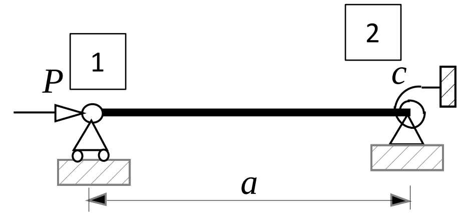
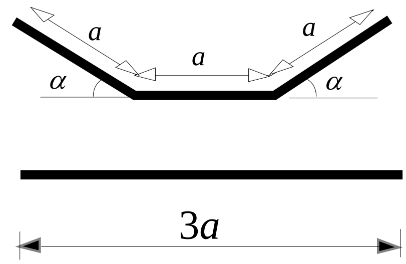
CIV-E4100 Stability of Structures

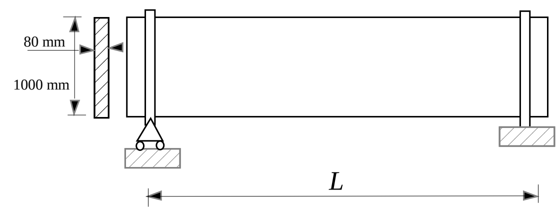
Examination April 5th 2018



1. A straight beam is simply supported at one end, and supported by a rotational spring, with spring constant $c = \alpha EI / a$, at the other. Its length is *a,* and bending stiffness *EI*. Determine the critical compressive load of the beam, when $\alpha = 1$. Show further that the result is covering the cases where the right hand end of the beam is simply supported and clamped by varying the coefficient α.

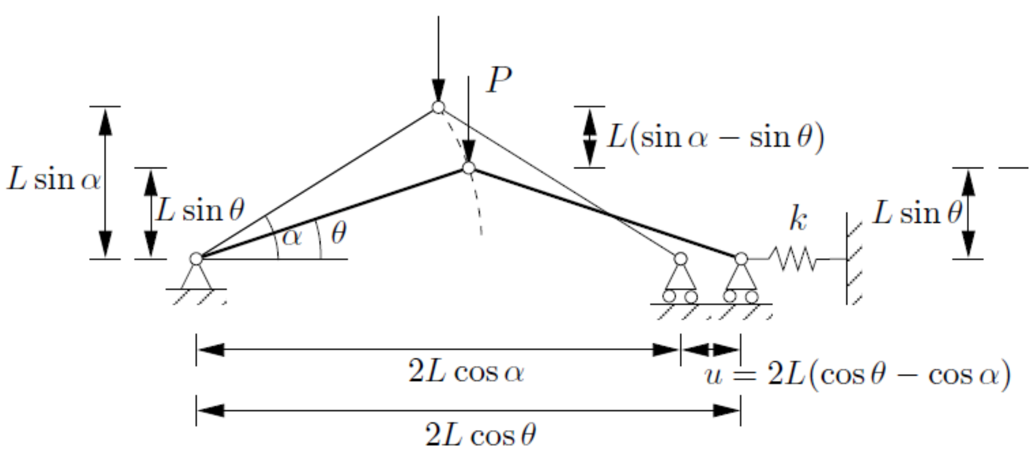


2. The cross-section of a beam is formed so that a plate of width *3a* is bent on the sides by an angle of α according to the figure. Determine the dependence of the coordinates of the shear center, and of the torsional and warping constants I_t and I_ω on the angle α. The wall thickness is constant *t*.



3. What is the critical length of a simply supported beam with respect to lateral buckling, when its cross-section is a narrow rectangle ($80\,\text{mm} \times 1000\,\text{mm}$)? The Young's modulus and the shear modulus are $E = 36\,\text{kN/mm}^2$ and $G = 15,4\,\text{kN/mm}^2$ respectively. The loading due to the own weight is $g = 24\,\text{kN/mm}^3$.

4. The truss is constructed of two stiff bars $(EI, EA = \infty)$ which are hinged together. The truss is supported by an elastic horizontal spring with spring constant *k*. The truss is loaded by a concentrated load *P* on the top. Determine and draw all the equilibrium paths of this system. Determine further the type of the equilibrium on different paths.



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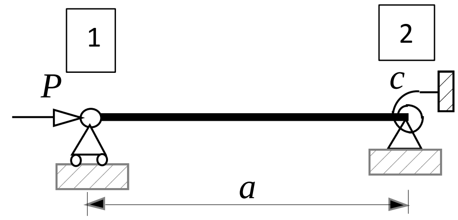
Solutions:



1. Easiest way is to apply the slope-deflection method. Thus the equilibrium equation is $M_{21} + M_{2s} = 0 \Rightarrow (A^{o}_{21} + c)\varphi_2 = 0$.

