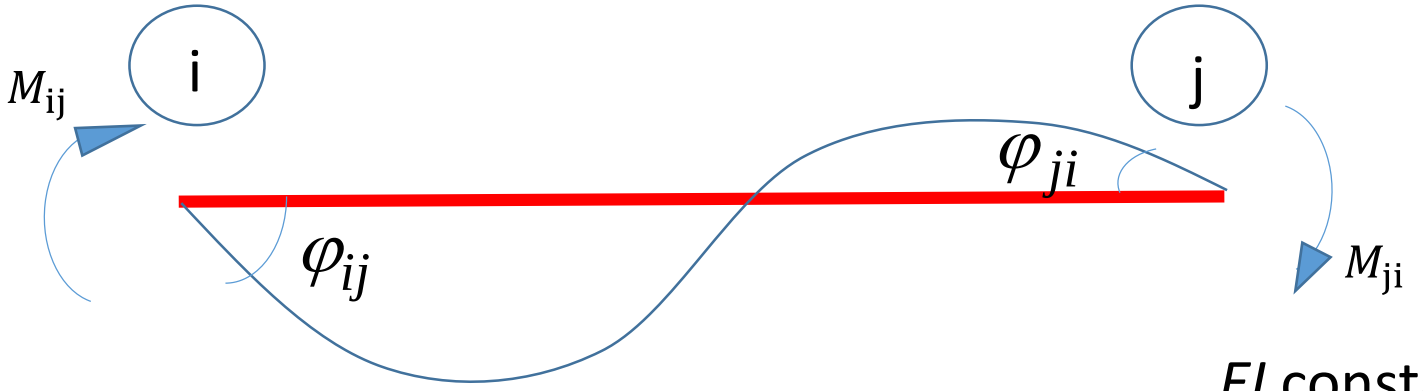
Buckling of Frames



$$\psi_{ij} = \psi_{ji}$$

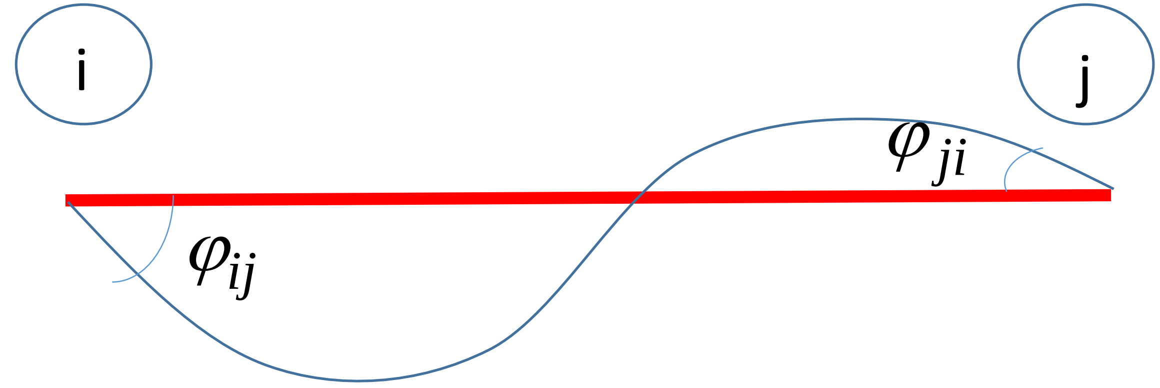
EI constant

Frame analysis without compressive load

$$\begin{cases} \varphi_{ij} = \dfrac{L}{3EI} M_{ij} - \dfrac{L}{6EI} M_{ji} + \psi_{ij} + \alpha_{ij}^{o} \\ \varphi_{ji} = \dfrac{L}{3EI} M_{ji} - \dfrac{L}{6EI} M_{ij} + \psi_{ji} + \alpha_{ji}^{o} \end{cases}$$

$$\begin{cases} M_{ij} = \dfrac{4EI}{L} \varphi_{ij} + \dfrac{2EI}{L} \varphi_{ji} - \dfrac{6EI}{L} \psi_{ij} + (MK)_{ij} \\ M_{ji} = \dfrac{4EI}{L} \varphi_{ji} + \dfrac{2EI}{L} \varphi_{ij} - \dfrac{6EI}{L} \psi_{ji} + (MK)_{ji} \end{cases}$$

and with compressive load



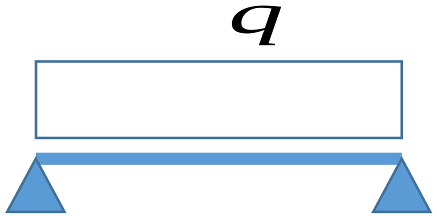
EI constant

$$\begin{cases} \varphi_{ij} = \dfrac{L}{3EI}\Psi(kL)M_{ij} - \dfrac{L}{6EI}\Phi(kL)M_{ji} + \psi_{ij} + \alpha_{ij}^{o} \\ \varphi_{ji} = \dfrac{L}{3EI}\Psi(kL)M_{ji} - \dfrac{L}{6EI}\Phi(kL)M_{ij} + \psi_{ji} + \alpha_{ji}^{o} \end{cases}$$

