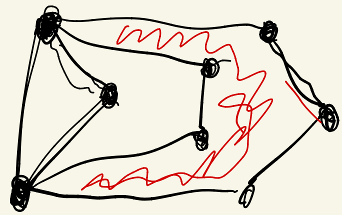
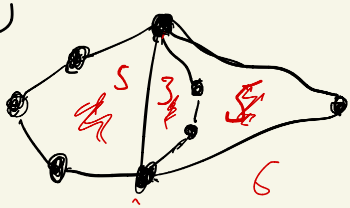
7-cycle Face



$v - e + f = 2$

SAME GRAPH



No 7-cycle face.

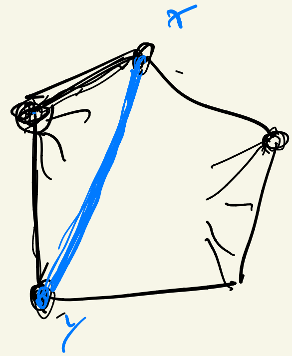
in particular, the <u>number</u> of faces in a plane graph only depend on the underlying comb. graph!

So the <u>set</u> of face cycles depends on the drawing.

<u>**Thm:**</u> IF G is planar 3-conn, then a cycle C bounds a face $\iff$ C is induced and non-separating

"<u>Corr</u>": "3-conn planar graphs can essentially only be drawn in one way"

Pf: Face-bounding $\Rightarrow$ Induced:

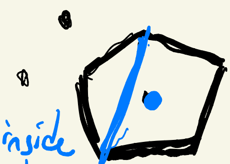


Assume xy chord. Then, nothing outside cycle $\Rightarrow$ $\{x,y\}$ separator. $\blacksquare$

Face-bounding $\Rightarrow$ Non-separating:

Because if C separator, then all 3 paths $u \leftrightarrow v$ would go through C. Then they would intersect ↯

$\Leftarrow$ Induced & non-separating.



If not face-bd, then either verts inside & outside (separating)

or chord inside (so non-induced. $\blacksquare$

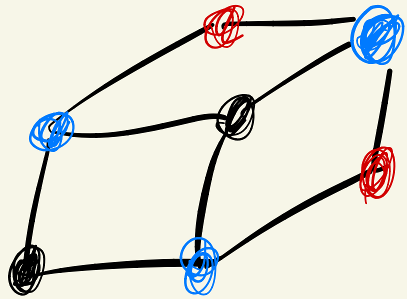
Graph k-colourings:

$V \xrightarrow{\gamma} \{1 \ldots k\}$

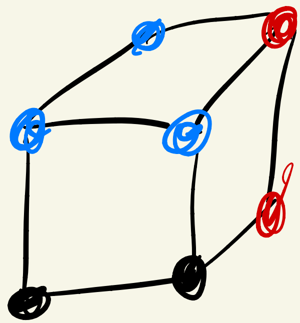
s.t. if

$uv \in E$

then $\gamma(u) \neq \gamma(v)$



OK



NOT OK

Chromatic number

$\chi(G)$ is the smallest number k s.t. G has a k-colouring.

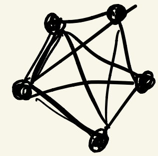
In other words, partition V into k independent sets.

$\alpha(G)$: size of largest ind. set

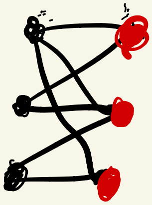
$\omega(G)$: size of largest clique.

Note: A clique Q intersects an ind.t set I in 0 or 1 pts.

