Probability primer

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Agenda

- First half: Group work in breakout rooms
  - MyCourses/Worksheets: Worksheet 2 according to your breakout room number
  - Copy and discuss your questions
  - Discuss questions prepared by me
  - The group can decide whether to write down solutions
- Break
- Second half: Lecture on connecting probability and algebra we have studied so far

Last time

Def: Let $S \subset \mathbb{K}[p]$ be a set of polynomials. The variety defined by S is

$$V(S) = \{a \in \mathbb{K}^r : f(a) = 0 \quad \forall f \in S\}.$$

Def: Let $W \subseteq \mathbb{K}^r$. The vanishing ideal of W is

$$I(W) = \{f \in \mathbb{K}[p] : f(a) = 0 \quad \forall a \in W\}.$$

Def: Let S be a set of polynomials. The ideal generated by S is

$$\langle S \rangle = \{ \sum_{i=1}^{k} h_i f_i : f_i \in S, h_i \in \mathbb{K}[p]\}.$$

Binomial random variables

- Consider the polynomial map $\phi : \mathbb{C} \to \mathbb{C}^{r+1}$ whose ith coordinate function is

$$\phi_i(t) = \binom{r}{i} t^i (1-t)^{r-i}, \quad i = 0,1,\ldots,r\,.$$

- Let $\theta \in [0,1]$ and let X be the discrete random variable that counts the number of heads in r coin tosses where the coin shows heads with the probability θ. In other words, $X \sim Bin(r, \theta)$.

- The value $\phi_i(\theta)$ is the probability $P(X = i)$.

- The vector $\phi(\theta)$ is the distribution of X.

Implicitization problem

- Goal: Describe the image of the real interval $[0,1]$ under the map ϕ.
- Simplified goal: Describe $\phi(\mathbb{C})$.

Question 1: Find a finite set of polynomials that generate $I(\phi(\mathbb{C}))$.

- If $W = \phi(\mathbb{K}^d)$ for some rational map ϕ, then finding a finite set of polynomials that generate $I(W)$ is called an implicitization problem.
- A rational map is a map $\phi : \mathbb{K}^d \to \mathbb{K}^r$, where $\phi_i = f_i/g_i$ and $f_i, g_i \in \mathbb{K}[t_1, \ldots, t_d]$.

Question 2: Is $\phi(\mathbb{C})$ a variety?

Image of parametrization

Theorem: Let $\mathbb{K}$ be algebraically closed, let $V \subseteq \mathbb{K}^d$ be a variety, and let ϕ be a rational map $\phi : V \to \mathbb{K}^r$. Then there is a finite sequence of varieties $W_1 \supset W_2 \supset \cdots \supset W_k$ in $\mathbb{K}^r$ such that $\phi(V) = W_1 \backslash (W_2 \backslash (\cdots (\backslash W_k) \cdots))$.

Example 1: $\phi : \mathbb{C} \to \mathbb{C}^2, t \mapsto (\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2})$

Example 2: $\phi : \mathbb{R} \to \mathbb{R}, t \mapsto t^2$

$\Longrightarrow$ The theorem holds only when $\mathbb{K}$ is algebraically closed.

Zariski closure

- Let $\mathbb{K}$ be a field and let $W \subseteq \mathbb{K}^r$ be a set.

Prop: The set $V(I(W))$ is called the Zariski closure of W. It is the smallest variety containing W.

- A set W is called Zariski closed if it is a variety.
- Zariski closed sets are also closed in the standard Euclidean topology.

Question 1: Let $W \subset \mathbb{R}^2$ be the p_1-axis with the origin $(0,0)$ removed. What is the Zariski closure of W?

Question 2: What is the Zariski closure of $[0,1]$ inside $\mathbb{R}$?

Graph of a map

- Let $\phi : \mathbb{K}^d \to \mathbb{K}^r$ be a polynomial map.
- The graph of the map is the set of points

$$\Gamma_\phi = \{(\phi(\theta), \theta) : \theta \in \mathbb{K}^d\} \subset \mathbb{K}^{r+d}.$$

- If $\mathbb{K}$ is an infinite field, the vanishing ideal of the graph is

$$I(\Gamma_\phi) = \langle p_1 - \phi_1(q), \ldots, p_r - \phi_r(q) \rangle \subseteq \mathbb{K}[p_1, \ldots, p_r, q_1, \ldots, q_d].$$

Sum of ideals

- Define the sum of two ideals $I, J \subseteq \mathbb{K}[p]$ as

$$I + J := \{f + g : f \in I \text{ and } g \in J\}.$$

- If $I = \langle \mathcal{F} \rangle$ and $J = \langle \mathcal{G} \rangle$, then $I + J = \langle \mathcal{F} \cup \mathcal{G} \rangle$.