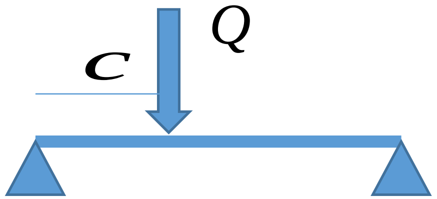
$$\psi_{ij} = \psi_{ji}$$

$$\Psi(kL) = \frac{3}{kL}\left(\frac{1}{kL} - \frac{1}{\tan kL}\right), \qquad \Phi(kL) = \frac{6}{kL}\left(\frac{1}{\sin kL} - \frac{1}{kL}\right), \quad k = \sqrt{\frac{P}{EI}}$$

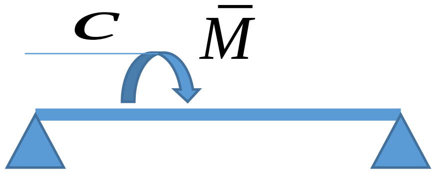
Transverse loading terms



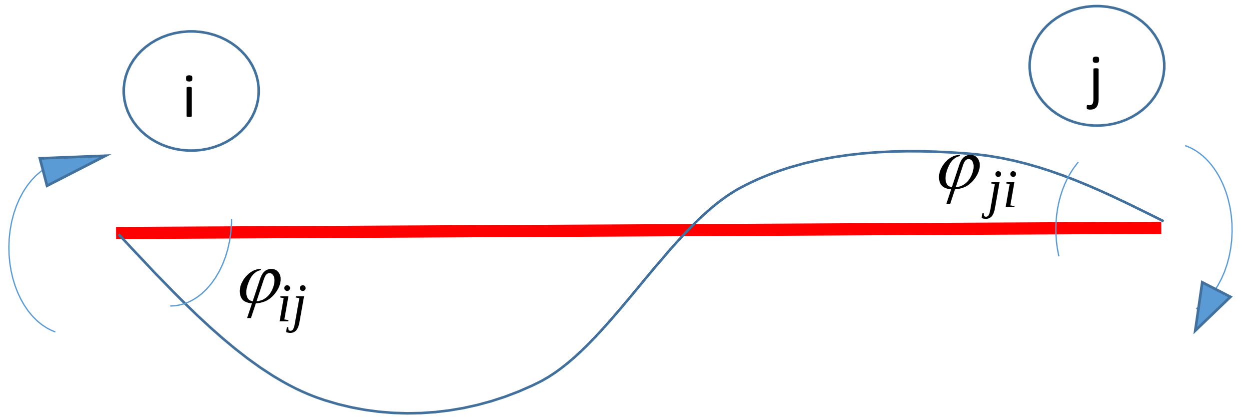
$$\alpha_{ij}^{o} = -\alpha_{ji}^{o} = \frac{qL^3}{24EI}\mathrm{X}(kL), \qquad \mathrm{X}(kL) = \frac{24(\tan(kL/2)-kL/2}{(kL)^3}$$



$$\alpha_{ij}^{o} = \frac{Q\sin k(L-c)}{P\sin kL} - \frac{Q(L-c)}{PL}, \qquad \alpha_{ji}^{o} = -\frac{Q\sin kc}{P\sin kL} + \frac{Qc}{PL}$$



$$\alpha_{ij}^{o} = -\frac{\bar{M}k\cos k(L-c)}{P\sin kL} + \frac{\bar{M}}{PL}, \qquad \alpha_{ji}^{o} = -\frac{\bar{M}k\cos kc}{P\sin kL} + \frac{\bar{M}}{PL}$$



$$\begin{cases} M_{ij} = A_{ij}\varphi_{ij} + B_{ji}\varphi_{ji} - C_{ij}\psi_{ij} + (MK)_{ij} \\ M_{ji} = A_{ji}\varphi_{ji} + B_{ij}\varphi_{ij} - C_{ji}\psi_{ji} + (MK)_{ji} \end{cases}$$