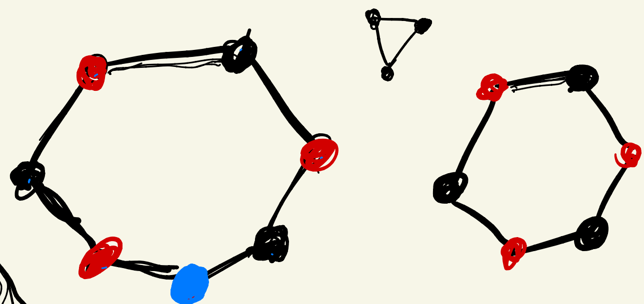
Ex: $\chi(K_n) = n$



B bipartite $\chi(B) = 2$



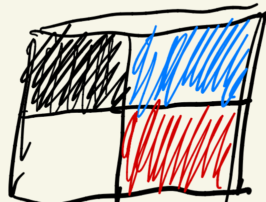
$$\chi(C_n) = \begin{cases} 2 & \text{if } n \text{ even} \\ 3 & \text{if } n \text{ odd.} \end{cases}$$



If $|Q| = \omega(G)$ max clique, all of Q have different colours, so

$\omega(G) \le \chi(G)$

(not always tight, as seen by odd cycles of length ≥ 5.)



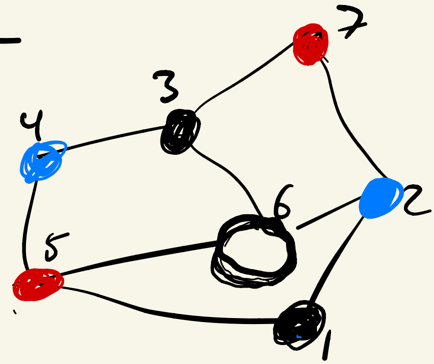
$\frac{|V|}{\alpha} \le \chi$

$$|V| \le \chi\alpha$$

$$\sum_{\text{colors } c} |c| \le \#\text{colors} \cdot \alpha$$

Greedy colouring:

Order nodes arbitrarily.



For $i = 1 \ldots n$

$$\gamma(i) = \min \underbrace{\{\text{colours not used on } N(i)\}}_{\text{at most } d(i) \text{ forbidden colours,}}$$

so $\gamma(i) \le d(i) + 1$

So $\chi(G) \le \Delta(G) + 1$

max degree.

Brook's Theorem

If G not complete, not odd cycle, then $\chi(G) \le \Delta(G)$

(clever ordering + greedy alg)

Thm: Every planar graph is 4-colourable.

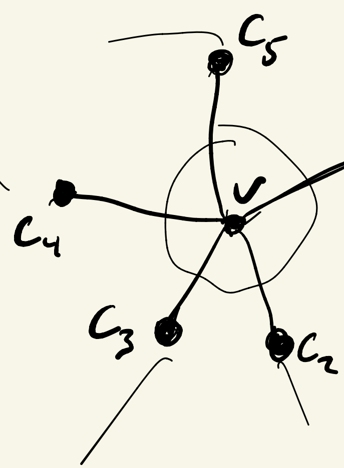
Proof by exhaustive computer search in the 70's. Still no nice proof.

Thm: Every planar graph is 5-colourable. $\chi \leq 5$

Pf: Some vertex $\boxed{v \text{ of degree} \leq 5.}$

$e < 3v$

so avg. deg < 6.

