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School of Science

Lecture 4:
Mathematical treatment of plasma, Fluid approach (in more sense than one)

Today's menu

- From 6D *equilibrium* plasma to 3D fluid
- How to get macroscopic = measurable quantities from the 6D distribution function
- How to get the dynamic equations for fluid quantities: concept of *velocity moments*
- More drifts: *diamagnetic drift*
- *…* and a consequence: *diamagnetic current*

What do we actually want to know about plasma?

Luckily kinetic approach gives unnecessarily detailed description: we want to know the dynamics of the *macroscopic* = *measurable* quantities.

The distribution function $f(\mathbf{r},\mathbf{v};t)$ contains all the information.

How to get the macroscopic quantities from it?

‘Easily’ – at least as long as the plasma is in *equilibrium* ➔ velocity/energy distribution given by the Maxwellian

Plasma as ’stuff’ (mömmö)

But what are we really looking for?

- Density
- Temperature
- Flow
- pressure

➡

Measurable quantities are ***macroscopic quantities****, given by contributions from individual particles summed up together*

Put the distribution function to work…

Here it is most natural to think of $f_s(\boldsymbol{r}, \boldsymbol{v}; t)$ as phase space density.

- Usual density: we want to know the # of pcles in a volume element dV
  - We do not care about the velocity of the particles ➔ integrate over the entire velocity space
- Flow: net motion of the plasma
  - mv. f(r,v;t) = momentum times # of guys having it. Sum them up for net momentum!
- Temperature: temperature is the measure of average energy
  - ½ mv^2. f(r,v;t) = kinetic energy times # of guys having it. Sum them up!
- Pressure: a kind of measure of uneven distribution of energy
  - m(v-V) (v-V) f(r,v;t) = energy related to deviations from flow V times its probability

And mathematically

Fluid quantities are thus obtained as integrals over velocity space of f_s multiplied by different functions of $\boldsymbol{v}$, called velocity *moments* of f_s

Zeroth-order gives the particle density n_s: $n_s = \iiint d^3v\, f_s(\boldsymbol{r}, \boldsymbol{v}, t)$

➔ charge density, $\rho_c = \sum q_s n_s$, and

➔ mass density, $\rho_m = \sum m_s n_s$

First-order gives the plasma flow $\boldsymbol{V_s}$: $n_s \boldsymbol{V_s} = \iiint d^3v\, \boldsymbol{v} f_s(\boldsymbol{r}, \boldsymbol{v}, t)$

➔ current density $\boldsymbol{j}_c = \sum q_s n_s \boldsymbol{V}_s$

… and more 'special moments'

Second-order moment(s):

- Average energy: $n_s \cdot < E_{kin} > = \iiint d^3v\ \frac{1}{2} m v^2 f_s(\boldsymbol{r}, \boldsymbol{v}, t)$
- Pressure: $n_s \boldsymbol{P}_s = \iiint d^3v\, m(\boldsymbol{v} - \boldsymbol{V}_s)(\boldsymbol{v} - \boldsymbol{V}_s) f_s(\boldsymbol{r}, \boldsymbol{v}, t)$; *a tensor !!*
  - Off-diagonal terms: physically shear viscosity
  - Diagonal terms: these give what we normally consider pressure
  - Isotropy ➔ all diagonal terms identical ➔ scalar pressure: $p_s = \frac{1}{3} tr(\boldsymbol{P}_s)$

And even a third-order moment:

- Heat flux density: $\boldsymbol{q}_s = \iiint d^3v \frac{1}{2} m(v - V_s)^2 (\boldsymbol{v} - \boldsymbol{V}_s) f_s(\boldsymbol{r}, \boldsymbol{v}, t)$

How to get dynamical equations for the macroscopic quantities?

Recall last lecture: the dynamical equation for distribution fct f_s is

- The Liouville equation (most general)
- The Boltzmann equation (separate fluctuations into collision term)
- Vlasov equation (for phenomena fast compared to collisinal time scales)

Most macroscopic phenomena, such as a multitude of waves, occur on time scales fast compared to collisions

➔ Let's start with the Vlasov equation !

From kinetic to fluid equations

Since we got the macroscopic quantities by integrating away the velocity dependences, let's do the same for the Vlasov equation!

This is called '*taking velocity moments*' of the Vlasov equation.

Like earlier, different moments correspond to multiplying the Vlasov equation by different powers of the velocity $\boldsymbol{v}$ **:**

- Zeroth-order velocity moment: multiply by $\boldsymbol{v}^{\mathbf{0}} = 1$.
- First-order moment: multiply by $\boldsymbol{v}^{\mathbf{1}} = \boldsymbol{v}$.
- Second-order moment: multiply by $\boldsymbol{vv}$ ... or something of 2nd order

… and integrating over the velocity space.