Implicitization problem

Prop: Let $I \subseteq \mathbb{K}[q_1, \ldots, q_d]$ be the vanishing ideal of a variety $V \subseteq \mathbb{K}^d$. Let $\phi : \mathbb{K}^d \to \mathbb{K}^r$ be a polynomial map. Then the vanishing ideal of the Zariski closure is the elimination ideal

$$I(\phi(V)) = (I + I(\Gamma_\Phi)) \cap \mathbb{K}[p_1, \ldots, p_r].$$

Computations

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Macaulay2, version 1.14
--loading configuration for package "FourTiTwo" from file /Users/kubjask1/Librar→
--loading configuration for package "Topcom" from file /Users/kubjask1/Library/A→
with packages: ConwayPolynomials, Elimination, IntegralClosure, InverseSystems,
               LLLBases, PrimaryDecomposition, ReesAlgebra, TangentCone,
               Truncations

i1 : R = QQ[t,p0,p1,p2]

o1 = R

o1 : PolynomialRing

i2 : J = ideal(p0-(1-t)^2,p1-2*t*(1-t),p2-t^2)

                2                       2                 2
o2 = ideal (- t  + 2t + p0 - 1, 2t  - 2t + p1, - t  + p2)

o2 : Ideal of R

i3 : K = eliminate(t,J)

                                   2                2
o3 = ideal (p0 + p1 + p2 - 1, p1  + 4p1*p2 + 4p2  - 4p2)

o3 : Ideal of R

i4 :
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Behind the scenes a Gröbner basis is used for elimination.

Binomial random variables

Is $\phi(\mathbb{C}) = V(p_0 + p_1 + p_2 - 1, 4p_0p_2 - p_1^2)$?

- $p_0 = 1 - p_1 - p_2$
- $4p_0p_2 - p_1^2 = 0 \quad \iff \quad 4(1 - p_1 - p_2)p_2 - p_1^2 = 0 \quad \iff \quad p_1^2 + 4p_2p_1 + 4p_2^2 - 4p_2 = 0$
- Quadratic formula: $p_1 = -2p_2 \pm 2\sqrt{p_2}$
- Substitution: $p_0 = 1 + p_2 \mp 2\sqrt{p_2}$

