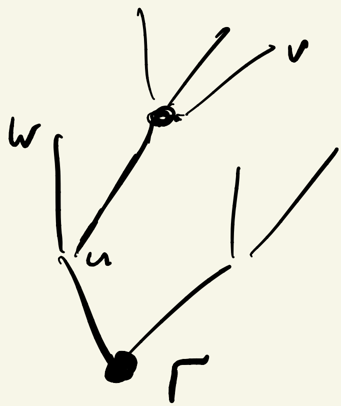
$$G \text{ tree} \iff \boxed{|E| = |V| - 1}$$

Thm: If $G=(V,E)$ connected, then it has a spanning tree $T=(V,F)$ $F \subseteq E$.

A tree with a prescribed root $r \in V$, is a rooted tree, yields a partial order on V.

$u \leq v$ if the r–v path goes through u.

TREE ORDER



$u \leq v$

v & w incomparable.

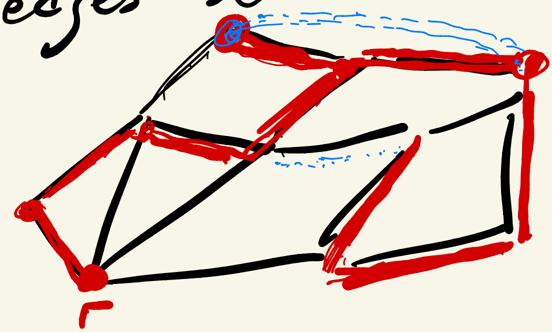
NORMAL TREE (**DEPTH FIRST SEARCH TREE**)

is a rooted spanning tree of G s.t.

G has no edges between incomp. nodes

exists for all conn. G

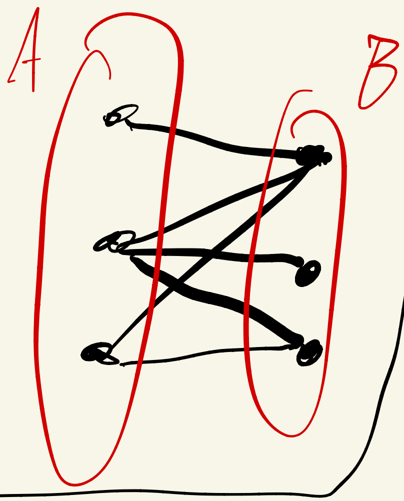
all roots r.



Def

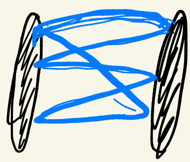
$(V,E)=G$ **bipartite** if $V = A \sqcup B$

s.t. $E\cap\binom{A}{2} = E\cap\binom{B}{2} = \emptyset$



Thm: G bipartite $\iff$ G has no odd cycles.

Pf: $\Longrightarrow$

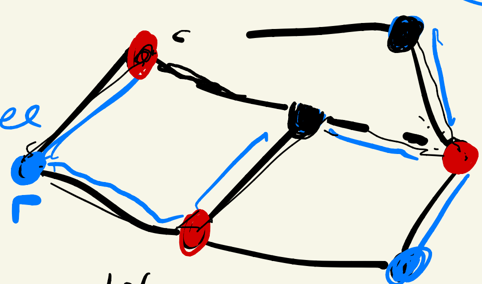


a cycle in bip. visits A & B equally many times.

$\Longleftarrow$ (normal)

T spanning tree

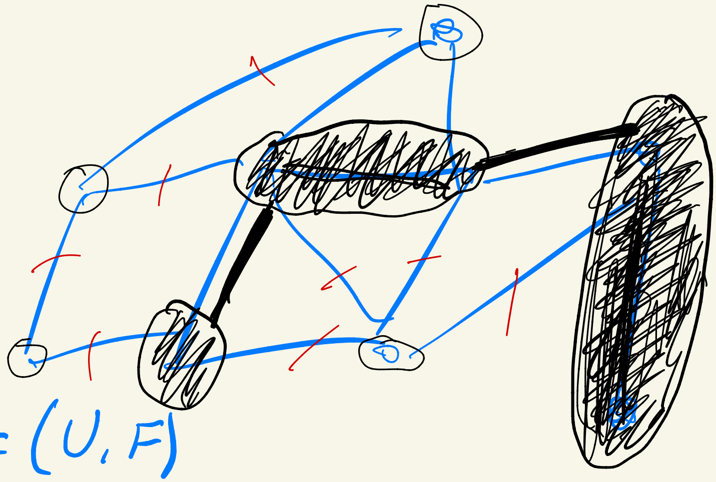
$r \in V$



$A = \{v \in V :$ $v-r$ path has even length.$\}$

$(A, \bar{A})$ bipartition.