From kinetic to fluid equations

- Zeroth-order moment ➔ continuity equation:

$$\frac{\partial n_s}{\partial t} + \nabla \cdot (n_s \boldsymbol{V}_s) = 0$$

- First-order moment ➔ momentum conservation:

$$m_s n_s \frac{d\boldsymbol{V}_s}{dt} + \nabla \cdot \boldsymbol{P}_s - q_s n_s(\boldsymbol{E} + \boldsymbol{V}_s \times \boldsymbol{B}) = 0$$

- Second-order moment ➔ energy conservation

$$\frac{3}{2}\frac{dp_s}{dt} + \frac{3}{2} p_s \nabla \cdot \boldsymbol{V}_s + \boldsymbol{P}_s : \nabla \boldsymbol{V}_s + \nabla \cdot \boldsymbol{q}_s = 0$$

New physics brought about by fluid approach …

Can we recover the same physics we found for single particles?

In particular, are the drifts still there…?

Let's look at the fluid equation of motion:

$$m_s n_s \frac{d\boldsymbol{V}}{dt} = -\nabla p_s + q_s n_s(\boldsymbol{E} + \boldsymbol{V}_s \times \boldsymbol{B})$$

The drifts are just… drifts, no acceleration. So let's ignore the LHS.

Take cross product with $\boldsymbol{B}$

➔ $0 = -\nabla p \times \boldsymbol{B} + qn(\boldsymbol{E} \times \boldsymbol{B} + (\boldsymbol{v}_\perp \times \boldsymbol{B}) \times \boldsymbol{B})$

$$= -\nabla p \times \boldsymbol{B} + qn(\boldsymbol{E} \times \boldsymbol{B} + \boldsymbol{v}_\perp B^2)$$

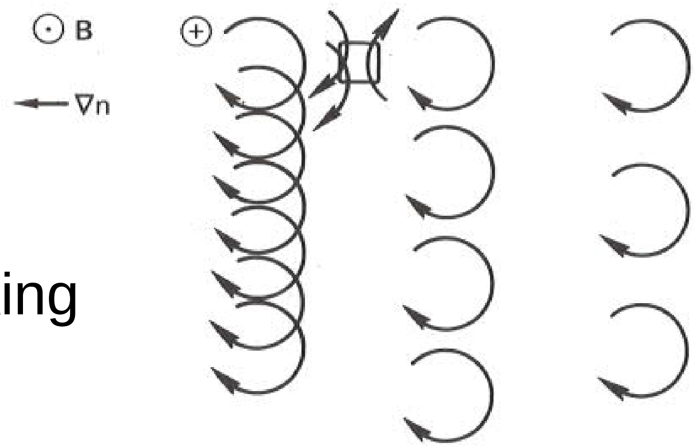
… is the diamagnetic drift !

So for the perpendicular drift we get $\boldsymbol{v}_{\perp} = \boldsymbol{v}_{ExB} + \boldsymbol{v}_D$

Where $\boldsymbol{v}_{ExB}$ is our familiar ExB drift, but we also get a new drift:

$$\boldsymbol{v}_D = -\frac{\boldsymbol{\nabla} p \times \boldsymbol{B}}{qnB^2},$$ *the diamagnetic drift !*

Not present in the single particle picture because it needs a collection of particles building up a pressure gradient, $\boldsymbol{\nabla} p$.



The origin of the drift is easy to understand by looking at the special case where $\boldsymbol{\nabla} p = T\boldsymbol{\nabla} n$

And we get…. *Diamagnetic current !*

Diamagnetic drift depends on the charge ➔ electrons and ions drift in opposite directions

Let's assume isothermal plasma ➔ $\nabla p = T\nabla n$; always ∃ density gradient

$$\Rightarrow \boldsymbol{j}_{dia} = ne(\boldsymbol{v_i} - \boldsymbol{v_e}) = (T_i - T_e)\,\frac{\boldsymbol{B}\times\boldsymbol{\nabla} n}{B^2}$$

A problem is observed…

The equations do not ’close’: each equation obtained by a velocity moment will require the knowledge of the higher-order moment !!

This problem is eliminated by various approaches of *closure.*

The most common closure is to introduce the equation of state:

$$\frac{d}{dt}\left(\frac{p}{\rho^{\gamma}}\right) = 0$$

From a collection of species to a plasma

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From several species to single plasma

We now have something called the *two-fluid* model:

dynamical equations separately for the ion fluid and the electron fluid.

For many applications it is reasonable (and sufficient) to consider the plasma as a *single fluid*.

What are the macroscopic quantities relevant for a single fluid plasma?