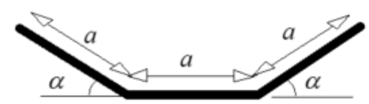
$$A^{o}_{21} + c = -\frac{1}{\Psi(ka)}\frac{3EI}{a} + \alpha\frac{EI}{a} = 0 \Rightarrow \Psi(ka) = \frac{3}{\alpha}.\ \text{Jos}\ \Psi(ka) = \frac{3}{ka}\left(\frac{1}{ka} - \frac{1}{\tan ka}\right)$$

$$\Rightarrow \tan ka = \frac{\alpha ka}{\alpha + (ka)^2}\ \text{ If } \alpha = 1 \Rightarrow \tan ka = \frac{ka}{1+(ka)^2} \Rightarrow ka = 3.405 \Rightarrow P_{cr} = 1.175\frac{\pi^2 EI}{a^2}$$

If $\alpha = 0 \Rightarrow \tan ka = 0 \Rightarrow ka = n\pi \Rightarrow P_{cr} = \dfrac{\pi^2 EI}{a^2}$. If $\alpha = \infty \Rightarrow \tan ka = ka \Rightarrow P_{cr} = 2.046\dfrac{\pi^2 EI}{a^2}$.

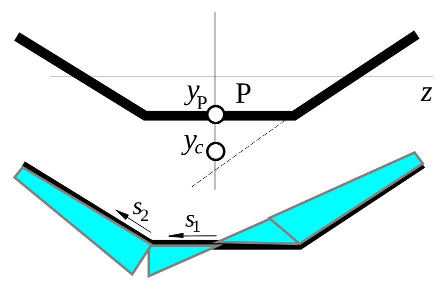
From differential equation, the solution is $v(x) = C_1 \sin kx + C_2 \cos kx + C_3 x + C_4$ where $k^2 = P/EI$ and the boundary conditions $v(0) = v''(0) = v(a) = 0,\ cv'(a) = -EIv''(a)$ yielding $C_2 = C_4 = 0,$ $C_3 = -C_1 \sin ka / a$ and the condition $c(k\cos ka - \sin ka / a) = P\sin ka,$ yielding the same result.

2. The shear center is located on the axis of symmetry, and also the origin of *s*-coordinate. The center of gravity is $y_{\mathrm{P}} = \dfrac{a}{3}\sin\alpha$



The sectorial coordinate with respect to the point P is

$$\omega(s) = \pm\int_0^s \frac{a}{2}\sin\alpha \mathrm{d}s = \pm\frac{as}{2}\sin\alpha \ \text{ and } \ z(s) = \pm(\frac{a}{2} + s\cos\alpha).$$



Thus $I_y = \dfrac{a^3 t}{12} + 2\left(at\dfrac{a^2}{4}(1+\cos\alpha)^2 + \dfrac{(a\cos\alpha)^3 t}{12\cos\alpha}\right) =$

$= \dfrac{a^3 t}{12}\left(7 + 12\cos\alpha + 8\cos^2\alpha\right)$ and

$$I_{\omega z} = 2\int_0^a \frac{as}{2}\sin\alpha(\frac{a}{2} + s\cos\alpha)t\mathrm{d}s = \frac{a^4 t}{12}\sin\alpha(3 + 4\cos\alpha)$$

The coordinate of the shear center is

$$y_c = y_{\mathrm{P}} + \frac{I_{\omega z}}{I_y} = \frac{a}{3}\sin\alpha + \frac{a\sin\alpha(3+4\cos\alpha)}{(7+12\cos\alpha+8\cos^2\alpha)} = \frac{8a\sin\alpha}{3}\frac{2+3\cos\alpha+\cos^2\alpha}{7+12\cos\alpha+8\cos^2\alpha}$$

$$\omega(s) = \begin{cases} \pm(y_c - y_{\mathrm{P}})s_1 = \pm\dfrac{a\sin\alpha(3+4\cos\alpha)s_1}{7+12\cos\alpha+8\cos^2\alpha} & 0 \le s_1 \le \pm\dfrac{a}{2} \\ \mp[(y_c - y_{\mathrm{P}})(\dfrac{a}{2} + s_2\cos\alpha) - \dfrac{as_2}{2}\sin\alpha] = \mp\dfrac{a\sin\alpha}{2}(\dfrac{a(3+4\cos\alpha) - s_2(7+6\cos\alpha)}{7+12\cos\alpha+8\cos^2\alpha}) & 0 \le s_2 \le \pm a \end{cases}$$