$$\begin{cases} A_{ij} = A_{ji} = \dfrac{2\Psi(kL)}{4\Psi^2(kL) - \Phi^2(kL)} \dfrac{6EI}{L} \\ B_{ij} = B_{ji} = \dfrac{\Phi(kL)}{4\Psi^2(kL) - \Phi^2(kL)} \dfrac{6EI}{L} \\ C_{ij} = A_{ij} + B_{ji} \end{cases}$$

$$(MK)_{ij} = -A_{ij}\alpha_{ij}^{o} - B_{ij}\alpha_{ji}^{o}, \qquad (MK)_{ji} = -B_{ij}\alpha_{ij}^{o} - A_{ij}\alpha_{ji}^{o}$$

$$\Psi(kL) = \frac{3}{kL}\left(\frac{1}{kL} - \frac{1}{\tan kL}\right), \qquad \Phi(kL) = \frac{6}{kL}\left(\frac{1}{\sin kL} - \frac{1}{kL}\right), \quad k = \sqrt{\frac{P}{EI}}$$

$$\begin{cases} M_{ij} = A_{ij}\varphi_{ij} + B_{ji}\varphi_{ji} - C_{ij}\psi_{ij} + (MK)_{ij} \\ M_{ji} = A_{ji}\varphi_{ji} + B_{ij}\varphi_{ij} - C_{ji}\psi_{ji} + (MK)_{ji} \end{cases}$$

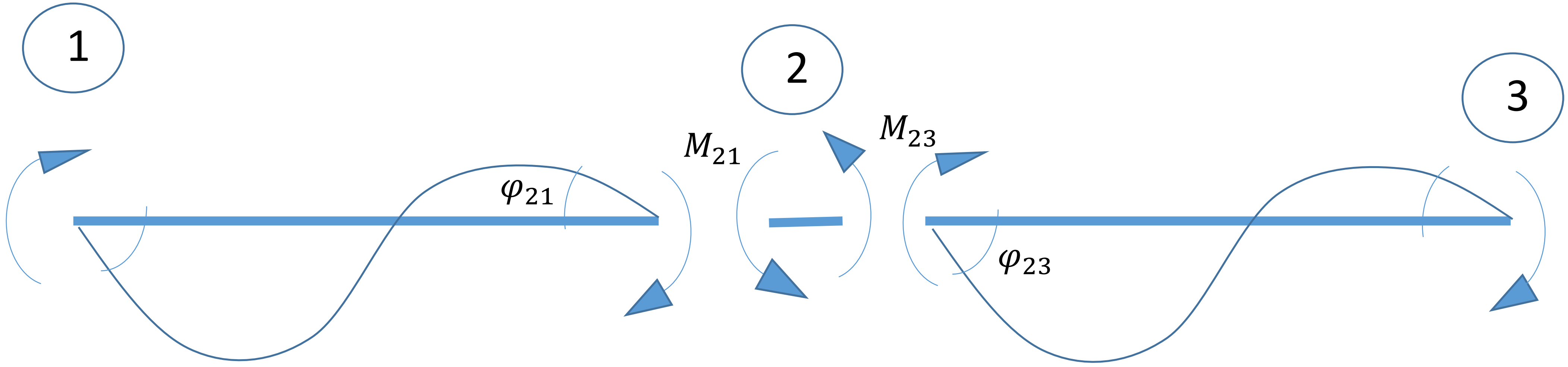
If $M_{ji} = 0, \Rightarrow \varphi_{ji} = -\frac{B_{ij}}{A_{ji}}\varphi_{ij} + \frac{C_{ji}}{A_{ji}}\psi_{ji} - \frac{(MK)_{ji}}{A_{ji}} \Rightarrow$

$$M_{ij} = A_{ij}^{o}(\varphi_{ij} - \psi_{ij}) + (MK)_{ij}^{o}$$

$$A_{ij}^{o} = A_{ij} - \frac{B_{ij}^{2}}{A_{ji}} = \frac{1}{\Psi(kL)}\frac{3EI}{L}$$

$$(MK)_{ij}^{o} = (MK)_{ij} - \frac{B_{ij}}{A_{ji}}(MK)_{ji} = -A_{ij}^{o}\alpha_{ij}^{o} = -\frac{1}{\Psi(kL)}\frac{3EI}{L}\alpha_{ij}^{o}$$

