Worksheet 1

MS-E1621, Algebraic Statistics

September 9, 2020

Group members: Write your names here.

1. Which points in $\mathbb{R}^2$ satisfy
   (a) the equation $p_1^2 + p_2^2 - 1 = 0$?
   (b) the equation $p_1 - p_2^2 = 0$?
   (c) both equations $p_1 - p_2^2 = 0$ and $p_1^2 + p_2^2 - 1 = 0$?

   These solution sets of systems of polynomial equations are *algebraic varieties*. If S is a set of polynomials, then then the variety defined by S is denoted $V(S)$.

2. How does the answer in Exercise 1(c) change if one considers only points in $\mathbb{Q}^2$? If one wants to emphasize over which field one considers the variety, then one can write $V_{\mathbb{K}}(S)$ instead of $V(S)$.

3. Consider the polynomial map $\phi : \mathbb{C} \to \mathbb{C}^3$ given by
$$t \mapsto (t, t^2, t^3).$$
   Why is the image $\phi(\mathbb{C}) = \{\phi(t) : t \in \mathbb{C}\}$ an algebraic variety? In this exercise, a variety is presented as a *parametric set*.

4. Let ϕ be as in Exercise 3. What do you think are all polynomials $f \in \mathbb{C}[p_1, p_2, p_3]$ such that $f(a) = 0$ for all $a \in \phi(\mathbb{C})$? The set of such polynomials is the *vanishing ideal* of $\phi(\mathbb{C})$ or the *defining ideal* of $\phi(\mathbb{C})$, and it is denoted $I(\phi(\mathbb{C}))$.

5. Let $S = \{p_1^2, p_2\} \subset \mathbb{R}[p_1, p_2]$. Consider the variety
$$V(S) = \{a \in \mathbb{R}^2 : a_1^2 = 0, a_2 = 0\}.$$
   Is the vanishing ideal $I(V(S))$ equal to the ideal $\langle S \rangle$ generated by S? The *ideal generated by* S consists of polynomials $f \in \mathbb{R}[p_1, p_2]$ that can be written as $f = h_1 p_1^2 + h_2 p_2$, where h_1, h_2 are arbitrary polynomials in $\mathbb{R}[p_1, p_2]$. The vanishing ideal $I(V(S))$ always contains the ideal $\langle S \rangle$.

6. An ideal I is called *radical* if $f^k \in I$ for some polynomial f and positive integer k implies $f \in I$. Is the ideal $\langle S \rangle$ in Exercise 5 radical? Is the ideal $I(V(S))$ in Exercise 5 radical?

7. The *radical of an ideal* I, denoted $\sqrt{I}$, is the smallest radical ideal that contains I:
$$\sqrt{I} = \{f \in \mathbb{K}[p] : f^k \in I \text{ for some } k \in \mathbb{N}\}.$$
What is the radical of the ideal $\langle S \rangle$ in Exercise 5?

   A fundamental result in algebraic geometry is *Nullstellensatz* due to Hilbert in 1893. It states that if $\mathbb{K}$ is algebraically closed, then the vanishing ideal of the variety of an ideal is the radical of the ideal, i.e. $I(V(I)) = \sqrt{I}$.

8. Together with the Nullstellensatz, the identity $V(I(V)) = V$ establishes the *ideal-variety correspondence*: In an algebraically closed field, there is an inclusion-reversing bijection between the set of varieties and the set of radical ideals. What is the maps from the set of varieties to the set of radical ideals, and vice versa, that gives the bijection?

9. Let $S = \{p^6 - p, p^4 - p\} \subset \mathbb{R}[p]$ and $f = p^3 - p$. Is f in the ideal $\langle S \rangle$?

   Divide the polynomial f by S. Dividing an univariate polynomial by a finite set S of polynomials is similar to dividing it by one polynomial: The only difference is that at each step one is allowed to divide by any polynomial in the set S whose highest degree term divides a term of the remainder.

   Let $I \subset \mathbb{K}[p]$ be an ideal. A finite subset $\mathcal{G}$ of I is called a *Gröbner basis* if dividing f by $\mathcal{G}$ gives remainder 0 for all $f \in I$. Is S a Gröbner basis of the ideal $\langle S \rangle$?

   Division algorithm and Gröbner bases can be defined for multivariate polynomial rings. Almost any computation with ideals requires finding a Gröbner basis behind the scenes. We will not cover the multivariate case here, but you can read more on Gröbner bases from Chapter 3.3.

10. Let $S = \{p - q, p^2 - q\} \subset \mathbb{R}[p, q]$. Let $\pi : \mathbb{R}^2 \to \mathbb{R}$ be the coordinate projection
$$(a, b) \mapsto a.$$
What is the vanishing ideal of the image $\pi(V(S))$ in $\mathbb{R}[p]$?

    Let $V \subseteq \mathbb{K}^{r_1+r_2}$ be a variety and let $I := I(V) \subset \mathbb{K}[p_1, \ldots, p_{r_1}, q_1, \ldots, q_{r_2}]$ be its vanishing ideal. Then $I(\pi(V)) = I \cap \mathbb{K}[p]$, where π is the coordinate projection to the first r_1 coordinates. The ideal $I \cap \mathbb{K}[p]$ is called an *elimination ideal*. Why in this exercise we can consider $\langle S \rangle \cap \mathbb{K}[p]$ instead of $I(V(S)) \cap \mathbb{K}[p]$?