Def



$H = (U, F)$

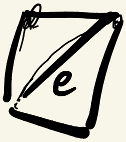
$G = (V, E)$ minor of H if

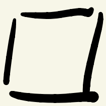
there are disjoint sets $U_x \subseteq U$, $x \in V$

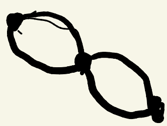
s.t.

$H[U_x]$ connected,

$\{xy\} \in G \iff$ there is an edge $U_x - U_y$ in H.

G 

$G \setminus e$ 

G/e  (multigraph)

G minor of $H \iff$ there are edges $d_1 \dots d_k$ s.t. $c_1 \dots c_l$

$H = (U, F)$

$v\left(H \setminus \{d_1 \dots d_k\}\right)/(c_1 \dots c_l) \cong G$ union isolated nodes.

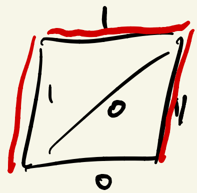
$\mathbb{F}_2 = \{0,1\}$ (with addition $1+1=0$)

Edge space $\mathcal{E}(G) = \{ f : E \to \mathbb{F}_2 \}$

(subset of edges)

Addition in $\mathcal{E}(G)$ $\Leftrightarrow$ symm. difference.

$$\langle f, f' \rangle = \sum_{e \in E} f(e) \cdot f'(e)$$



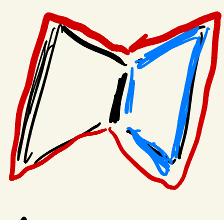
$$= \begin{cases} 0 & \text{if } f \cap f' \text{ even} \\ 1 & \text{if } f \cap f' \text{ odd} \end{cases}$$

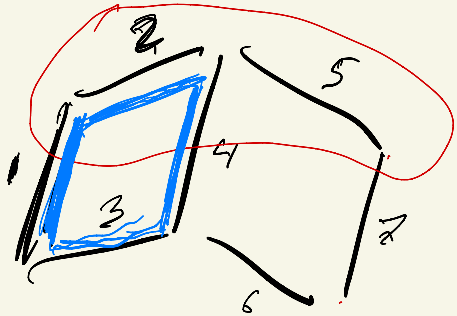
(seen as edge sets)

$\mathcal{C}(G) \subseteq \mathcal{E}(G)$: spanned by (edge sets of) cycles

$\mathcal{B}(G) \subseteq \mathcal{E}(G)$

contains all edge sets between A and $\bar{A}$, for subsets $A \subseteq V$.





$c = \{1,2,3,4\} \longleftrightarrow (1111000) \in \mathcal{C}(G)$

$b \;\; \{1,4,7\} \longleftrightarrow (1001001) \in \mathcal{B}(G)$

$$\langle c, b \rangle = 1\cdot 1 + 1\cdot 0 + 1\cdot 0 + 1\cdot 1 + 0\cdot 0 + 0\cdot 0 + 0\cdot 1$$
$$= 0$$

Note: every cycle intersects every cut an even number of times, so $\mathcal{C} \perp \mathcal{B}$

Obvious

$$\dim \mathcal{E}(G) = m = |E|$$

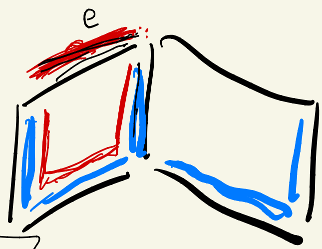
Claim

$$\dim \mathcal{C}(G) = m-n+1 = |E|-|V|+1 \quad \boxed{\text{CONNECTED}}$$

$$\dim \mathcal{B}(G) = n-1 = |V|-1$$

Span. tree

$T \subseteq G$



WTS: $\boxed{\mathcal{C}(G)}$ has a basis with one elt for each edge in $G-T$

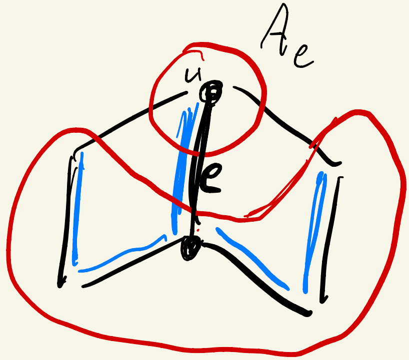
$\boxed{\mathcal{B}(G)}$ has a basis with one elt for each edge in T.

So for $e = \{u,v\}$ not in T, its fundamental cycle C_e is $\{e\} \cup$ the u-v path in T.

Clearly, $\{C_e : e \in E \setminus E(T)\}$ lin. independent

so

$\dim \mathcal{C} \geq \underline{m-n+1}$



Let u be one of the endpts of e, let A_e be the set of nodes reachable in $T-e$ from u.

Let $B_e = \{$edges between A_e and $\bar{A}_e\}$

fundamental cut.

$e \in B_e$

$e \notin B_f$ if $f \in T_{-e}$.

so $\{B_e\}$ is a lin. indep set

so $\dim \mathcal{B} \geq n-1$.