Chapter 4 Buckling of Frames

In chapters before, individual members with very idealized boundary conditions. In a framework, the members are rigidly connected to one another at the joints.

4.1 Introduction

4.1.1 • Classification according to sidesway:

1. Sidesway is prevented as shown in **Fig. 4-1a • 4-1b •** the frames give the symmetric buckling.

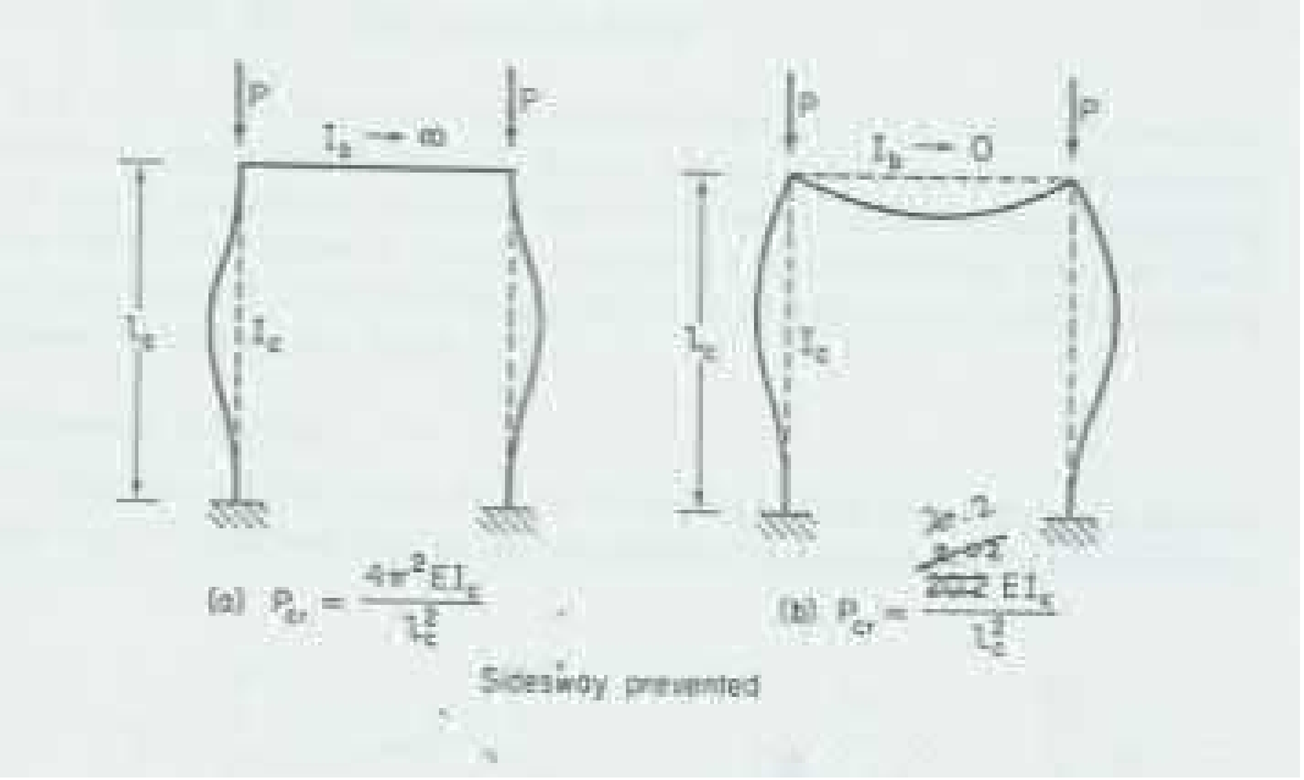


Fig.4-1 buckling modes of single-story frame

2. Sidesway is permitted as shown in **Fig.4-1c • 4-1d,** the frames give the sidesway buckling.