Single fluid quantities for the dynamical equations

Recall from the first lecture: plasma is *quasineutral*

➔ Plasma $n_e \approx n_i \equiv n$

How about plasma flow? For the net flow, take the center-of-mass flow:

- plasma flow: $\boldsymbol{V} = \frac{m_i \boldsymbol{V}_i + m_e \boldsymbol{V}_e}{m_i + m_e}$

But the ions and electrons have different charges ➔ electrical currents!

- Define plasma current: $\boldsymbol{j} = -ne(\boldsymbol{V_e} - \boldsymbol{V_i})$

How about pressure? Must be the sum of electron and ion pressures:

- Plasma pressure: $p = p_e + p_i$

Equations for the plasma fluid

The equations of motion for the single-fluid plasma we get by appropriately summing up/substracting the equations for different species

Start with the 0th-order moment: $\frac{\partial n_s}{\partial t} + \nabla \cdot (n_s \boldsymbol{V}_s) = 0$

Define mass density of the plasma fluid: $\rho \equiv M n_i + m n_e$

- Multiply electron equation with m, ion equation with M and add (HW?)

$$\Rightarrow \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{V}) = 0$$

- Multiply electron and ion equations with their charges and add (HW?)

$$\nabla \cdot \boldsymbol{V} = 0$$

Equation of motion for plasma fluid

Take the 1st moments, multiply by masses and add (HW?) ➔

$$n\frac{\partial}{\partial t}(M\boldsymbol{V}_i + m\boldsymbol{V}_e) = en(\boldsymbol{V}_i - \boldsymbol{V}_e) \times \boldsymbol{B} - \nabla p$$

Where we have assumed scalar pressure $p = p_i + p_e$.

Note that the electric field has disappeared!

And we get the single-fluid equation of motion:

$$\rho\frac{\partial \boldsymbol{V}}{\partial t} = \boldsymbol{j} \times \boldsymbol{B} - \boldsymbol{\nabla} p$$

But that's not all, folks…

Use the 1st moment again, this time by *substracting* the two equations, cross-multiplied by *each others*' masses…

$$Mmn\frac{\partial}{\partial t}(\boldsymbol{V}_i - \boldsymbol{V}_e) = en(M+m)\boldsymbol{E} + en(m\boldsymbol{V}_i + M\boldsymbol{V}_e)\times\boldsymbol{B} - m\nabla p_i + M\nabla p_e$$

$$\frac{Mmn}{e}\frac{\partial}{\partial t}\left(\frac{\boldsymbol{j}}{n}\right) = e\rho\boldsymbol{E} + en(m\boldsymbol{V}_i + M\boldsymbol{V}_e)\times\boldsymbol{B} - m\nabla p_i + M\nabla p_e$$

Be clever:

$$m\boldsymbol{V}_i + M\boldsymbol{V}_e = M\boldsymbol{V}_i + m\boldsymbol{V}_e - M(\boldsymbol{V}_i - \boldsymbol{V}_e) + m(\boldsymbol{V}_i - \boldsymbol{V}_e) = \frac{\rho}{n}\boldsymbol{V} - (M-m)\frac{\boldsymbol{j}}{ne}$$

… and we get …

Rearrange the terms :

$$\boldsymbol{E} + \boldsymbol{V} \times \boldsymbol{B} = \frac{1}{e\rho}\left[\frac{Mmn}{e}\frac{\partial}{\partial t}\left(\frac{\boldsymbol{j}}{n}\right)\right] + (M-m)\boldsymbol{j} \times \boldsymbol{B} + m\nabla p_i - M\nabla p_e$$

Now we will drop out terms that are generally small:

- Usually $p_i \approx p_e$ ➔ drop $m\nabla p_i$
- The time derivative of current density typically not important in MHD phenomena

➔ $\boldsymbol{E} + \boldsymbol{V} \times \boldsymbol{B} \approx \frac{1}{en}(\boldsymbol{j} \times \boldsymbol{B} - \nabla p_e)$

Terms on the RHS are typically much smaller (… or cancel!) ➔

$\boldsymbol{E} + \boldsymbol{V} \times \boldsymbol{B} \approx 0$; *Ohm's law …*

Full set of single-fluid equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{V}) = 0$$

$$\nabla \cdot \boldsymbol{V} = 0$$

$$\rho \frac{\partial \boldsymbol{V}}{\partial t} = \boldsymbol{j} \times \boldsymbol{B} - \boldsymbol{\nabla} p$$

$$\boldsymbol{E} + \boldsymbol{V} \times \boldsymbol{B} \approx 0$$

$$\frac{d}{dt}\left(\frac{p}{\rho^{\gamma}}\right) = 0$$

Magnetohydrodynamics, MHD

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Dynamics of the EM fields

But there are also electric and magnetic fields in the equations?

