Aalto University
School of Science

Lecture 3: Mathematical treatment of plasma, sadist… statistical approach

Today’s Menu

- Plasma as a statistical system
- Why phase space?
- What is a distribution function?
- Review of Maxwell-Boltzmann distribution
- Liouville, Boltzmann & Vlasov equations
- Concept of a collision operator

From single particles to plasma

Plasma as a collection of individual particles

Plasmas of interest consist of an enormous # of particles, N >>> 1

➔ Impractical to solve equations of motion for all pcles

➔ actually *impossible* due to 'infinite' # of interactions

But who is interested in the trajectory of an individual charge if there are, for instance, 10^{23} of them?

What matters is, e.g.,

- how *many* of them is in a given region ➔ density
- How *many* of them are moving at a given velocity ➔ (possible) flow

Statistical approach

We are not interested in the *identity* of 10^{23} particles.

An interesting = relevant quantity: particle density $n(\boldsymbol{r})$

$n(\boldsymbol{r})d^3r$ = # of particles in an infinitesimal volume d^3r @ $\boldsymbol{r}$

It does not matter who the particles @ $\boldsymbol{r}$ are.

Similarly:

- Mass density $n_m = m * n(\boldsymbol{r})$
- charge density $n_q = q * n(\boldsymbol{r})$

But how about stuff involving motion? Energy? Flow? Current?

Phase space and distribution function

Even the *dynamical* state of a plasma can be contained if we generalize the spatial density into the so-called

distribution function: $f(\boldsymbol{r}, \boldsymbol{v}, t)$

3D real space (x, y, z) ←→ 6D ***phase space*** (x, y, z, v_x, v_y, v_z)

Particle density $n(\boldsymbol{r}, t)$ ←→ distribution function: $f(\boldsymbol{r}, \boldsymbol{v}, t)$

$N = \iiint_{-\infty}^{\infty} n(\boldsymbol{r}) d^3r.$ How about integrals of $f(\boldsymbol{r}, \boldsymbol{v}, t)$?

What does the distribution function mean?

- The dynamical state of each plasma particle is given by its location $\boldsymbol{r}$ and its velocity (momentum) $\boldsymbol{v}$
- Thus each particle occupies some point in the six-dimensional *phase space* with its coordinate $\boldsymbol{z} = (\boldsymbol{r}, \boldsymbol{v})$
- The *distribution function* $f_s(\boldsymbol{r}, \boldsymbol{v}, t)$ (species *s*) ≡ the # of particles per unit (phase space) volume around point $\boldsymbol{z} = (\boldsymbol{r}, \boldsymbol{v})$

➔ $[f_s(\boldsymbol{r}, \boldsymbol{v}, t)] = \mathrm{m}^{-3} \left(\frac{m}{s}\right)^{-3}$

➔ $f_s(\boldsymbol{r}, \boldsymbol{v}, t)\ d^3v d^3r$ is the number of particles in the volume element $d^3v d^3r$ surrounding the point $(\boldsymbol{r}, \boldsymbol{v})$ at time t

Distribution function can be thought of also in more 'QM' way …

Two interpretations (*particle* vs *probability* distribution):

1. $f(\boldsymbol{r},\boldsymbol{v})$ = *6D phase space density:* $N = \iiint_{-\infty}^{\infty} d^3v \iiint_{-\infty}^{\infty} d^3r\, f(\boldsymbol{r},\boldsymbol{v})$

Then

$$\int f_s(\boldsymbol{r},\boldsymbol{v},t)\,\mathrm{d}^3\boldsymbol{v} = n_s(\boldsymbol{r},t)$$

2. $f(\boldsymbol{r},\boldsymbol{v})$ = *probability function:* $1 = \iiint_{-\infty}^{\infty} d^3v \iiint_{-\infty}^{\infty} d^3r\, f(\boldsymbol{r},\boldsymbol{v})$

Here,

$$f(\boldsymbol{r},\boldsymbol{v}) = \textit{probability} \text{ to find particles in a phase space element } d^3r d^3v$$

Moving around in velocity space …

The concept of *particle density* in *real space* = easy & comfortable

The velocity space distribution is, in principle, analogous: it simply tells how particles are distributed in *velocity space*.