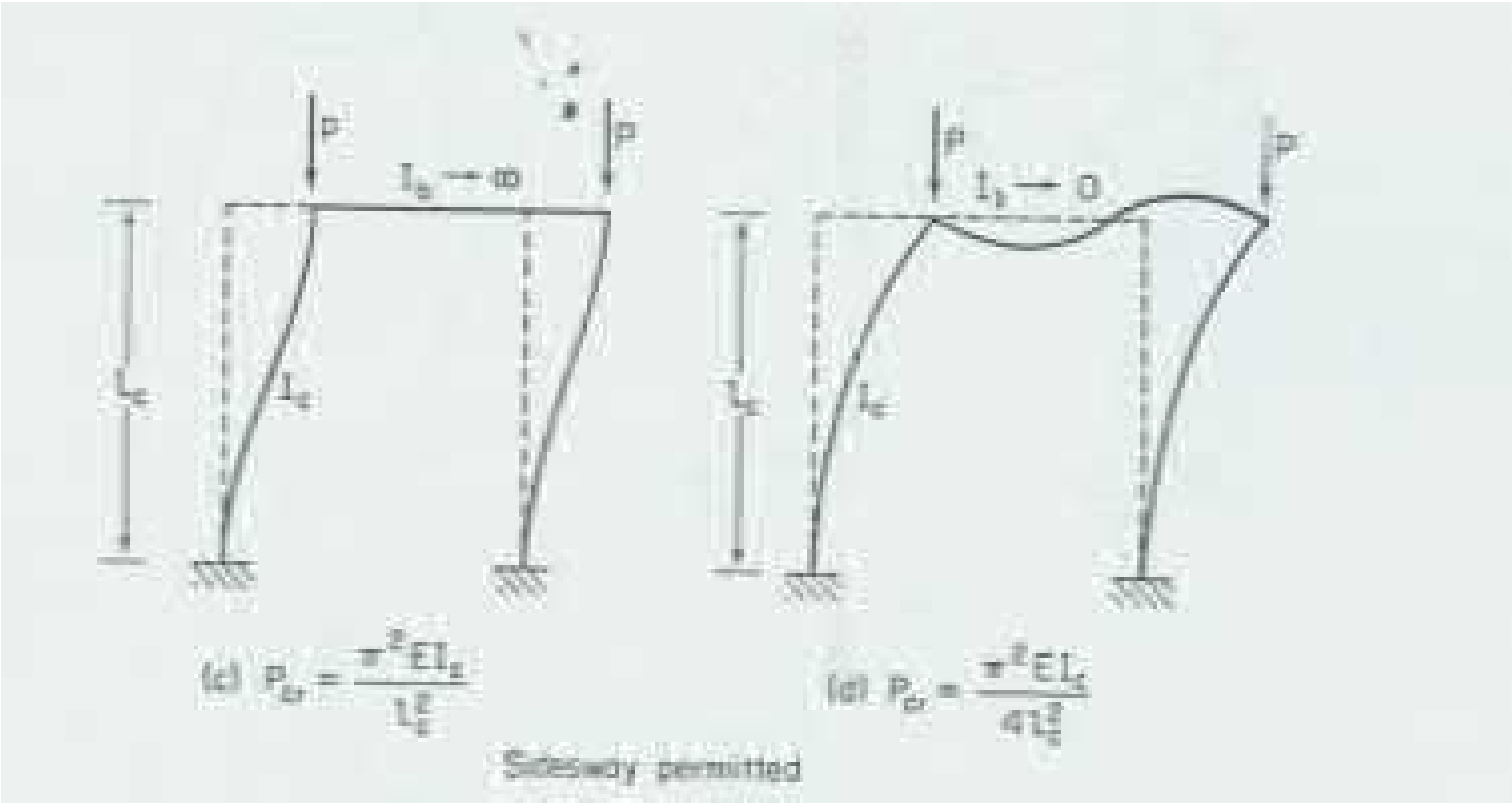


Fig.4-1c • 4-1d

4.1.2 • Classification according to the beam•s stiffness:

1. the beam is infinitely rigid and remain straight with no deflection while the frame deforms as shown in **Fig.4-1a • c**.

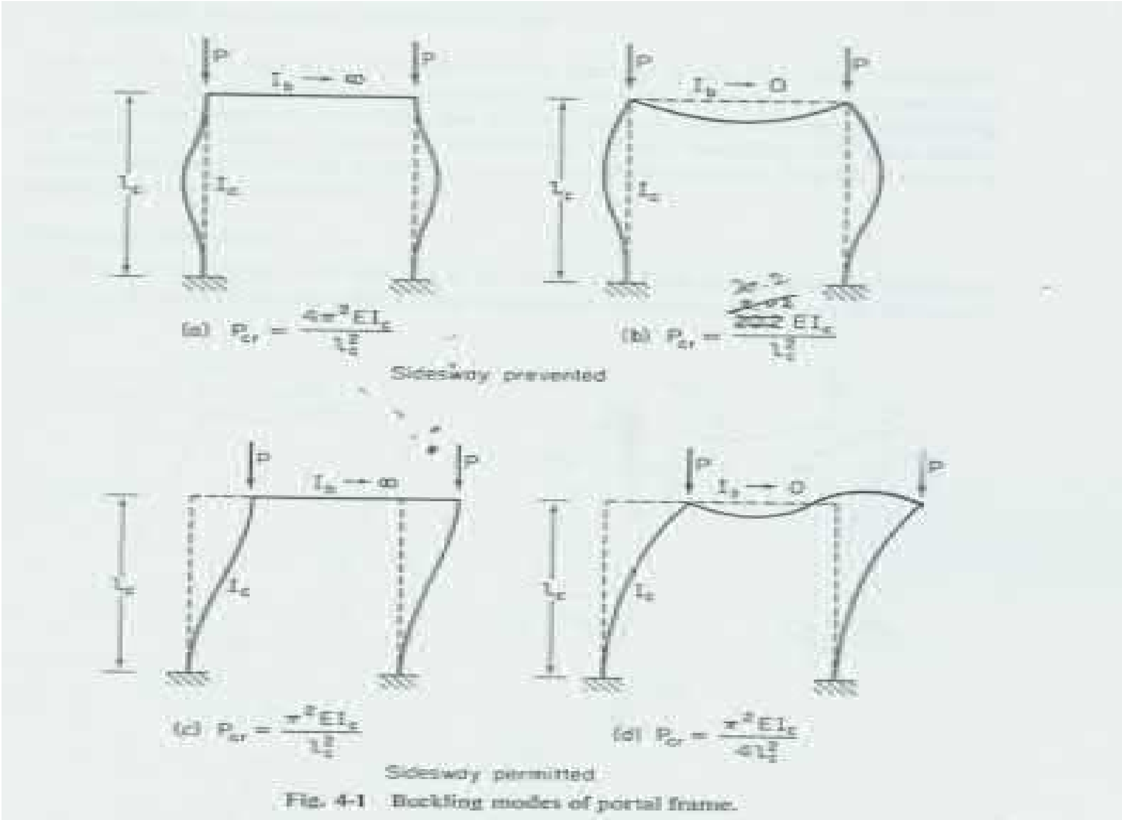


Fig. 4-1 Buckling modes of portal frame.

Fig.4-1a • c

2. The beam is infinitely flexible and can not restrain the rotation of the upper end of the column as shown in **Fig.4-1b • d**.

