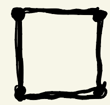
Extremal graph theory:

How many edges can a graph w n nodes have without containing H as a <u>subgraph</u> ((minor) later today)

$ex(n, H)$

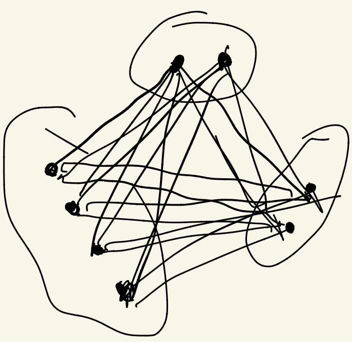
Ex: $ex(4, \Delta) = 4$



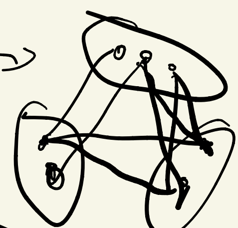
Note: If $m > ex(n, K_r)$, then every graph w. n nodes, m edges have every r node graph H as a subgraph.

What graphs have no K_r?

Among others, $(r-1)$-partite graphs.



no K_4 $\rightarrow$



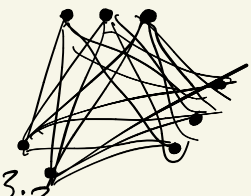
edges in $K_{s_1 s_2 \dots s_{r-1}}$

$$= \sum_{i \neq j} s_i s_j$$

$T_r(n)$ = Complete r-partite graph on n nodes, divided in parts of size $\lceil \frac{n}{r} \rceil$ or $\lfloor \frac{n}{r} \rfloor$

$t_r(n)$ = # edges in $T_r(n)$

Ex $T_3(8)$:



$t_3(8) = 3\cdot 3 + 3\cdot 2 + 3\cdot 2 = 21$

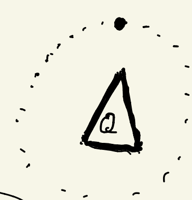
Turan's theorem: $ex(n, K_r) = t_{r-1}(n)$.

Pf: • $ex(n, K_r) \geq t_{r-1}(n)$ because $T_{r-1}(n)$ has no K_r.

• Nts if G (edge-maximal) has no K_r then G has at most t_{r-1} edges.

G has a $Q \cong K_{r-1}$ copy.

Consider $G \setminus Q = H$



$$|E(G)| \leq \underbrace{\binom{r-1}{2}}_{Q-Q} + \underbrace{ex(n-r+1, K_r)}_{H-H} + \underbrace{(r-2)(n-r+1)}_{Q-H}$$

$$\overset{\text{I.H.}}{\leq} \binom{r-1}{2} + (r-2)(n-r+1) + \underline{t_{r-1}(n-r+1)}$$

$$\parallel$$

$$t_{r-1}(n)$$

$\blacksquare$

Note: $\chi(H) \leq r \iff H \subseteq T_r(rn)$ if $|H| = n$

$$T_r(rn) = K_{\underbrace{n,n,\dots n}_{r \text{ parts}}}$$

Erdős-Stone thm: For all ε, m exists N s.t.

"Every graph with $n \geq N$ nodes and $\geq t_{r-1}(n) + \varepsilon n^2$ edges contains $T_r(m)$ as a subgraph"

Edge density $= \dfrac{\#\text{edges}}{\binom{n}{2}} \approx 2\dfrac{\#\text{edges}}{n^2}$

Edge density of Turan graph $T_r(n)$ $\approx \dfrac{r-1}{r}$

So morally, all large graphs with higher edge density than $\frac{r-1}{r}$ contains all small r-colourable graphs as subgraphs.

Erdős - Stone is proven via Szemeredi's regularity lemma.

(on monday)

<u>Cor</u> : For every graph H,

$$\lim_{n\to\infty} \frac{ex(n,H)}{\binom{n}{2}} = \frac{\chi(H)-2}{\chi(H)-1}$$

So $ex(n,H) \sim n^2$ for graphs with $\chi(H) \geq 3$

$ex(n,H) = o(n^2)$ for bipartite graph

Proof of corollary:

$\chi(H)=k$. Then $H \not\subseteq T_{r-1}(n)$ for all n

but $H \subseteq T_r(m)$ for $m = r|H|$

So for all n we have

$$t_{r-1}(n) \leq ex(n,H) \leq ex(n, T_r(m)).$$

By Erdős-Stone, $\underbrace{\forall \varepsilon \, \exists N_0 :}$ $ex(n, T_r(m)) < t_{r-1}(n) + \varepsilon n^2$

for all $n \geq N_0$.