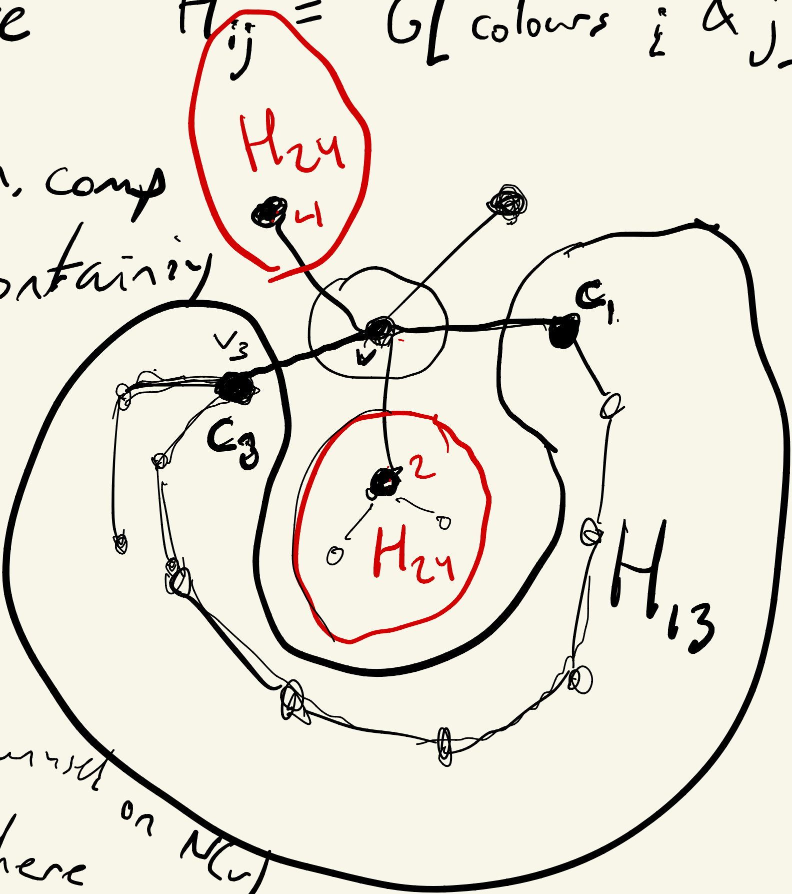


Assume $G \setminus \{v\}$ already 5-coloured.

If only 4 colours used on $N(v)$, then use fifth colour on v.

Now assume n'bours use colours $c_1 \ldots c_5$, ordered around v clockwise.

Denote $H_{ij} = G[\text{colours } i \,\&\, j]$



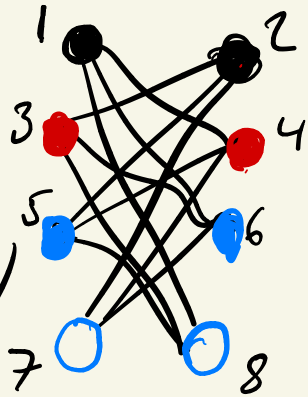
In the conn. comp of H_{13} containing v_3, swap colours 1&3.

This leaves colour 3 unused on $N(v)$ unless there is a path from v_3 to v_1 in H_{13}.

But then v_2 would be separated from v_4 in H_{24}, so I can change colours in the H_{24}-component of v_2, leaving colour 2 to v. $\square$

Greedy alg can be arbitrarily bad:

$G = K_{n,n}$ − matching



$\chi = 2$

Greedy alg uses n colours

Thm (Erdős)

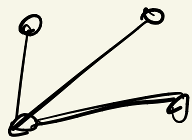
For any $k, \ell \in \mathbb{N}$, there exist graphs with girth $\geq \ell$, chrom. # $\geq k$

shortest cycle length.

"G can have large χ even if it looks like a tree locally everywhere"

"Pf:" Let $G = G(n,p)$ have vertex set $\{1, \dots, n\}$ & edges $ij \in E$ with prob. $p = n^{1/\ell - 1 + \varepsilon}$ independently for all pairs

With prob $\xrightarrow[n \to \infty]{} 1$



This graph has

$\chi \geq \frac{n}{\alpha} \geq k$

and $< \frac{n}{2}$ short cycles $(\leq \ell)$

Take such a graph, delete one node from each short cycle

left with $\geq \frac{n}{2}$ nodes $\boxtimes$

"When is graph colouring purely local"

- When is $\chi(H) = \omega(H)$ for all induced $H \subseteq G$?

Such graphs are called perfect.