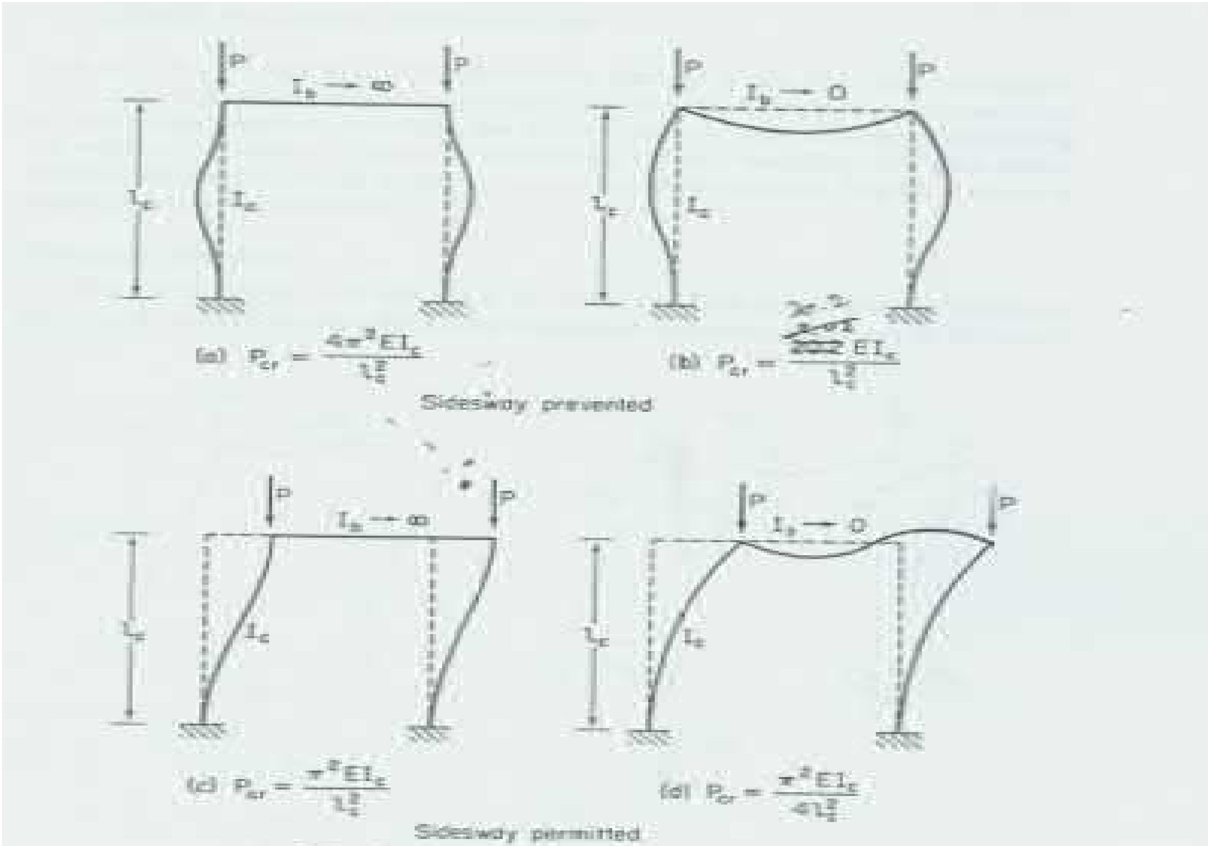


Fig. 4-1 Buckling modes of portal frame.

As the **Fig.4-1a** is concerned, the columns behave as if they were fixed at both extremities, and the critical load of the frame is equal to four times the Euler load of the columns:

$$N_{cr} = 4N_e$$

in which N_e • • the Euler load

N_{cr} • • the critical load

As the **Fig.4-1b** is concerned, the columns behave as if they were fixed at one end and hinged at the other, and the critical load of the frame is equal to twice times the Euler load of the columns:

$$N_{cr} = 2N_e$$

For an actual frame the flexibility of the beam must lie somewhere between the two extreme conditions just considered. The critical load of such a frame as show in Fig. a , b, can be bracketed as follows:

$$2N_e < N_{cr} < 4N_e \qquad , \; \mathbf{4.1}\, f$$

As the **Fig.4-1c** is concerned, the sidesway is permitted and the beam is infinitely rigid, so the upper ends of the columns are free to translate but not to rotate, and the critical load of the frame is equal to the Euler load of the columns:

$$N_{cr} = N_e$$

As the **Fig.4-1d** is concerned, the sidesway is permitted and the beam is infinitely flexible, so the upper ends of the columns are free to translate and rotate, and the critical load of the frame is equal to one fourth times the Euler load of the columns:

$$N_{cr} = \frac{1}{4} N_e$$

The two critical loads coming from **Fig.c • d** give the upper limit and lower limit in the sidesway mode respectively, namely:

$$\frac{1}{4} N_e < N_{cr} < N_e \qquad , \mathbf{4.2}\ \boldsymbol{f}$$

The expressions in (4.1)，(4.2) indicate that no matter how the stiffness of the members is, the critical load of the symmetric buckling is bigger than that of sidesway buckling. It can be therefore concluded that **the portal frame will always buckle in the sidesway mode** unless it is laterally braced, in which case it must buckle in the symmetric mode, this conclusion is always valid for multistory frames as well as for single-story frames.

In the code, the lateral bending rigidity of the brace system is five times as that of the frame system, it is assumed that the sidesway is prevented.