So for any $\varepsilon > 0$ we get

$$\frac{t_{r-1}(n)}{\binom{n}{2}} \leq \frac{ex(n,H)}{\binom{n}{2}} \leq \frac{t_{r-1}(n)}{\binom{n}{2}} + \frac{\varepsilon n^2}{\binom{n}{2}}$$

$$\frac{t_{r-1}(n)}{\binom{n}{2}} \xrightarrow{n \to \infty} \frac{r-2}{r-1} \qquad \frac{t_{r-1}(n)}{\binom{n}{2}} + \frac{\varepsilon n^2}{\binom{n}{2}} \xrightarrow{n \to \infty} \frac{r-2}{r-1} + 2\varepsilon.$$

Letting $\varepsilon \to 0$, we get

$$\frac{ex(n,H)}{\binom{n}{2}} \xrightarrow[n \to \infty]{} \frac{r-2}{r-1} \qquad \blacksquare$$

What is the growth rate of $ex(n, H)$ for bipartite graphs?

Thm: $c_1 n^{2-\frac{2}{r-1}} \leq ex(n, K_{r,r}) \leq c_2 n^{2-\frac{1}{r}}$ for some universal constants c_1, c_2.

Conj: (Erdős-Sós) For any tree T with k edges $ex(n, T) = \frac{1}{2}(k-1)n$

How many edges can a graph have that has no K_r minor?

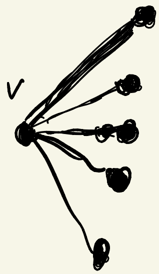
It is enough to have large enough average degree to force K_r minors

Thm: If $G = (V, E)$ has average degree $\geq 2^{r-2}$, then G has a K_r minor

$$\frac{|E|}{n} \geq 2^{r-3}$$

Pf: Assume that G minor-minimal with average degree $\geq 2^{r-2}$, no K_r minor.

By induction on r, assume that all graphs w. av. degree $\geq 2^{r-3}$ has a K_{r-1} minor



Either $N(u) \cap N(v)$ $\geq 2^{r-3}$ nodes

or G/uv has one fewer nodes and $< 2^{r-3}$ fewer edges, so avg degree $\geq 2^{r-2}$

So $G[N[v]]$ has min degree $\geq 2^{r-3}$, so K_{r-1} minor, M, then $M \cup v$ is a K_r minor.

$\boxtimes$

Thm (Kostochka)

There exists $c > 0$ s.t. any graph with avg degree $\geq c r \sqrt{\log r}$ has a K_r minor.

Note: If $\chi(G) \geq d$, then G has a subgraph with minimum degree $\geq d-1$

So large chromatic number $\Longrightarrow$ large minors

$\approx r\sqrt{\log r}$ $\qquad$ K_r

In other words, if G has no K_r minor, then $\chi(G) \leq c\, r\sqrt{\log r}$.

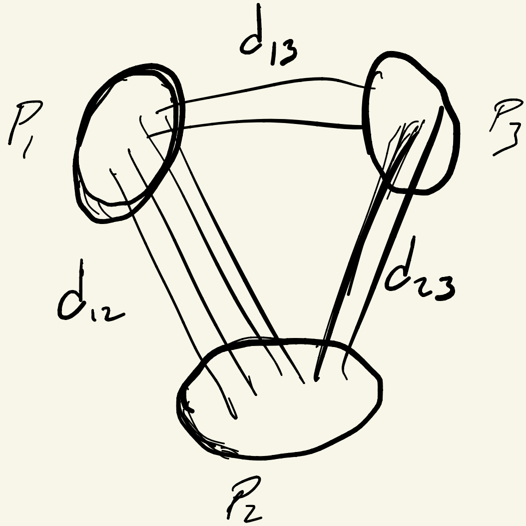
<u>Conj (Hadwiger)</u>

If G has no K_r minor, then $\chi(G) \leq r-1$

For monday:

Read Szemeredi's regularity ~~lemma~~ theorem 7.4, 7.5

- Statement
- Proof of Erdős-Stone theorem via SRT
- Proof of SRT



$\forall \varepsilon \ \exists m$

you can partition any graph in m parts s.t the edges between parts i, j "look like a random bipartite graph with density d_{ij}"

(ε-regular pair)