… and plasma current, which supposedly generates magnetic field …

So we need also *dynamic* equations for the fields = Maxwell's equations:

$$\nabla \cdot \boldsymbol{B} = 0$$

$$\nabla \cdot \boldsymbol{E} \approx 0 \quad ; \quad \text{plasma is quasineutral}$$

$$\nabla \times \boldsymbol{B} = \mu_0 \mathbf{j}$$

$$\nabla \times \boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t}$$

Where we have neglected the so-called *displacement current* $\propto \frac{\partial \boldsymbol{E}}{\partial t}$

Complete set of *ideal* MHD equations

Fluid equations:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{V}) = 0$$

$$\nabla \cdot \boldsymbol{V} = 0$$

$$\rho \frac{\partial \boldsymbol{V}}{\partial t} = \boldsymbol{j} \times \boldsymbol{B} - \boldsymbol{\nabla} p$$

$$\boldsymbol{E} + \boldsymbol{V} \times \boldsymbol{B} \approx 0$$

$$\frac{d}{dt}\left(\frac{p}{\rho^{\gamma}}\right) = 0$$

Field equations:

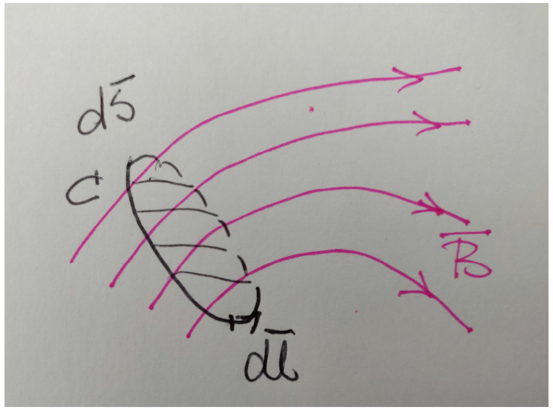
$$\nabla \cdot \boldsymbol{B} = 0$$

$$\nabla \cdot \boldsymbol{E} = 0$$

$$\nabla \times \boldsymbol{B} = \mu_0 \mathbf{j}$$

$$\nabla \times \boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t}$$

Forced marriage between B and plasma: Frozen-in condition



- consider fieldlines through a surface *S* in plasma

➔ magnetic flux $\Psi = \int \boldsymbol{B} \cdot d\boldsymbol{S}$

- Let the surface move together with the plasma

➔ time rate of change of Ψ has two parts:

  - Explicit time dependence of $\boldsymbol{B}$: $\frac{\partial \Psi}{\partial t} = \int \frac{\partial \boldsymbol{B}}{\partial t} \cdot d\boldsymbol{S} = -\int \boldsymbol{\nabla} \times \boldsymbol{E} \cdot d\boldsymbol{S}$
  - Change due to moving plasma: $\frac{\partial \boldsymbol{S}}{\partial t} = \boldsymbol{V} \times d\boldsymbol{l}$ ➔ $\frac{\partial \Psi}{\partial t} = \int \boldsymbol{B} \cdot \boldsymbol{V} \times d\boldsymbol{l} = \int \boldsymbol{B} \times \boldsymbol{V} \cdot d\boldsymbol{l}$
  - Use Stokes ➔ Total change: $\frac{d\Psi}{dt} = \int -\nabla \times (\boldsymbol{E} + \boldsymbol{V} \times \boldsymbol{B}) \cdot d\boldsymbol{S} = 0$!! (Ohm's law)

- Fieldlines are stuck with the plasma! ➔ Concept of a *flux tube*

Including the effect of collisions

The ideal MHD equations we derived from the Vlasov equation, which neglects the effect of collisions.

However, many important phenomena, the least of which is not the *resistivity* itself, require taking into account also the collisions.

Ideal MHD is valid only when the phenomenon is very fast – compared to the collisional time scale.

In more general case we take the moments of the *Boltzmann equation …*

More complicated math ➔ skipped in this course

Complete set of *resistive* MHD equations

Fluid equations:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{V}) = 0$$

$$\nabla \cdot \boldsymbol{V} = 0$$

$$\rho \frac{\partial \boldsymbol{V}}{\partial t} = \boldsymbol{j} \times \boldsymbol{B} - \boldsymbol{\nabla} p$$

$$\boldsymbol{E} + \boldsymbol{V} \times \boldsymbol{B} \approx \eta \boldsymbol{j}$$

$$\frac{d}{dt}\left(\frac{p}{\rho^{\gamma}}\right) = 0$$

Field equations:

$$\nabla \cdot \boldsymbol{B} = 0$$

$$\nabla \cdot \boldsymbol{E} = 0$$

$$\nabla \times \boldsymbol{B} = \mu_0 \mathbf{j}$$

$$\nabla \times \boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t}$$

So the difference to ideal MHD appears miniscule: just resistivity popping up in Ohm's law. But its effect is dramatic, e.g., fieldlines can now reconnect and diffuse …