4.2 Critical load of single-story frame using neutral equilibrium

4.2.1 • Sidesway Buckling

Sidesway buckling is happened when the sidesway is permitted. when the frame buckles, the external load，boundary conditions and the deformations are shown in **Fig.4-2**.

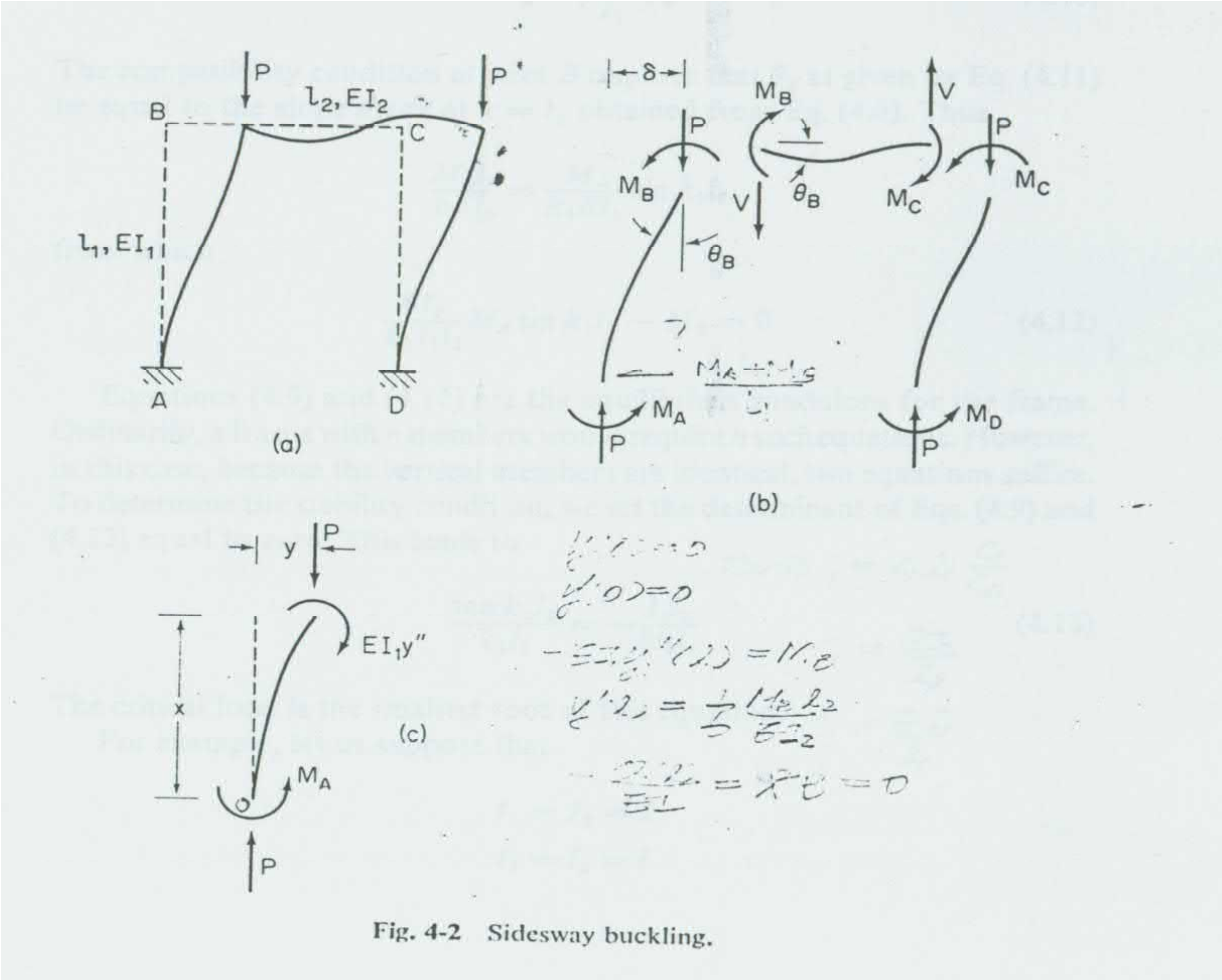


Fig.4-2 sidesway buckling

Suppose:

1 , no primary bending present in the frame prior to buckling;

2 , material behaves according to Hooke•s law $\sigma = E\varepsilon$;

3 , the deformations remain small $y'' = 1/\rho$;

4 , not considering the influence of the shear Q for compressed vertical columns

Obtaining the critical load needs to solve two equations:

1 , the equilibrium equation for the vertical member represented by the moment at the upper ends of the member φ

2 , the compatibility equation of the horizontal member represented by **Slope-deflection equation** at the upper ends of the member;

First equation „

$y'' > 0$ **,** the equation of moment equilibrium for vertical member **(Fig.4-2c)** is:

$$EI_1 \frac{d^2y}{dx^2} + Ny = M_A \qquad , \ \mathbf{4.3}\ f$$

or

$$\frac{d^2y}{dx^2} + k_1^2 y = \frac{M_A}{EI_1} \qquad , \ \mathbf{4.4}\ f$$

$$k_1^2 = \frac{N}{EI_1}$$

The solution of Eq.(4.4) is:

$$y = Asimk_1 x + B\cos k_1 x + \frac{N_A}{N} \qquad , \ \mathbf{4.5}\ f$$

from boundary conditions: **x=0…** **y=0**

and **x=0…** **y¢=0**

We obtain:

$$\mathbf{B=-M_A/N} \quad \text{and} \quad \mathbf{A=0}$$

Thus:

$$y = \frac{M_A}{N}\left(1 - \cos k_1 x\right) \qquad , \ \mathbf{4.6}\ f$$

at $x = l_1$:

$$d = \frac{M_A}{N}\left(1 - \cos k_1 l_1\right)$$

also moment equilibrium:

$$d = \frac{M_A + M_B}{N}$$

so

$$M_A \cos k_1 l_1 + M_B = 0 \qquad , \ \mathbf{4.9}\ f$$

Second equation „

the compatibility equation of the horizontal member represented by

Slope-deflection equation

$$M_B = \frac{2EI_2}{l_2}(2\theta_B + \theta_C)$$

Since $\theta_B = \theta_C$

$$M_B = \frac{6EI_2}{l_2}\theta_B$$

$$\theta_B = y'\Big|_{x=l_1}$$

So

$$\frac{6I_2}{k_1 I_1 l_2} M_A \sin k_1 l_1 - M_B = 0 \qquad (4.12)$$

Equations (4.9) and (4.12) lead to

$$\frac{\tan k_1 l_1}{k_1 l_1} = -\frac{I_1 l_2}{6 I_2 l_1} \tag{4.13}$$

For example, let us suppose that

$$I_1 = I_2 = I$$

$$l_1 = l_2 = l$$

$$\frac{\tan kl}{kl} = -\frac{1}{6}$$

$$kl = 2.71$$

$$P_{cr} = \frac{7.34 EI}{l^2}$$

4.2.2 • Symmetric Buckling

If the frame is prevented from translating laterally at the top, buckling will occur in the symmetric mode.

when the frame buckles, the external load，boundary conditions and the deformations are shown in **Fig.4-3.**

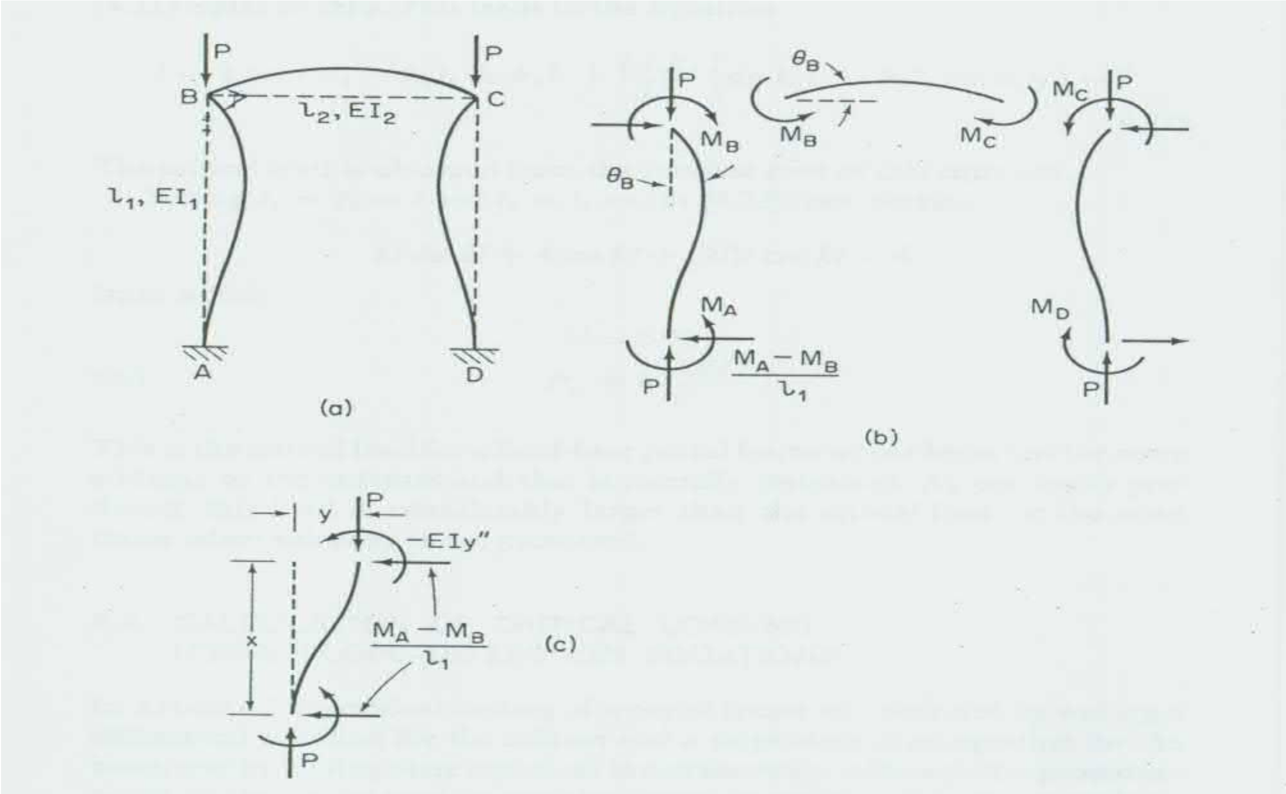


Fig. 4-3 Symmetric buckling.