- Weak Perfect Graph Theorem

G perfect $\iff$ $\bar{G}$ perfect

(prove on monday)

- Strong PGT

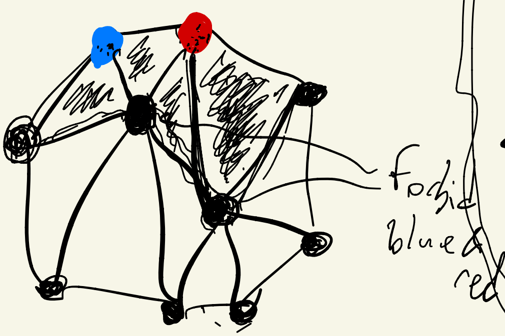
G perfect $\iff$ G has no induced C_n or $\bar{C}_n$ where $n \geq 5$ odd.

5-colour theorem à la Carsten Thomassen

If in a plane graph, every node v has a list $C(v)$ of allowed colours s.t.

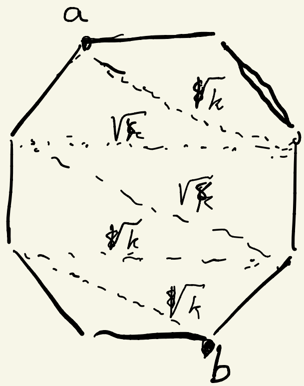
- $|C(v)| \le 5$
- $|C(v)| \le 3$ if v on the outer face
- $C(x) = \{1\}$, $C(y) = \{2\}$ if x, y two given n'bours on outer face.

<u>Then</u> there is a graph colouring where each node gets a colour from his list.



Assume wlog triangulation (except outer face.)

Comments on exercises:



outer cycle length $\sqrt{k}$

longest cycle length $3\sqrt{k}$

longest a-b-path $\approx k$

Recall: χ chromatic number is truly a global invariant

for example Erdős theorem: there are graphs with huge girth and huge chromatic #.

"Graphs G where χ is a local invariant"

$$\chi(H) = \omega(H) \quad \text{for all } H \underset{\text{induced}}{\subseteq} G$$

are called <u>perfect</u>.

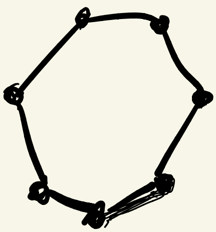
SPGT

G perfect

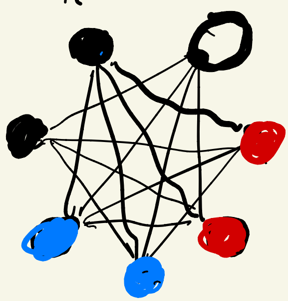
$\iff$

G has no induced C_n or $\overline{C_n}$ for $n \geq 5$ odd

C_n



$\overline{C_n}$



$\alpha(\overline{C_n}) = 2$ (if $n \geq 4$)

$\omega(C_n) = 2$ if $n > 4$

$\chi(C_n) = \begin{cases} 2 & \text{if } n \text{ even} \\ 3 & \text{if } n \text{ odd} \end{cases}$

$\omega(\overline{C_n}) = \lfloor \frac{n}{2} \rfloor$

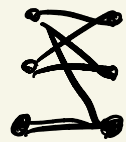
$\chi(\overline{C_n}) = \lceil \frac{n}{2} \rceil$

so C_n, $\overline{C_n}$ not perfect if $n \geq 5$ odd.

Examples of perfect graphs

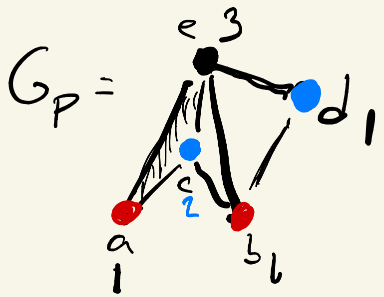
- Graphs with no edges 

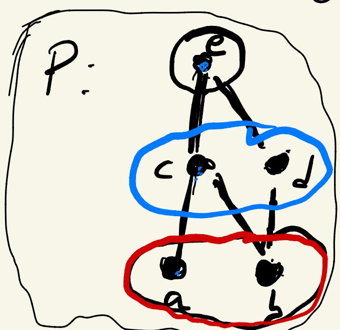
$$\omega(G) = 1 = \chi(G)$$

- Bipartite graphs 

$$\omega(G) = 2 = \chi(G)$$

- Comparability graphs (of partially ordered sets)