But there ***is*** an important difference: not all velocities are ’born equal’! This is because velocity is related to energy, $E = \frac{1}{2}mv^2$, and there are laws of nature that govern the *energy distribution*…

Plasma in thermodynamic equilibrium
-- *revisiting the Maxwell-Boltzmann distribution*

Recall from StaFy lectures …

In thermodynamical equilibrium at temperature *T*, the energy state ε_i is occupied with probability $P(\varepsilon_i)$:

$$P(\varepsilon_i) = \frac{\exp(-\varepsilon_i/T)}{\sum_j \exp(-\varepsilon_j/T)}$$

In a *regular gas*, $E = \frac{1}{2}mv^2$, and the energy states are continuous

$$\rightarrow P(\varepsilon_i) \to f(\boldsymbol{v}) \;\; \& \;\; \sum_j \exp(-\varepsilon_j/T) \to \iiint_{-\infty}^{\infty} \exp\left(-\frac{\frac{1}{2}mv^2}{T}\right) dv_x dv_y dv_z$$

Find the normalization (HW) ➔ "*Maxwellian" distribution*

$$f(\boldsymbol{v}) = \left(\frac{m}{2\pi T}\right)^{3/2} \exp(-(v_x^2 + v_y^2 + v_z^2)/T)$$

… and apply to plasmas…

Note: in $\exp(-\varepsilon_i/T)$ the energy is the *total energy*.

For ***plasmas***, the charged particles frequently move in *electrostatic potential*, and the energy has to include also that:

$$e^{-\varepsilon_i/T} = e^{-\left(\frac{1}{2}mv^2 + q\Phi(\boldsymbol{r})\right)/T}$$

➔ the distribution function no longer is a straightforward product of 'real space density' and 'velocity space density'

From velocity distribution

Most of the time equilibirum plasmas are *isotropic* = all directions are equally likely ➔ only the *speed*, $v = |\boldsymbol{v}|$ is of interest.

Let's denote this *one-dimensional* distribution function by *g*(v).

But now *extremely careful!!!*

$$g(v) \neq \left(\frac{m}{2\pi T}\right)^{3/2} \exp(-\left(\frac{1}{2}mv^2\right)/T) \text{ !!!}$$

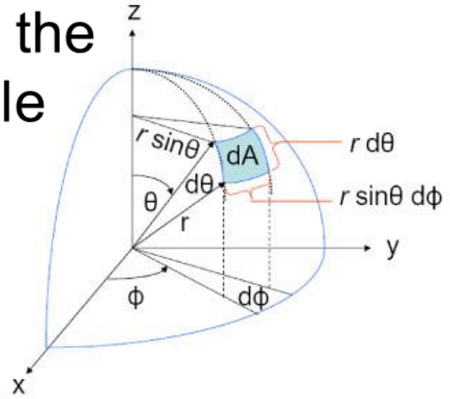
Even the dimensions are wrong!

What should remain intact is $g(v)dv = f(\boldsymbol{v})dv_x dv_y dv_z$

… to speed distribution and…

Directions do not matter ➔ the *natural* coordinate system is the spherical one: $d^3v = dv(v\sin\vartheta d\varphi)(vd\vartheta) \to 4\pi v^2 dv$

So the velocity space unit element replacing $dv_x dv_y dv_z$ has to include the terms $4\pi v^2$ -- not surprisingly, this is the surface area of a sphere of radius v, i.e., all the possible velocity *vectors* corresponding to the speed v.



$$➔ \; g(v) = 4\pi\left(\frac{m}{2\pi T}\right)^{3/2} v^2 \exp(-\left(\frac{1}{2}mv^2\right)/T)$$

… to (kinetic) energy distribution!

In plasma physics, one is mainly interested in the *kinetic energy*, not the speed (like in molecular physics, for instance).

