Taloustieteen matemaattiset menetelmät
31C01100
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Problem Set 1: Solutions

1. **Solution** Let R_1, R_2 and R_3 stand for rows 1–3, respectively.

   (1) Substract $2R_1$ from the second row and $3R_1$ from the third row

   (2) Multiply the third row by 4

   (3) Substract $5R_2$ from the third row

$$\begin{bmatrix} 1 & 2 & -3 & 0 \\ 2 & 4 & -2 & 2 \\ 3 & 6 & -4 & 3 \end{bmatrix} \xrightarrow{(1)} \begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 5 & 3 \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 20 & 12 \end{bmatrix}$$

$$\xrightarrow{(3)} \begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

   The matrix is now in its row echelon form and has 3 nonzero rows $\implies$ Rank is 3.

2. **Solution** Let R_1, R_2 and R_3 stand for rows 1–3, respectively.

   (1) Substract R_1 from the second row and $2R_1$ from the third row

   (2) Divide the second row by 3

   (3) Add $2R_2$ to the first row and substract $9R_2$ from the third row

   (4) Divide the third row by 2

   (5) Add $\frac{1}{3}R_3$ to the first row and $\frac{2}{3}R_3$ to the second row

$$\begin{bmatrix} 1 & -2 & 3 & 1 & 2 \\ 1 & 1 & 4 & -1 & 3 \\ 2 & 5 & 9 & -2 & 8 \end{bmatrix} \xrightarrow{(1)} \begin{bmatrix} 1 & -2 & 3 & 1 & 2 \\ 0 & 3 & 1 & -2 & 1 \\ 0 & 9 & 3 & -4 & 4 \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} 1 & -2 & 3 & 1 & 2 \\ 0 & 1 & \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \\ 0 & 9 & 3 & -4 & 4 \end{bmatrix}$$

$$\xrightarrow{(3)} \begin{bmatrix} 1 & 0 & \frac{11}{3} & -\frac{1}{3} & \frac{8}{3} \\ 0 & 1 & \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 2 & 1 \end{bmatrix} \xrightarrow{(4)} \begin{bmatrix} 1 & 0 & \frac{11}{3} & -\frac{1}{3} & \frac{8}{3} \\ 0 & 1 & \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 1 & \frac{1}{2} \end{bmatrix} \xrightarrow{(5)} \begin{bmatrix} 1 & 0 & \frac{11}{3} & 0 & \frac{17}{6} \\ 0 & 1 & \frac{1}{3} & 0 & \frac{2}{3} \\ 0 & 0 & 0 & 1 & \frac{1}{2} \end{bmatrix}$$

   The matrix is now in its reduced row echelon form and has 3 nonzero rows $\implies$ Rank is 3.

3. **Solution** Let R_1, R_2 and R_3 stand for rows 1–3, respectively. Write the system of equations in augmented coefficient matrix form and perform the following elementary row operations:

   (1) Add $(R_2 - 3R1)$ to the second row and $(R_3 - 5R_1)$ to the third row

   (2) Substract $5R_2$ from the third row

$$\left[\begin{array}{ccccc|c} 2 & -5 & 3 & -4 & 2 & 4 \\ 3 & -7 & 2 & 5 & 4 & 9 \\ 5 & -10 & -5 & -4 & 7 & 22 \end{array}\right] \xrightarrow{(1)} \left[\begin{array}{ccccc|c} 2 & -5 & 3 & -4 & 2 & 4 \\ 0 & 1 & -5 & 2 & 2 & 6 \\ 0 & 5 & 25 & 12 & 4 & 24 \end{array}\right]$$

$$\xrightarrow{(2)} \left[\begin{array}{ccccc|c} 2 & -5 & 3 & -4 & 2 & 4 \\ 0 & 1 & -5 & 2 & 2 & 6 \\ 0 & 0 & 0 & 2 & -6 & -6 \end{array}\right]$$

The matrix is now in its row echelon form. The corresponding system of equations is:

$$\begin{aligned} 2x - 5y + 3z - 4s + 2t &= 4 \\ y - 5z + 2s + 2t &= 6 \\ 2s - 6t &= -6 \end{aligned}$$

Solve for the leading unknowns (x, y, and s):

$$\begin{aligned} x &= 26 + 11z - 15t \\ y &= 12 + 5z - 8t \\ s &= -3 + 3t \end{aligned}$$

which is the general solution in the free-variable form (z and t are the free variables). The parametric form of the general solution is

$$\begin{aligned} x &= 26 + 11a - 15b \\ y &= 12 + 5a - 8b \\ s &= -3 + 3b \\ z &= a \\ t &= b \end{aligned}$$

where $a, b \in \mathbb{R}$.

4. **Solution** Let R_1, R_2 and R_3 stand for equations 1–3, respectively.

$$\begin{aligned} x + 2y - 3z &= a \\ 2x + 6y - 11z &= b \\ x - 2y + 7z &= c \end{aligned}$$

(1) Substract $2R_1$ from the second equation and R_1 from the third equation:

$$\begin{aligned} x + 2y - 3z &= a \\ 2y - 5z &= b - 2a \\ -4y + 10z &= c - a \end{aligned}$$

(2) Add $2R_2$ to the third equation:

$$\begin{aligned} x + 2y - 3z &= a \\ 2y - 5z &= b - 2a \\ 0 &= c - 5a + 2b \end{aligned}$$

Now we see that if $c - 5a + 2b \neq 0$ the system will have no solution. Thus, if $c - 5a + 2b = 0$, the system will have at least one solution. Because the rank of the corresponding coefficient matrix is less than the number of unknowns, the system has infinitely many solutions.

5. **Solution**

1.

$$A^T = \begin{bmatrix} 2 & 1 & -3 \\ -1 & 0 & 4 \end{bmatrix} \qquad B^T = \begin{bmatrix} 1 & 3 \\ -2 & 4 \\ -5 & 0 \end{bmatrix}$$

2.

$$AB = \begin{bmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} 1 & -2 & -5 \\ 3 & 4 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \times 1 + (-1) \times 3 & 2 \times (-2) + (-1) \times 4 & 2 \times (-5) + (-1) \times 0 \\ 1 \times 1 + 0 \times 3 & 1 \times (-2) + 0 \times 4 & 1 \times (-5) + 0 \times 0 \\ (-3) \times 1 + 4 \times 3 & (-3) \times (-2) + 4 \times 4 & (-3) \times (-5) + 4 \times 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -8 & -10 \\ 1 & -2 & -5 \\ 9 & 22 & 15 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & -2 & -5 \\ 3 & 4 & 0 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \times 2 + (-2) \times 1 + (-5) \times (-3) & 1 \times (-1) + (-2) \times 0 + (-5) \times 4 \\ 3 \times 2 + 4 \times 1 + 0 \times (-3) & 3 \times (-1) + 4 \times 0 + 0 \times 4 \end{bmatrix}$$

$$= \begin{bmatrix} 15 & -21 \\ 10 & -3 \end{bmatrix}$$