Fig.4-3 symmetric buckling

First equation „

the equation of moment equilibrium for vertical member **(Fig.4-3c)** is:

$$EI_1 \frac{d^2 y}{dx^2} + Ny = M_A - (M_A - M_B)\frac{x}{l_1} \quad , \ \mathbf{4.14} \ f$$

or

$$\frac{d^2 y}{dx^2} + k_1^2 y = \frac{M_A}{EI_1}(1 - \frac{x}{l_1}) + \frac{M_B}{EI_1}(\frac{x}{l_1}) \quad , \ \mathbf{4.15} \ f$$

$$k_1^2 = \frac{N}{EI_1}$$

The solution of Eq.(4.4) is:

$$y = A\sin k_1 x + B\cos k_1 x + \frac{M_A}{N}(1 - \frac{x}{l}) + \frac{M_B}{N}(\frac{x}{l}) \quad , \ \mathbf{4.16} \ f$$

from boundary conditions: **x=0…** **y=0**

and **x=0…** **y¢=0**

We obtain:

B=-M$_{\mathbf{A}}$/N and $A = \dfrac{M_A - M_B}{k_1 N l_1}$

Thus: $$y = \frac{M_A}{N}\left(\frac{1}{k_1 l_1}\sin k_1 x - \cos k_1 x + 1 - \frac{x}{l_1}\right)$$

$$+\frac{M_B}{N}\left(\frac{x}{l_1} - \frac{1}{k_1 l_1}\sin k_1 x\right) \quad , \ \mathbf{4.17}\ f$$

at $x = l_1$: $y = 0$

so $$M_A(\sin k_1 l_1 - k_1 l_1 \cos k_1 l_1) + M_B(k_1 l_1 - \sin k_1 l_1) = 0 \quad , \ \mathbf{4.18}\ f$$

Second equation „

the compatibility equation of the horizontal member represented by

Slope-deflection equation

$$M_B = \frac{2EI_2}{l_2}(2\theta_B + \theta_C)$$

Since $\theta_B = -\theta_C$

$$M_B = \frac{2E\ I_2}{l_2}\theta_B$$

$$-\theta_B = y'\Big|_{x=l_1}$$

So

$$M_A(\cos k_1 l_1 + k_1 l_1 \sin k_1 l_1 - 1) + M_B(1 - \cos k_1 l_1 + \frac{I_1 l_1 k_1^2 l_2}{2I_2}) = 0$$

(4.21)

Equations (4.18) and (4.21) lead to

$$2-2\cos k_1 l_1 - k_1 l_1 \sin k_1 l_1 + \frac{l_2 I_2 k_1}{2I_2}(\sin k_1 l_1 - k_1 l_1 \cos k_1 l_1) = 0 \tag{4.13}$$

