Worksheet 3

MS-E1621, Algebraic Statistics

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Group members: Write your names here.

1 Conditional independence axioms

Fill out blanks in the proof of Proposition 1.0.1. We start by recalling some definitions from the lecture.

Definition 1.0.1. *Let $A \subseteq [m]$. The marginal density $f_A(x_A)$ of X_A is obtained by integrating out $x_{[m]\setminus A}$*

$$f_A(x_A) := \int_{x_{[m]\setminus A}} f(x_a, x_{[m]\setminus A}) dx_{[m]\setminus A}$$

for all x_A.

Let $A, B \subseteq [m]$ be pairwise disjoint subsets and let $x_B \in \mathcal{X}_B$. The conditional density of X_A given $X_B = x_B$ is defined as

$$f_{A|B}(x_A|x_B) := \begin{cases} \frac{f_{A\cup B}(x_A, x_B)}{f_B(x_B)} & \text{if } f_B(x_B) > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 1.0.2. *Let $A, B, C \subseteq [m]$ be pairwise disjoint subsets. We say that X_A is conditionally independent of X_B given X_C if and only if*

$$f_{A\cup B|C}(x_A, x_B|x_C) = f_{A|C}(x_A|x_C) f_{B|C}(x_B|x_C)$$

for all x_A, x_B, x_C.

Proposition 1.0.1. *Let $A, B, C, D \subseteq [m]$ be pairwise disjoint subsets. Then*

(i) (symmetry) $X_A \perp\!\!\!\perp X_B | X_C \implies X_B \perp\!\!\!\perp X_A | X_C$

(ii) (decomposition) $X_A \perp\!\!\!\perp X_{B\cup D} | X_C \implies X_A \perp\!\!\!\perp X_B | X_C$

(iii) (weak union) $X_A \perp\!\!\!\perp X_{B\cup D} | X_C \implies X_A \perp\!\!\!\perp X_B | X_{C\cup D}$

(iv) (contraction) $X_A \perp\!\!\!\perp X_B | X_{C\cup D}$ *and* $X_A \perp\!\!\!\perp X_D | X_C \implies X_A \perp\!\!\!\perp X_{B\cup D} | X_C$

Proof. (i) The proof of the symmetry axiom follows from __________ (select one: associativity / commutativity / distributivity) of multiplication.
(ii) Assume that $X_A \perp\!\!\!\perp X_{B\cup D}|X_C$ holds. By Definition 1.0.2, this is equivalent to the factorization of densities

$$\underline{\hspace{6cm}}. \tag{1}$$

Marginalizing this expression over X_D (i.e. integrating out x_D from both sides of the equation, see the first part of Definition 1.0.1) gives

$$\underline{\hspace{6cm}}.$$

This is equivalent to the conditional independence statement $X_A \perp\!\!\!\perp X_B|X_C$.
(iii) As in (ii), the conditional independence statement $X_A \perp\!\!\!\perp X_{B\cup D}|X_C$ is equivalent to Equation (1). Conditioning on X_D (i.e. dividing through by $f_{D|C}(x_D|x_C)$, see the second part of Definition 1.0.1) gives

$$\underline{\hspace{6cm}}.$$

This is equivalent to the conditional independence statement $X_A \perp\!\!\!\perp X_B|X_{C\cup D}$.
(iv) Let x_C be such that $f(x_C) > 0$. By $X_A \perp\!\!\!\perp X_B|X_{C\cup D}$, we have (use Definition 1.0.2)

$$\underline{\hspace{6cm}}.$$

Multiplying by $f_{C\cup D}(x_C, x_D)$ gives

$$f_{A\cup B\cup C\cup D}(x_A, x_B, x_C, x_D) = \underline{\hspace{4cm}} \cdot f_{B|C\cup D}(x_B|x_C, x_D).$$

Dividing by $f(x_C) > 0$ we obtain

$$f_{A\cup B\cup D|C}(x_A, x_B, x_D|x_C) = \underline{\hspace{4cm}} \cdot f_{B|C\cup D}(x_B|x_C, x_D).$$

Using the conditional independence statement $X_A \perp\!\!\!\perp X_D|X_C$, we get

$$\begin{aligned} &f_{A\cup B\cup D|C}(x_A, x_B, x_D|x_C) \\ &\qquad = \underline{\hspace{4cm}} \cdot f_{B|C\cup D}(x_B|x_C, x_D) \\ &\qquad\ \ f_{A|C}(x_A|x_C) f_{B\cup D|C}(x_B, x_D|x_C), \end{aligned}$$

which means $X_A \perp\!\!\!\perp X_{B\cup D}|X_C$. $\square$

2 Conditional independence ideals

Write down the conditional independence ideals below. First we recall useful results from the lecture.

Proposition 2.0.1. *If X is a discrete random vector, then the conditional independence statement $X_A \perp\!\!\!\perp X_B|X_C$ holds if and only if*

$$p_{i_A,i_B,i_C,+} \cdot p_{j_A,j_B,i_C,+} - p_{i_A,j_B,i_C,+} \cdot p_{j_A,i_B,i_C,+} = 0$$

for all $i_A, j_A \in \mathcal{R}_A$, $i_B, j_B \in \mathcal{R}_B$ and $i_C \in \mathcal{R}_C$.

The notation $p_{i_A,i_B,i_C,+}$ denotes the probability $P(X_A = i_A, X_B = i_B, X_C = i_C)$ which can be written as

$$p_{i_A,i_B,i_C,+} = \sum_{j_{[m]\setminus A\cup B\cup C} \in \mathcal{R}_{[m]\setminus A\cup B\cup C}} p_{i_A,i_B,i_C,j_{[m]\setminus A\cup B\cup C}}.$$

Let $m = 3$.

- Consider the marginal independence statement $I_{1 \perp\!\!\!\perp 2}$. Suppose $r_3 = 2$, then the conditional independence ideal is

$$I_{1 \perp\!\!\!\perp 2} = \langle ... \rangle.$$

- Consider the conditional independence statement $I_{1 \perp\!\!\!\perp 2|3}$. The conditional independence ideal is

$$I_{1 \perp\!\!\!\perp 2|3} = \langle ... \rangle.$$