Therefore the most common Maxwellian distribution used is the *energy distribution (HW):*

$$h(E) = \frac{2}{\sqrt{\pi} T^{\frac{3}{2}}} \sqrt{E}\, e^{-E/T}$$

Special quantities (HW)

Most probable speed obtained at the extremum of $g(v)$: $\frac{dg}{dv} = 0$

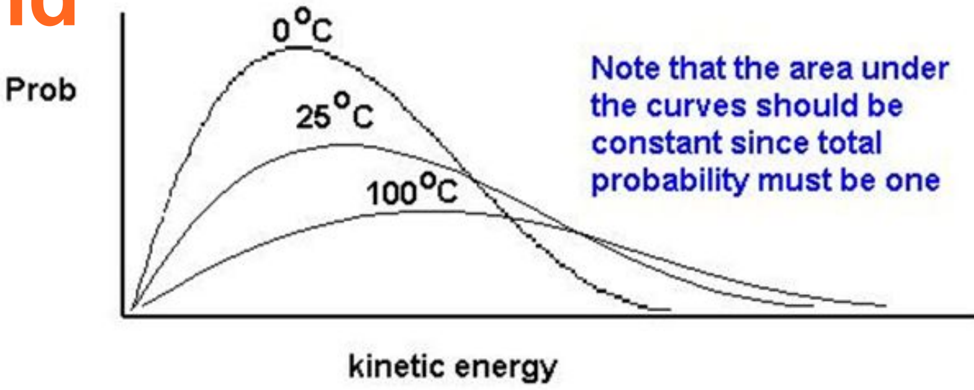
$$v_{MP} = \sqrt{2T/m}$$

Average speed: remember that $f(v)$ is a *probability* distribution…

$$v_{ave} = \sqrt{8T/\pi m} = \frac{2}{\sqrt{\pi}} v_{MP} > v_{MP}$$

Average (kinetic) energy: $E_{ave} = \frac{3}{2}T$

Things to keep in mind



- For a system in thermodynamical equilibrium, the most probable distribution of energies is given by the Maxwell-Boltzmann distribution
- the concept of temperature … only for Maxwellian systems!
- The temperature gives
  - *The width of the distribution*
  - *The average energy in the system*

Temperature curiosity in plasmas …

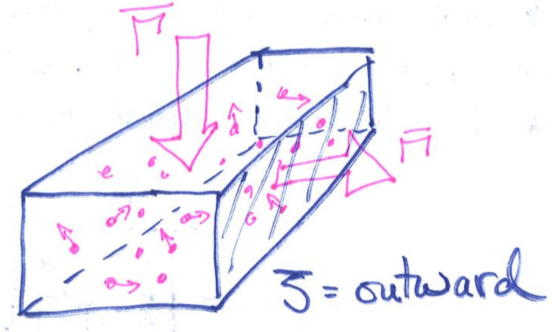
It is quite common that even an ’equilibrium’ plasma cannot be characterized with one single temperature…

1. In a magnetized plasma we can have $T_{\parallel} \neq T_{\perp}$
2. Different species can have different temperatures: $T_e \neq T_i$

This is to different rates of the *relaxation* processes.

Getting dynamical *Boltzmann equation*

Real space: continuity equatio



Particle *flux*, $\boldsymbol{\Gamma} = n\boldsymbol{v}$

No source, no sinks

➔ N in volume V can only change due to particles flowing in/out

$$\frac{\partial N}{\partial t} = -\int \boldsymbol{\Gamma} \cdot d\boldsymbol{S} \qquad \boldsymbol{Gauss'law}: \iint \boldsymbol{A} \cdot d\boldsymbol{S} = \iiint \nabla \cdot \boldsymbol{A}\, dV$$

$$\frac{\partial n(\boldsymbol{r},t)}{\partial t} = -\boldsymbol{\nabla} \cdot \boldsymbol{\Gamma}(\boldsymbol{r},t)$$

$$(\boldsymbol{\nabla} \cdot \boldsymbol{\Gamma} = \boldsymbol{\nabla} \cdot (n\boldsymbol{v}) = n\boldsymbol{\nabla} \cdot \boldsymbol{v} + \mathbf{v} \cdot \boldsymbol{\nabla} n)$$

Incompressible fluid

➔ $\boldsymbol{\nabla} \cdot \boldsymbol{v} = \boldsymbol{0}$

➔ $$\frac{\partial n(\boldsymbol{r},t)}{\partial t} + \boldsymbol{v} \cdot \nabla n(\boldsymbol{r},t) = 0$$

An alternative look at continuity equation …

The continuity equation introduces the concept of the *convective derivative:*

If the rate of change at the location of a *fluid element*, moving at speed $\boldsymbol{v}$ is $\frac{\partial n}{\partial t}$, then at a *fixed* position the rate of change has two parts:

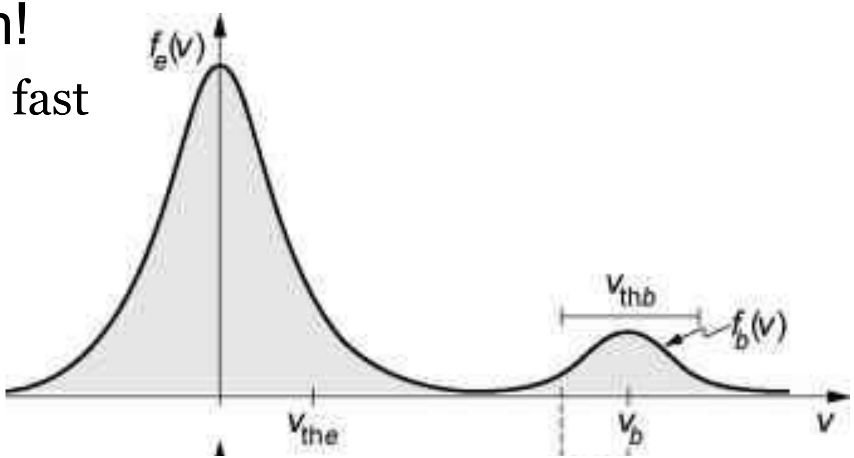
$$\frac{dn(\boldsymbol{r},t)}{dt} = \frac{\partial n(\boldsymbol{r},t)}{\partial t} + \boldsymbol{v} \cdot \nabla n$$

'no-sources, no-sinks': $\frac{dn(\boldsymbol{r},t)}{dt} = 0 \;\rightarrow\; \frac{\partial n(\boldsymbol{r},t)}{\partial t} + \boldsymbol{v} \cdot \nabla n(\boldsymbol{r},t) = 0$

Getting disturbed

Systems are not always in equilibrium!

- Fusion plasmas are typically heated with fast ions → 'bump-on-tail' distribution



Systems try to relax towards thermodynamic equilibrium

→ Where to find the dynamical equation for the distribution function???

’Continuity equation’ in phase space

Move to phase space:

$$n(\boldsymbol{r},t) \rightarrow f(\boldsymbol{r},\boldsymbol{v},t)$$

Generalize also the *convective* derivative:

$$\nabla_v \equiv \hat{v}_x \frac{\partial}{\partial v_x} + \hat{v}_y \frac{\partial}{\partial v_y} + \hat{v}_z \frac{\partial}{\partial v_z}$$

3D: $\boldsymbol{v} \cdot \nabla = \frac{d\boldsymbol{r}}{dt} \cdot \nabla$

6D: $\frac{d\boldsymbol{r}}{dt} \cdot \nabla + \frac{d\boldsymbol{v}}{dt} \cdot \nabla_v = \boldsymbol{v} \cdot \nabla + \boldsymbol{a} \cdot \nabla_v$; $\boldsymbol{a} = \frac{\boldsymbol{F}}{m} = \frac{q}{m}(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$

➔ ***Liouville equation:***

$$\frac{\partial f(\boldsymbol{r},\boldsymbol{v},t)}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} f(\boldsymbol{r},\boldsymbol{v},t) + \boldsymbol{a} \cdot \boldsymbol{\nabla}_v f(\boldsymbol{r},\boldsymbol{v},t) = 0$$

Or simply mathematically:

If we have a function with several variables (like $\boldsymbol{r}, \boldsymbol{v}$ and t) where some variables depend on some other (like $\boldsymbol{r}(t)$ and $\boldsymbol{v}(t)$) then the total derivative can be obtained as a sum of the partial ones.

Here:

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{\partial \boldsymbol{r}}{\partial t}\frac{\partial}{\partial \boldsymbol{r}} + \frac{\partial \boldsymbol{v}}{\partial t}\frac{\partial}{\partial \boldsymbol{v}} \equiv \frac{\partial}{\partial t} + \boldsymbol{v}\cdot\nabla + \boldsymbol{a}\cdot\nabla_v$$

Phase space ’continuity equation’ from scratch …

Move to phase space:

$$n(\boldsymbol{r},t) \rightarrow f(\boldsymbol{r},\boldsymbol{v},t)$$

Then the conservation of particles/probability implies:

$$\frac{\partial f(\boldsymbol{r},\boldsymbol{v},t)}{\partial t} + \boldsymbol{\nabla}\cdot\big(\boldsymbol{v}f(\boldsymbol{r},\boldsymbol{v},t)\big) + \boldsymbol{\nabla}_v\cdot\big(\boldsymbol{a}f(\boldsymbol{r},\boldsymbol{v},t)\big) = 0$$

flux in real space

flux in velocity space

… but this is not the same as the Liouville equation … ??