For example, let us suppose that

$$I_1 = I_2 = I$$

$$l_1 = l_2 = l$$

$$kl \sin kl + 4\cos kl + (kl)^2 \cos kl = 4$$

$$kl = 5.02$$

$$P_{cr} = \frac{25.2EI}{l^2}$$

4.3 critical loading using slope-deflection equations

Too complex

$$if K = EI / l,$$

$$M_{BA} = K(a_{n1} q_B)$$

$$M_{BE} = K(a_{n2} q_B + a_{f2} q_E)$$

$$M_{BC} = K(a_{n3} q_B + a_{f3} q_C)$$

$$M_{CB} = K(a_{n3} q_C + a_{f3} q_B)$$

$$M_{CD} = K(a_{n4} q_C + a_{f4} q_D)$$

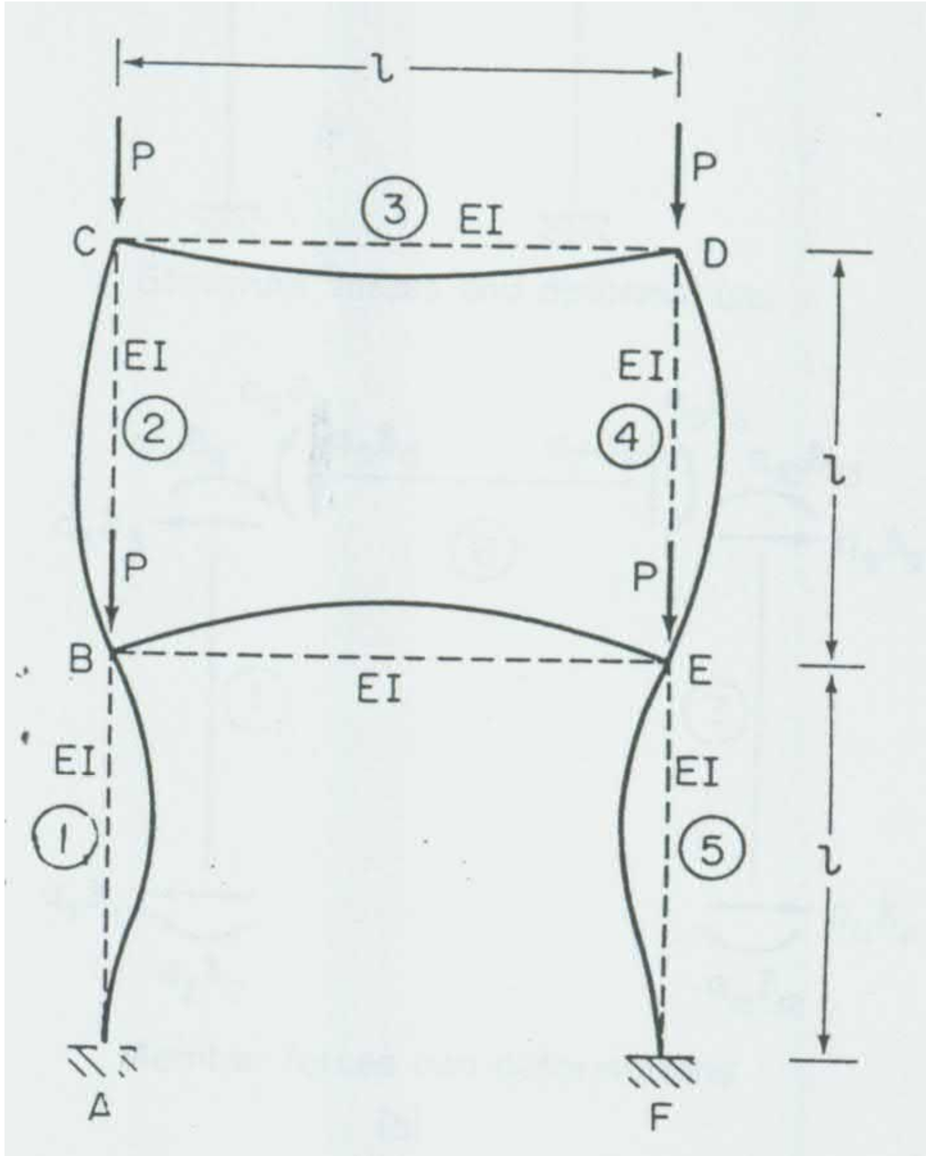


Fig 4-4 buckling of two-story frame

kl	α_n	α_f
0.00	4.0000	2.0000
0.20	3.9946	2.0024
0.40	3.9786	2.0057
0.60	3.9524	2.0119
0.80	3.9136	2.0201
1.00	3.8650	2.0345
1.20	3.8042	2.0502
1.40	3.7317	2.0696
1.60	3.6466	2.0927
1.80	3.5483	2.1199
2.00	3.4364	2.1523
2.04	3.4119	2.1589
2.08	3.3872	2.1662
2.12	3.3617	2.1737
2.16	3.3358	2.1814
2.20	3.3090	2.1893
2.24	3.2814	2.1975
2.28	3.2538	2.2059
2.32	3.2252	2.2146
2.36	3.1959	2.2236
2.40	3.1659	2.2328
2.44	3.1352	2.2424

kl	α_n	α_f
2.48	3.1039	2.2522
2.52	3.0717	2.2623
2.56	3.0389	2.2728
2.60	3.0052	2.2834
2.64	2.9710	2.2946
2.68	2.9357	2.3060
2.72	2.8997	2.3177
2.76	2.8631	2.3300
2.80	2.8255	2.3425
2.84	2.7870	2.3555
2.88	2.7476	2.3688
2.92	2.7073	2.3825
2.96	2.6662	2.3967
3.00	2.6243	2.4115
3.10	2.5144	2.4499
3.15	2.4549	2.4681
3.20	2.3987	2.4922
3.25	2.3385	2.5148
3.30	2.2763	2.5382
3.40	2.1463	2.5881
3.50	2.0084	2.6424
3.60	1.8619	2.7017
3.70	1.7060	2.7668
3.80	1.5400	2.8382
3.90	1.3627	2.9168
4.00	1.1731	3.0037
4.20	0.7510	3.2074
4.40	0.2592	3.4619
4.60	−0.3234	3.7866
4.80	−1.0289	4.2111
5.00	−1.9087	4.7845
5.25	−3.3951	5.8469
5.50	−5.6726	7.6472
5.75	−9.8097	11.2438
6.00	−20.6370	21.4534
6.25	−188.3751	188.4783
2π		
6.50	29.4999	−30.2318

$$\alpha_{n2} = \alpha_{n4} = 4$$

$$\alpha_{f2} = \alpha_{f4} = 2$$

$$\theta_C = -\theta_D = -\theta_B)$$

$$M_{BA} + M_{BE} + M_{BC} = 0$$

$$M_{CB} + M_{CD} = 0$$

$$\theta_B(\alpha_{n1} + \alpha_{n3} + 2) + \theta_C(\alpha_{f3}) = 0$$

$$\theta_B(\alpha_{f3}) + \theta_C(\alpha_{n3} + 2) = 0$$

Using the values of α, kl, it can be solved by trial and error for the critical loading.

since

$$k_1 = \sqrt{2P / EI}$$

$$k_3 = \sqrt{P / EI}$$

$$k_1 = k_3\sqrt{2}$$

$$k_3 l = 3.55$$

$$P_{cr} = \frac{12.6EI}{l^2}$$

4.4 effect of primary bending and plasticity on frame behavior

4.4.1 effect of primary bending on elastic buckling load

From the study, primary bending does not significantly lower the critical load of a frame as long as stresses remain elastic. The only exception occurs when the beam is exceptionally long but they are rarely encountered. So it appears safe that the effect of primary bending can be neglected in determining the elastic buckling load of a frame.

4.4.2 inelastic buckling load

If the proportional limit is exceeded before instability occurs, a rough estimate of the collapse load can be obtained if an interaction equation is used.