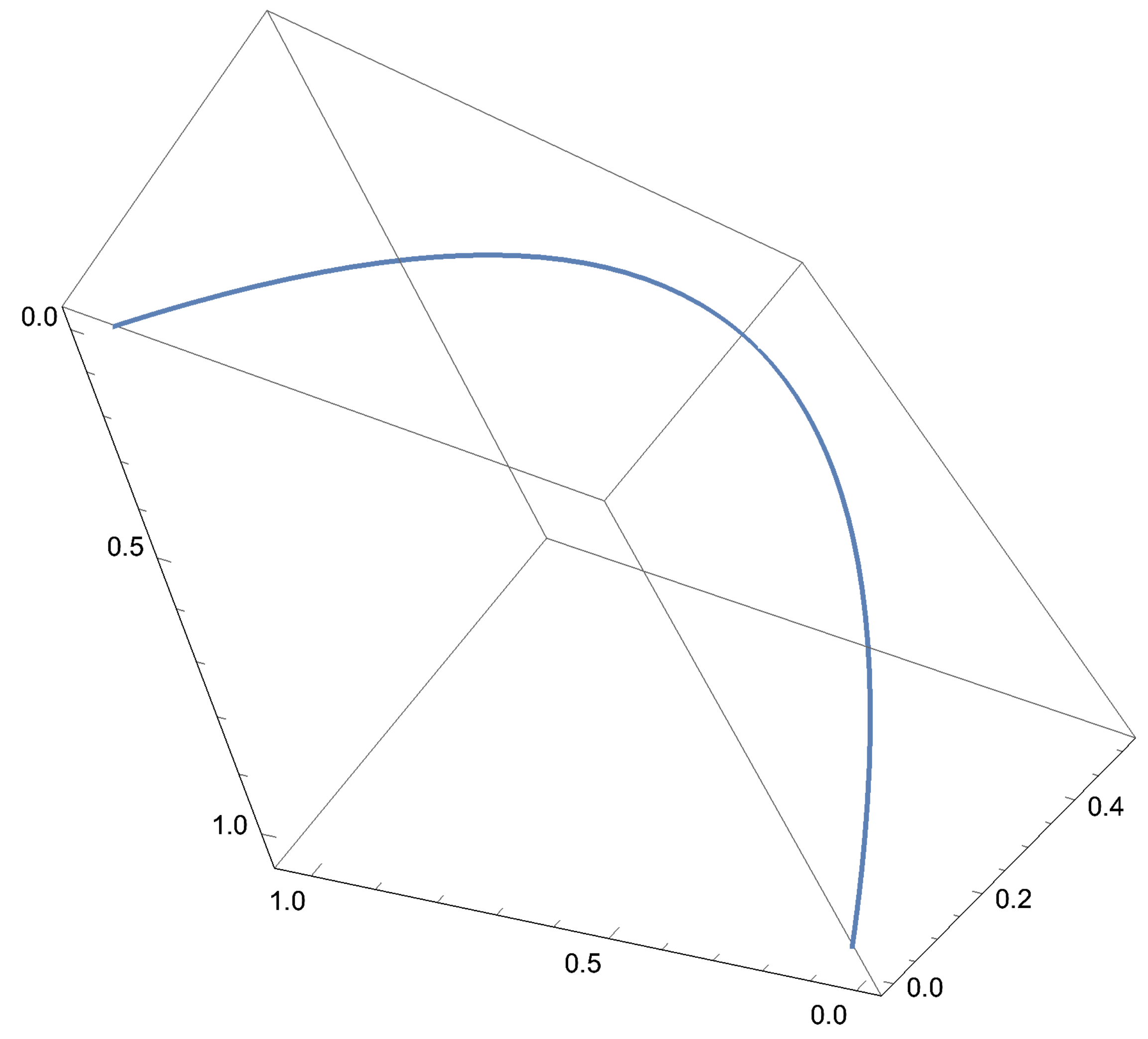
$\implies V(p_0 + p_1 + p_2 - 1, 4p_0p_2 - p_1^2) = \{(1 + p_2 \mp 2\sqrt{p_2}, -2p_2 \pm 2\sqrt{p_2}, p_2) : p_2 \in \mathbb{C}\}$

- $\phi(\sqrt{p_2}) = (1 + p_2 - 2\sqrt{p_2}, -2p_2 + 2\sqrt{p_2}, p_2), \phi(-\sqrt{p_2}) = (1 + p_2 + 2\sqrt{p_2}, -2p_2 - 2\sqrt{p_2}, p_2)$

$\implies \phi(\mathbb{C}) = V(p_0 + p_1 + p_2 - 1, 4p_0p_2 - p_1^2)$

Binomial random variables

- What is $\phi([0,1])$? (Extra homework)
- It is a semialgebraic set: It is a solution set of a system of polynomial equations and inequalities.
- Describing $\phi([0,1])$ is in general much more difficult than describing $\phi(\mathbb{C})$.
- Algebraic geometry vs real algebraic geometry.



Implicitization

Prop: Let $I \subseteq \mathbb{K}[q_1, \ldots, q_d]$ be the vanishing ideal of a variety $V \subseteq \mathbb{K}^d$. Let $\phi : \mathbb{K}^d \to \mathbb{K}^r$ be a rational map, with coordinate functions $\phi_i = f_i / g_i$. Then the vanishing ideal of the Zariski closure is the elimination ideal

$$I(\phi(V)) = (I + \langle g_1 p_1 - f_1, \ldots, g_r p_r - f_r, z g_1 \cdots g_r - 1 \rangle) \cap \mathbb{K}[p_1, \ldots, p_r].$$

- The ideal in the parentheses belongs to the ring $\mathbb{K}[p, q, z]$, where z is an extra indeterminate.

Next time: Conditional independence