Equilibrium and Compatibility



$$\varphi_{21} = \varphi_{23} \; (= \varphi_2)$$

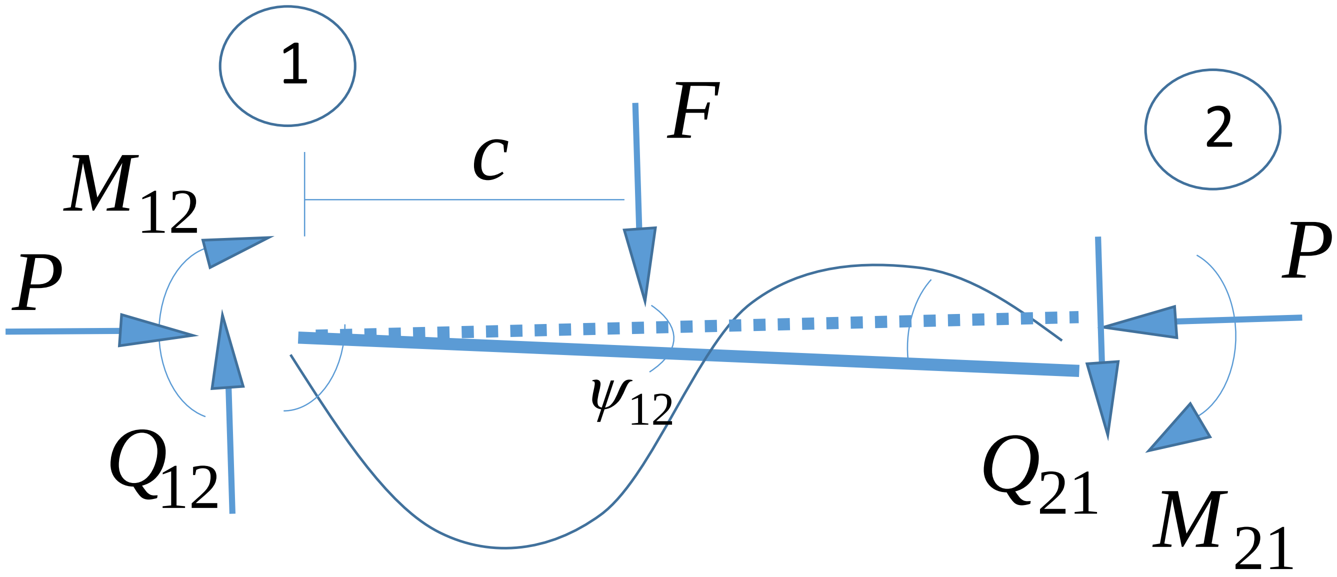
$$M_{21} + M_{23} = 0$$

More generally:

$$\varphi_i = \varphi_{ij} \quad j = 1, 2, \ldots N$$

$$\sum_{j=1}^{N} M_{ij} = 0$$

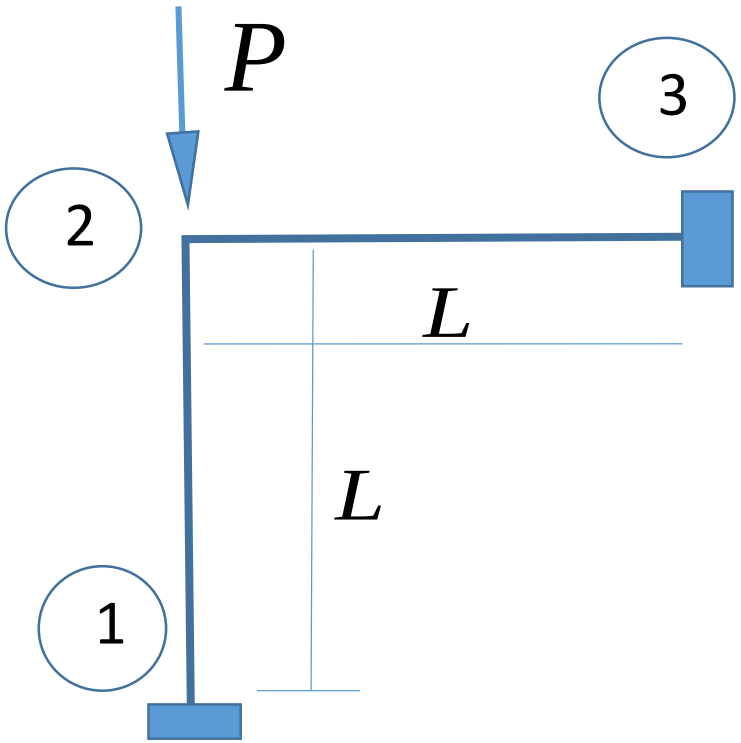
Swaying frame



$$Q_{21} = -\frac{c}{L}F - \frac{M_{12}+M_{21}}{L} - P\psi_{21}$$

$$Q_{12} = \frac{L-c}{L}F - \frac{M_{12}+M_{21}}{L} - P\psi_{21}$$

Examples



$$\varphi_{12} = 0 \qquad M_{12} = M_1$$

$$\varphi_{32} = 0 \qquad M_{32} = M_3$$

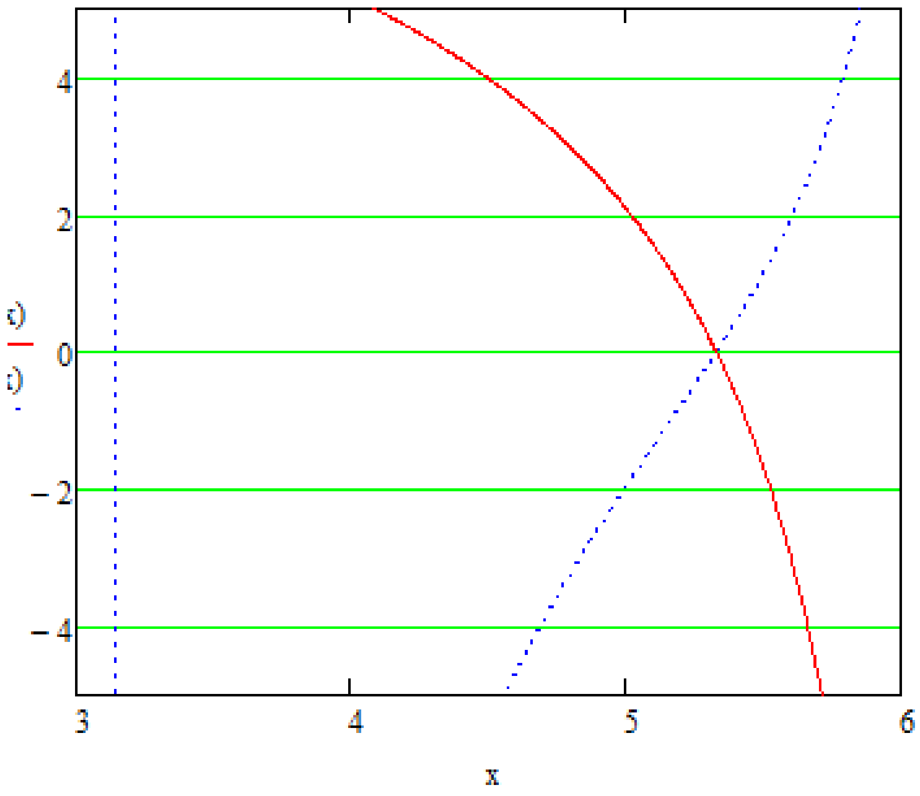
$$\varphi_{21} = \varphi_{23} \qquad M_{21} = -M_{23} = -M_2$$

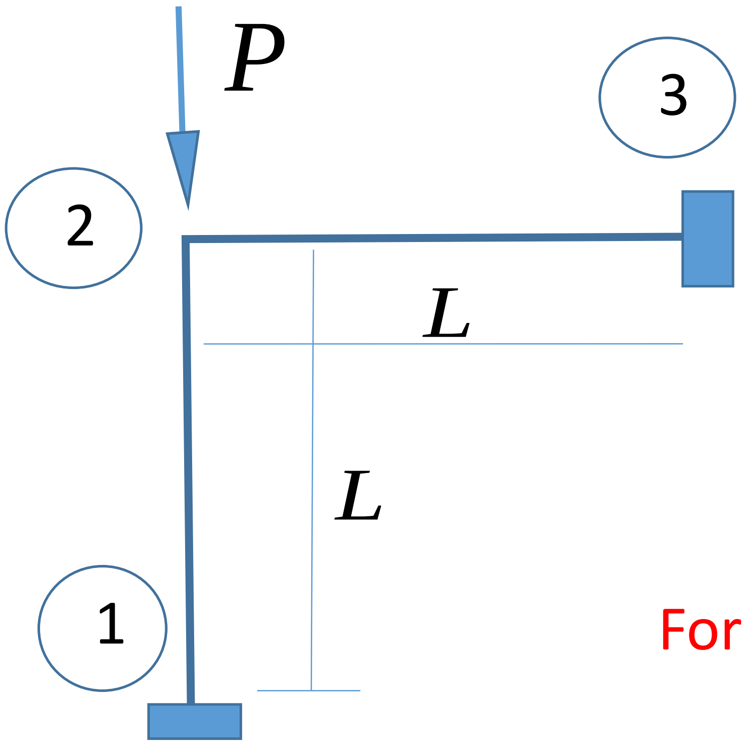
$$\begin{cases} \frac{L}{3EI}\Psi(kL)M_{12} - \frac{L}{6EI}\Phi(kL)M_{21} = 0 \\ \frac{L}{3EI}M_{32} - \frac{L}{6EI}M_{23} = 0 \Rightarrow 2M_3 - M_2 = 0 \\ \frac{L}{3EI}\Psi(kL)M_{21} - \frac{L}{6EI}\Phi(kL)M_{12} = \frac{L}{3EI}M_{23} - \frac{L}{6EI}M_{32} \end{cases}$$