$$\frac{P_f}{P_e} + \frac{P_f}{P_p} = 1.0$$

Where P_f =failure load

P_e =elastically determined critical load

P_p =plastic mechanism load

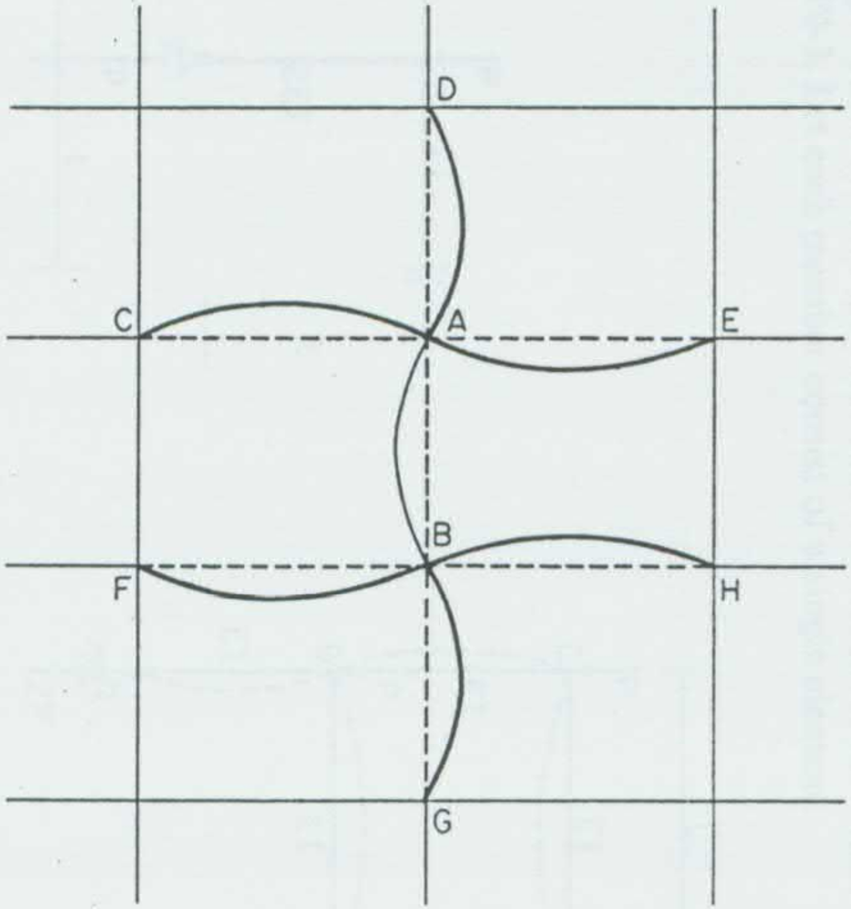
Conservative and reasonable accurate.

4.5 design of framed columns

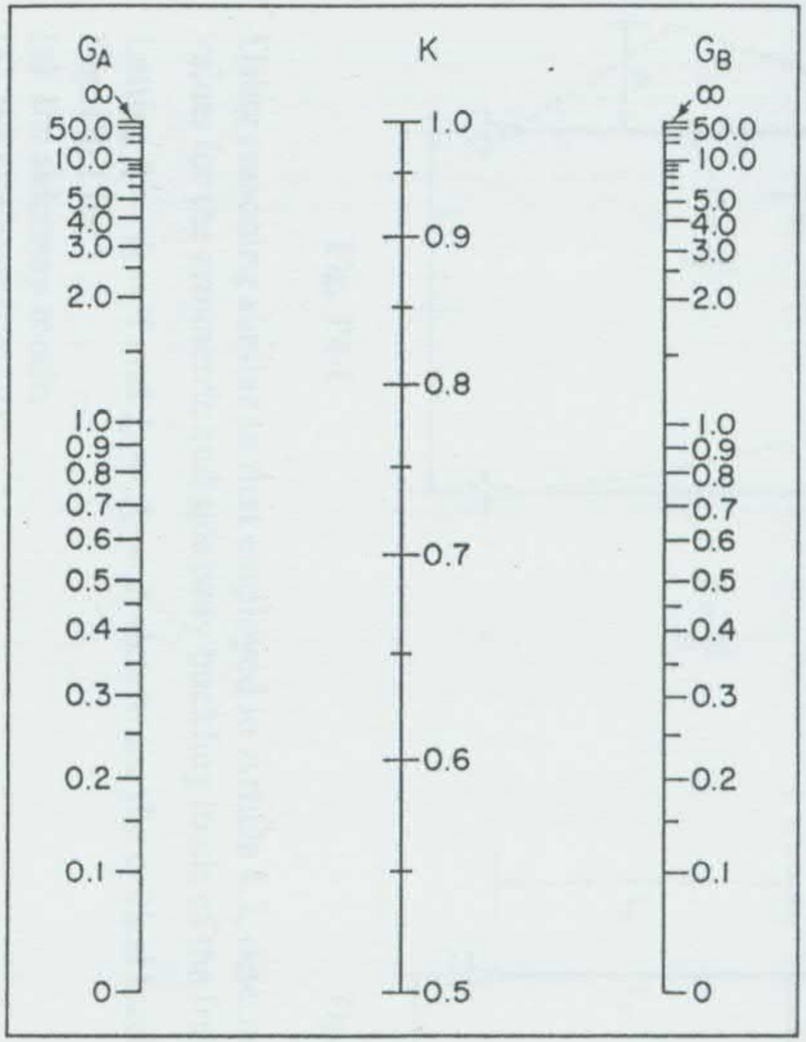
One way is to carry out a stability analysis of the entire frame. Too involved for routine design.

One very simple but quite crude method is to estimate the degree of restrain by interpolating between the idealized boundary conditions.

A one considerably more accurate was developed. It involves an exact analysis.however, only the member in question and the beams and columns that frame directly into it are considered..



(a) Framed column and subassemblage of adjacent members assumed to resist its deformation



Sidesway prevented

(b) Jackson and Moreland nomograph for effective length of framed columns

Fig. 4-8 Julian and Lawrence method for estimating effective length of framed column. (Adapted from Ref. 4.9.)

The parameters:

$$G_A = \frac{I_{AB}/L_{AB} + I_{AD}/L_{AD}}{I_{CA}/L_{CA} + I_{AE}/L_{AE}}$$

$$G_B = \frac{I_{AB}/L_{AB} + I_{BG}/L_{BG}}{I_{FB}/L_{FB} + I_{BH}/L_{BH}}$$

By the given stiffnesses of the adjacent members, the nomograph shown in Fig 4-8b allows one to determine directly the effective length of a framed column.