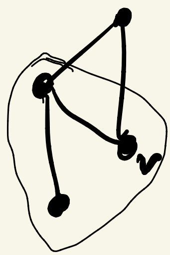
$\omega(G_P)$ = height of P (length of the longest chain)

$\chi(G_P)$ = height (colour vertex $v \in G_P$ by colour i if it has a chain of length i below it (but not of length $i+1$)

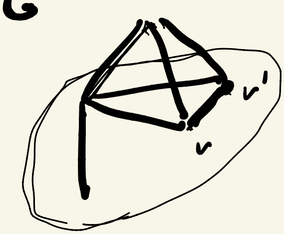
Operations that preserve perfectness:

1) Replicating a vertex (If $G=(V,E)$, $v \in V$, $G' = \left(V \cup \{v'\},\ E \cup \{v'u : vu \in E\} \cup \{v'v\} \right)$

G



G'



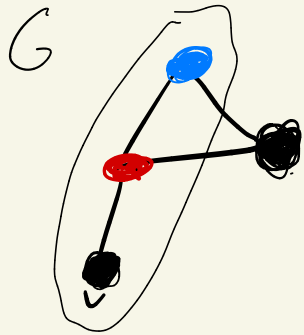
<u>Pf:</u> By induction on size, enough to show $\chi(G') \leq \omega(G')$.

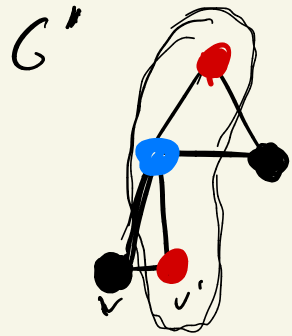
Case 1: If v in some max size clique of G, then

$$\begin{array}{rcl} \omega(G') = & \omega(G) + 1 \\ & \| \\ & \chi(G) + 1 \\ & \geq \\ & \chi(G') \end{array}$$

Case 2: If v not in any max size clique of G.

$\omega(G') = \omega(G)$.



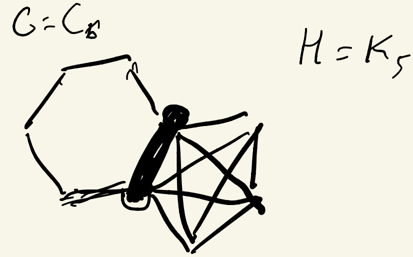


Let A be the colour class of v in some ω-colouring of G.

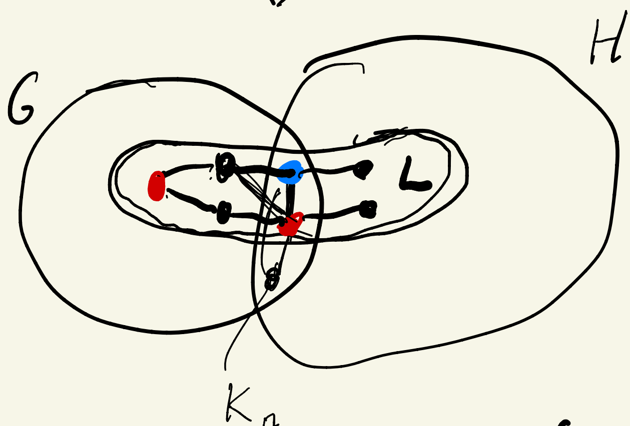
Then $G'-A$ is perfect and has clique number $\leq \omega(G)-1$. (because every ω-clique intersects $A-v$ in G