$$\begin{cases} 2\Psi(kL)M_1 + \Phi(kL)M_2 = 0 \\ 2M_3 - M_2 = 0 \\ 2(1+\Psi(kL))M_2 + \Phi(kL)M_1 - M_3 = 0 \end{cases} \Leftrightarrow \begin{bmatrix} 2\Psi(kL) & \Phi(kL) & 0 \\ \Phi(kL) & 2(1+\Psi(kL)) & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{Bmatrix} M_1 \\ M_2 \\ M_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow \Psi(kL)(6+8\Psi(kL)) - 2\Phi^2(kL) = 0$$

$$\Rightarrow kL \cong 5{,}33 \Rightarrow P = 5{,}33^2 \frac{EI}{L^2} = 28{,}4 \frac{EI}{L^2}$$





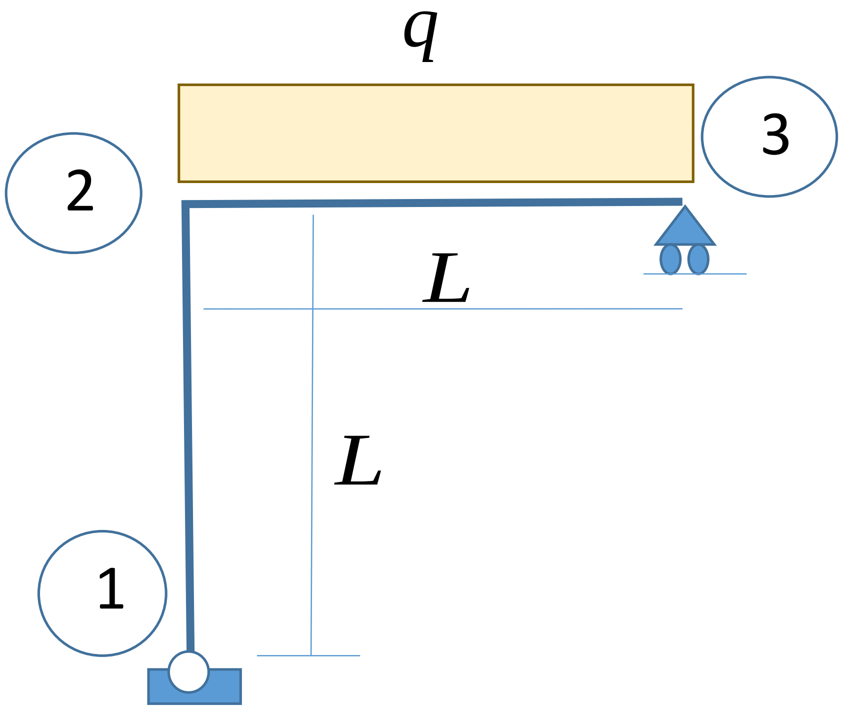
$$\begin{cases} M_{21} = A_{21}\varphi_{21} + B_{12}\not{\varphi_{12}} - C_{12}\not{\psi_{21}} + \not{(MK)_{21}} \\ M_{23} = a_{23}\varphi_{23} + b_{32}\not{\varphi_{32}} - c_{23}\not{\psi_{23}} + \not{(MK)_{23}} \end{cases}$$

$$M_{21} + M_{23} = 0 \Rightarrow (A_{21} + a_{23})\varphi_2 = 0$$

For a beam without compressive load

$$M_{ij} = a_{ij}\varphi_{ij} + b_{ji}\varphi_{ji} - c_{ij}\psi_{ij} + (MK)_{ij}$$

$$a_{ij} = \frac{4EI}{L}, \quad b_{ij} = \frac{2EI}{L}, \quad c_{ij} = \frac{6EI}{L}$$



$$M_{21} + M_{23} = 0$$

$$\begin{cases} M_{21} = A_{21}^{0}(\varphi_{21} - \psi_{21}) \\ M_{23} = a_{23}^{0}\varphi_{23} + (MK)_{23}^{o} \end{cases}$$