… or is it? Remember: r and v are independent variables …

Therefore …

$$\boldsymbol{\nabla} \cdot \big(\boldsymbol{v}\, f(\boldsymbol{r},\boldsymbol{v},t)\big) = f(\boldsymbol{r},\boldsymbol{v},t)\boldsymbol{\nabla} \cdot \boldsymbol{v} + \boldsymbol{v} \cdot \nabla f(\boldsymbol{r},\boldsymbol{v},t) = \boldsymbol{v} \cdot \nabla f(\boldsymbol{r},\boldsymbol{v},t)$$

and

$$\boldsymbol{v} \times \boldsymbol{B} = (v_y B_z - v_z B_y, v_z B_x - v_x B_z, v_x B_y - v_y B_x)$$

$$\Rightarrow \nabla_v \cdot (\boldsymbol{v} \times \boldsymbol{B}) = \frac{\partial}{\partial v_x}(v_y B_z - v_z B_y) + \frac{\partial}{\partial v_y}(v_z B_x - v_x B_z) + \frac{\partial}{\partial v_z}(v_x B_y - v_y B_x) = \mathbf{0}$$

We thus do not need any *assumption* of incompressibility in order to write Liouville equation in its convective form:

$$\frac{\partial f(\boldsymbol{r},\boldsymbol{v},t)}{\partial t} + \boldsymbol{v}\cdot\nabla f(\boldsymbol{r},\boldsymbol{v},t) + \boldsymbol{a}\cdot\nabla_{\boldsymbol{v}} f(\boldsymbol{r},\boldsymbol{v},t) = 0$$

Liouville equation thus tells that ’probability fluid’ in 6D phase space is incompressible!

From Liouville equation …

The Liouville equation looks innocent, but …

The *acceleration term* contains all the microscopic forces due to interparticle interactions

→ Impossible to track ☹

However…

It is possible (in advanced course) to separate the macroscopic *average fields*, $\boldsymbol{E}_{ave}$ & $\boldsymbol{B}_{ave}$, from the fluctuating fields !!

… to Boltzmann equation …

The *mean* fields are included in the acceleration term on the LHS.

The fluctuation contribution from inter-particle fields are mangled into a *collision term*, $C(f)$, appearing on the RHS:

$$\frac{\partial f(\boldsymbol{r},\boldsymbol{v},t)}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla}(\boldsymbol{r},\boldsymbol{v},t)f + \frac{q}{m}(\boldsymbol{E}_{ave} + \boldsymbol{v}\times\boldsymbol{B}_{ave})\cdot\boldsymbol{\nabla}_v f(\boldsymbol{r},\boldsymbol{v},t) = C(f)$$

… and to Vlasov equation!

If dynamics is faster than collisions

➔ Vlasov equation:

$$\frac{\partial f(\boldsymbol{r},\boldsymbol{v},t)}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla}(\boldsymbol{r},\boldsymbol{v},t) f + \frac{q}{m}(\boldsymbol{E}_{ave} + \boldsymbol{v} \times \boldsymbol{B}_{ave}) \cdot \boldsymbol{\nabla}_v f(\boldsymbol{r},\boldsymbol{v},t) = 0$$

About the collision term …

- A neutral gas:
  - ’head-on’ binary collisions → strong change in direction
- In a plasma,
  - ’collisions’ = scatterings in Coulomb potential due to surrounding particles → continuous small-angle scatterings

→ In plasma physics, the collision frequency is *not* the inverse of the time between collisions but the inverse of the time it takes a particle to change its direction by 90 deg, the so-called *90-degree scattering rate:*

$$\nu = \frac{e^4 ln\Lambda}{4\pi\varepsilon_0^2\sqrt{m}}\frac{n}{T^{3/2}}$$; introducing the *Coulomb logarithm*

Why collisions lead to transport