$G'-A$ can be $(\chi(G)-1)$-coloured, add A for one more colour, get a $\chi(G)$-colouring of G $\blacksquare$

2) If G perfect H perfect, $G \cap H$ a separating clique then $G \cup H$ perfect



Pf:



L induced subgraph of $G \cup H$,

Show $\chi(L) = \omega(L)$.

$$\omega(L) = \max(\omega(L \cap H), \omega(L \cap G))$$

Colour $L \cap G$ with $\omega(L \cap G)$ colours - wlog use colours $1 \dots k$ (injectively) on $L \cap G \cap H$. Colour $L \cap H$ w $\omega(L \cap H)$ colours - wlog use colours $1 \dots k$ on $L \cap G \cap H$

This colours the union $L = (L \cap G) \cup (L \cap H)$
(combined)

with $\max\begin{pmatrix}\omega(L \cap G), \\ \omega(L \cap H)\end{pmatrix}$ colours. $\square$

<u>Def</u>: The class of <u>chordal</u> graphs are defined by:

- Complete graphs are chordal
- If G, H chordal, $G \cap H$ sep. clique then $G \cup H$ chordal.

In particular, chordal graphs are perfect.

Prop: TFAE (the following are equivalent):

- G chordal
- All minimal separators are cliques
- All induced cycles are triangles.

Proof: by induction on $|G|$.

WPGT (weak perfect graph theorem): G perfect $\iff$ $\bar{G}$ perfect

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Corr: Any poset P can be partitioning into w chains, where w is the size of the largest antichain (set of pairwise non-comp elts).

Pf: $\bar{G}_P$ is perfect by WPGT

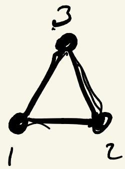
Colouring of $\bar{G}_P$: partition into chains

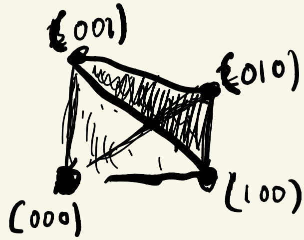
$\omega(\bar{G}_P) = w(P)$. ∎

Proof of WPGT: roadmap:

- Defining two polytopes
  $P(G) \supseteq P_I(G) \subseteq \mathbb{R}^{|V|}$
- G perfect $\Longrightarrow P_I = P$
- $P_I = P \Longrightarrow \bar{G}$ perfect.

$$P_I^{(G)} = \mathrm{Conv}\left\{ \sum_{i \in I} e_i \;:\; I \subseteq V \text{ independent} \right\}$$





$$P(G) = \left\{ x \in \mathbb{R}^V_+ \;:\; \sum_{i \in Q} x_i \leq 1 \text{ for all } Q \text{ cliques} \right\}$$

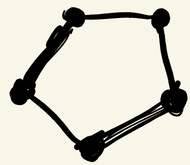
$0 \leq x_1 \quad 0 \leq x_2 \quad 0 \leq x_3$

$x_1 + x_2 + x_3 \leq 1$.

Certainly $P_I \subseteq P$.

Sometimes $P_I = P$, for example

$$\begin{array}{ccc} P_I(C_5) & \neq & P(C_5) \\ \not\ni & & \ni \\ & \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) & \end{array}$$



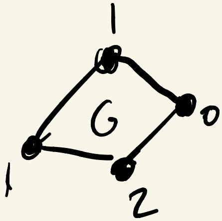
<u>Lemma</u>: G perfect $\Rightarrow P_I(G) = P(G)$

<u>Proof</u>: Assume $x \in P(G) \cap \mathbb{Q}^V$

Show $x \in P_I(G)$

Then choose N s.t.

$$Nx = y \in \mathbb{Z}^V$$



Consider the graph G_y that has y_i copies of the vertex i for all $i \in V(G)$.

$G_y =$ 

By replication lemma, G_y perfect. ...