$$A_{21}^{0} = \frac{3}{\Psi(kL)}\frac{EI}{L}$$

$$a_{23}^{0} = \frac{3EI}{L}$$

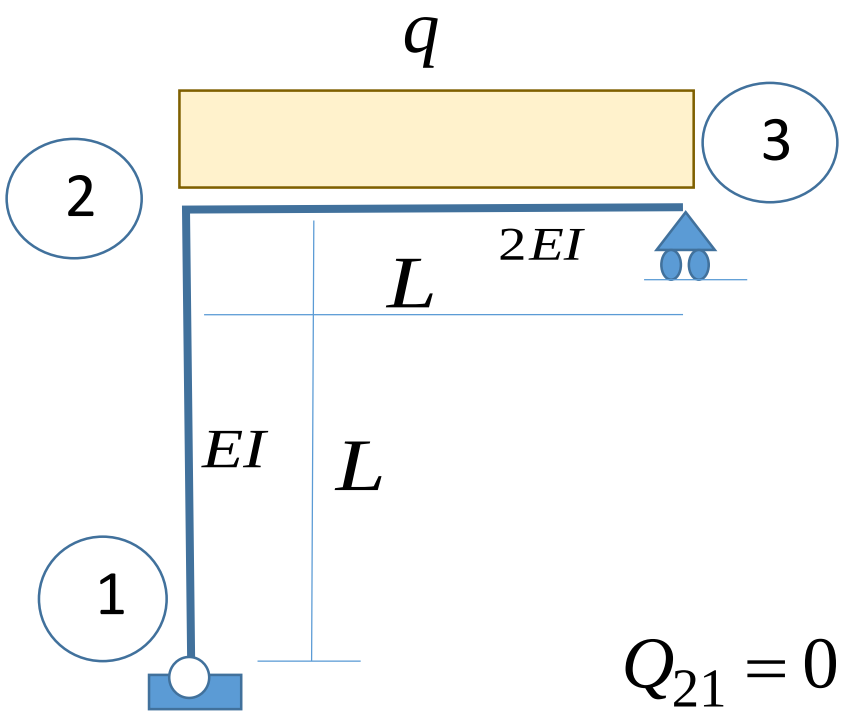
$$(MK)_{23}^{o} = -\frac{qL^{2}}{8}$$

$$\Rightarrow \boxed{(A_{21}^{0} + a_{23}^{o})\varphi_{2} - A_{21}^{0}\psi_{21} + (MK)_{23}^{o} = 0}$$

$$Q_{21} = 0 \Rightarrow -\frac{M_{21}}{L} - P\psi_{21} = 0 \qquad \Rightarrow \boxed{-A_{21}^{0}\varphi_{21} + (A_{21}^{0} - PL)\psi_{21} = 0}$$

$$\begin{cases} (A_{21}^{0} + a_{23}^{o})\varphi_{2} - A_{21}^{0}\psi_{21} + (MK)_{23}^{o} = 0 \\ -A_{21}^{0}\varphi_{21} + (A_{21}^{0} - PL)\psi_{21} = 0 \end{cases} \Leftrightarrow \begin{bmatrix} A_{21}^{0} + a_{23}^{o} & -A_{21}^{0} \\ -A_{21}^{0} & A_{21}^{0} - PL \end{bmatrix} \begin{Bmatrix} \varphi_{2} \\ \psi_{21} \end{Bmatrix} = \begin{Bmatrix} -(MK)_{23}^{o} \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} \dfrac{3}{\Psi(kL)} + 3 & \dfrac{-3}{\Psi(kL)} \\ \dfrac{-3}{\Psi(kL)} & \dfrac{3}{\Psi(kL)} - k^{2}L^{2} \end{bmatrix} \begin{Bmatrix} \varphi_{2} \\ \psi_{21} \end{Bmatrix} = \begin{Bmatrix} \dfrac{qL^{3}}{8EI} \\ 0 \end{Bmatrix}$$



$$\varphi_{21}=\varphi_{23}\Rightarrow\frac{L}{3EI}\Psi(kL)M_{21}+\psi_{21}=\frac{L}{6EI}M_{23}+\frac{qL^3}{48EI}$$

$$(1+2\Psi(kL))M_2+\frac{6EI}{L}\psi_{21}=\frac{qL^2}{8}$$

$$Q_{21}=0\ \Rightarrow\ -\frac{M_2}{L}-P\psi_{21}=0\Rightarrow\psi_{21}=-\frac{M_2}{PL}$$

$$(1+2\Psi(kL))M_2-\frac{3M_2}{k^2L^2}=\frac{qL^2}{8}$$

$$\Rightarrow\quad M_2=\frac{qL^2}{8}\frac{PL^2}{PL^2(1+2\Psi(kL))-6EI}$$

Here, we have the iteration procedure to determine the bending moment value at corner 2.