Warping constant is then $I_\omega = \int_A \omega^2(s)\mathrm{d}A = \frac{5}{12}a^5 t \frac{\sin^2\alpha}{7+12\cos\alpha+8\cos^2\alpha}$, $I_t = a^3 t$ (remains unchanged)

3. Let $L = 2\ell$, thus the bending moment due to the own weight is $M_z^{\mathrm{o}} = \frac{q\ell^2}{2}(1-(\frac{x}{\ell})^2)$, when the origin is located at the mid span. The energy integral is $\Pi = \int_0^\ell \left[EI_y(w'')^2 + GI_t(\phi')^2 + 2(M_z^{\mathrm{o}}\phi)'w' \right]\mathrm{d}x$ The beam is simply supported at each end when the approximations for the deflection and rotation can be of polynomial form, satisfying the boundary conditions $w'(0) = w(\pm\ell) = \phi'(0) = \phi(\pm\ell) = 0$ and are $w = w_{\mathrm{o}}(1-(\frac{x}{\ell})^2)$ and $\phi = \phi_{\mathrm{o}}(1-(\frac{x}{\ell})^2)$. Trigonometric functions $w = w_{\mathrm{o}}\cos(\frac{\pi x}{\ell})$ and $\phi = \phi_{\mathrm{o}}\cos(\frac{\pi x}{\ell})$ give better approximation.

$$\Pi = \int_0^\ell \left[EI_y(\frac{-2w_{\mathrm{o}}}{\ell^2})^2 + GI_t(\frac{-2x\phi_{\mathrm{o}}}{\ell^2})^2 + 2\left(\frac{q\ell^2}{2}\phi_{\mathrm{o}}(1-(\frac{x}{\ell})^2)^2\right)'(\frac{-2xw_{\mathrm{o}}}{\ell^2}) \right]\mathrm{d}x =$$

$$= \frac{4EI_y}{\ell^3}w_{\mathrm{o}}^2 + \frac{4GI_t}{3\ell}\phi_{\mathrm{o}}^2 + \frac{16q\ell}{15}w_{\mathrm{o}}\phi_{\mathrm{o}} \Rightarrow \begin{cases} \frac{\partial\Pi}{\partial w_{\mathrm{o}}} = \frac{8EI_y}{\ell^3}w_{\mathrm{o}} + \frac{16q\ell}{15}\phi_{\mathrm{o}} \\ \frac{\partial\Pi}{\partial\phi_{\mathrm{o}}} = \frac{8GI_t}{3\ell}\phi_{\mathrm{o}} + \frac{16q\ell}{15}w_{\mathrm{o}} \end{cases} \Rightarrow$$

$$\begin{bmatrix} \frac{8EI_y}{\ell^3} & \frac{16q\ell}{15} \\ \frac{16q\ell}{15} & \frac{8GI_t}{3\ell} \end{bmatrix} \begin{Bmatrix} w_{\mathrm{o}} \\ \phi_{\mathrm{o}} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \Rightarrow \ell^6 = \frac{75}{4}\frac{EI_yGI_t}{q^2} \Rightarrow L = 2\ell = 33.1\mathrm{m}$$

4. Energy formulation $\Pi = U + V = \frac{1}{2}k(2L)^2(\cos\theta - \cos\alpha)^2 - PL(\sin\alpha - \sin\theta)$ Along the equilibrium path $\delta\Pi = \frac{\partial\Pi}{\partial\theta}\delta\theta = 0 \Rightarrow -4kL^2\sin\theta(\cos\theta-\cos\alpha) + PL\cos\theta = 0$ From this we get the equilibrium path $\frac{P}{4kL} = \tan\theta(\cos\theta - \cos\alpha) = \sin\theta - \tan\theta\cos\alpha$. The stability/instability is determined by the second derivative when we get $\delta^2\Pi = \frac{\partial^2\Pi}{\partial\theta^2}\delta^2\theta = 0 \Rightarrow -4kL^2(\cos^2\theta - \cos\theta\cos\alpha - \sin^2\theta) - PL\sin\theta.$ Inserting here the

value of $P/4kL$ we get

$$\frac{\partial^2 \Pi}{\partial \theta^2} = -\cos^2\theta + \frac{\cos\alpha}{\cos\theta} = \begin{cases} > 0 \text{ when } \theta < -\arccos(\cos\alpha)^{1/3} \text{ or } \theta > \arccos(\cos\alpha)^{1/3} \text{ (stable)} \\ < 0 \text{ when } -\arccos(\cos\alpha)^{1/3} > \theta > \arccos(\cos\alpha)^{1/3} \text{ (unstable)} \\ = 0 \text{ when } \theta = \arccos(\cos\alpha)^{1/3} \text{(indifferent)} \end{cases}$$