for $i \in 1..10$

$$q \leftarrow 80\frac{kN}{m}$$

$$EI \leftarrow 2.1\cdot 10^{3}\cdot kN\cdot m^{2}$$

$$L \leftarrow 6m$$

$$a \leftarrow q\cdot\frac{L^{3}}{48\cdot EI}$$

$$P_{0} \leftarrow \frac{q\cdot L}{2}$$

$$M_{0} \leftarrow 0$$

$$P_{i} \leftarrow P_{0} - \frac{M_{i-1}}{L}$$

$$kL_{i} \leftarrow \sqrt{\frac{P_{i}\cdot L^{2}}{EI}}$$

$$ps_{i} \leftarrow \left(\frac{3}{kL_{i}}\right)\cdot\left(\frac{1}{kL_{i}} - \frac{1}{\tan(kL)_{i}}\right)$$

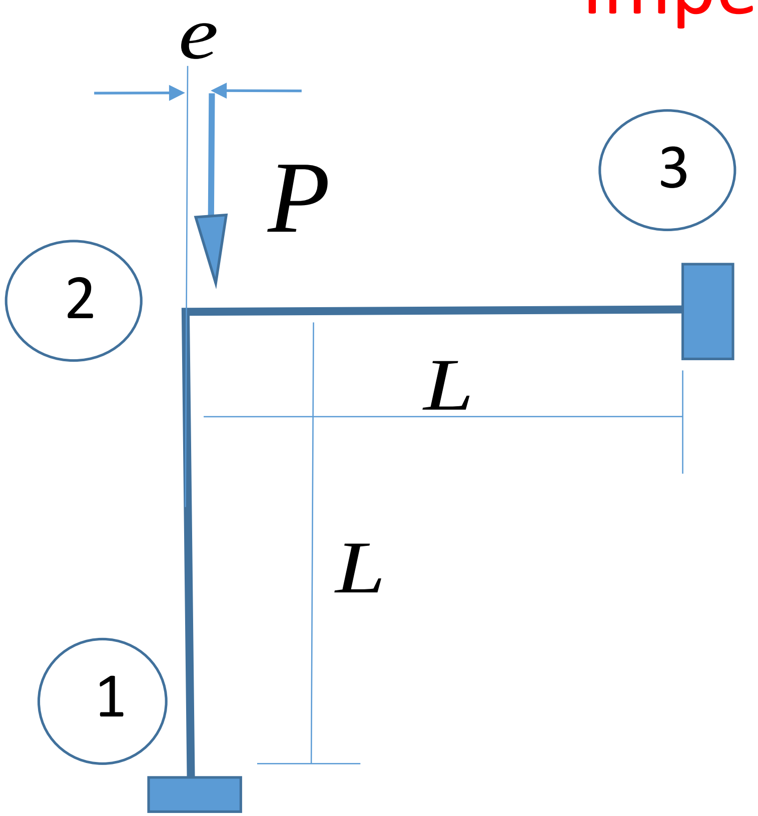
$$M_{i} \leftarrow \frac{6\cdot P_{i}\cdot L\cdot EI\cdot a}{P_{i}\cdot L^{2}\cdot\left[1 + 2(ps)_{i}\right] - 6\cdot EI}$$

M

= (J)

	0
0	0
1	1.465·105
2	1.678·105
3	1.716·105
4	1.723·105
5	1.724·105
6	1.724·105
7	1.724·105
8	1.724·105
9	1.724·105
10	1.724·105

Imperfection, eccentricity



$$M_{21}+M_{23}-Pe=0 \Rightarrow \varphi_2=\frac{Pe}{(A_{21}+a_{23})}$$

$$\begin{cases} M_{21}=A_{21}\varphi_2=Pe\dfrac{A_{21}}{(A_{21}+a_{23})} \\ M_{23}=a_{23}\varphi_2=Pe\dfrac{a_{23}}{(A_{21}+a_{23})} \\ M_{32}=b_{32}\varphi_2=\dfrac{M_{23}}{2} \end{cases}$$

$$Q_{23}=-\frac{M_{23}+M_{32}}{L}=-\frac{3}{2}\frac{M_{23}}{L}=-Pe\frac{3a_{23}}{2(A_{21}+a_{23})}$$

$$P=P+Q_{23}=P(1-e\frac{3a_{23}}{2(A_{21}+a_{23})})$$

$$P = P + Q_{23} = P(1 - e\frac{3a_{23}}{2(A_{21} + a_{23})}) = P(1 - e\frac{1}{\frac{\Psi(kL)}{4\Psi^2(kL) - \Phi^2(kL)} + \frac{2}{3}})$$

If now the eccentritcity *e* is negative, the value of the compressive load *P* is increasing all the time, and no convergence will be reached. If positive, the convergence is reached.