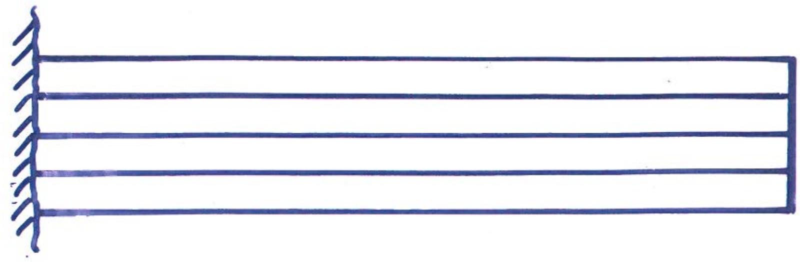
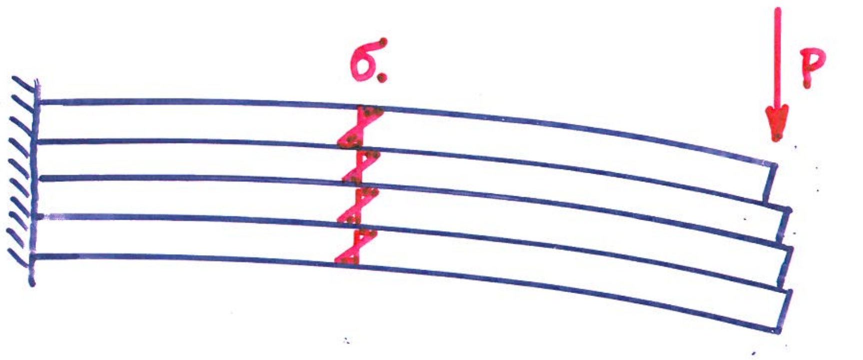
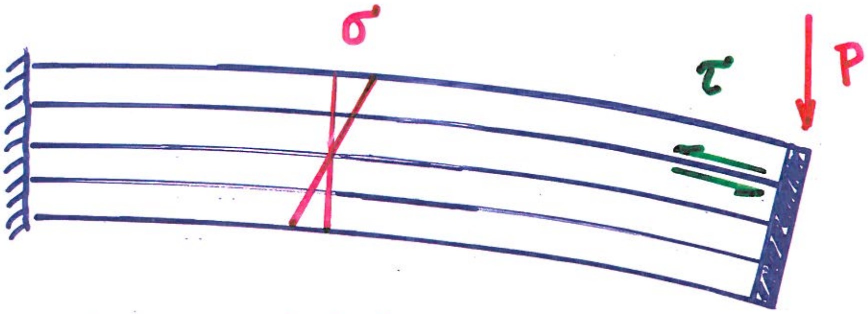
# Palkin toiminta:



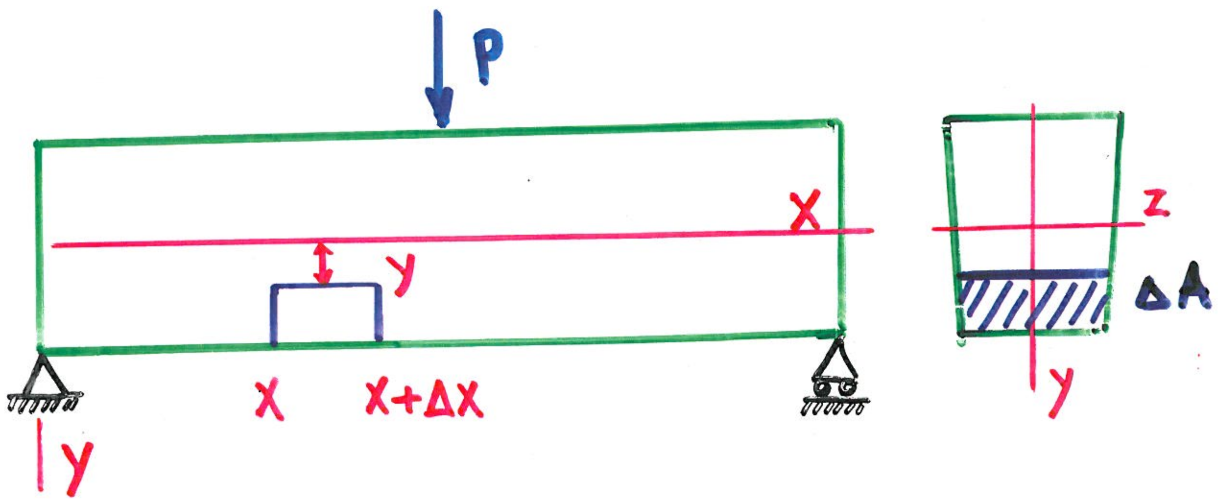


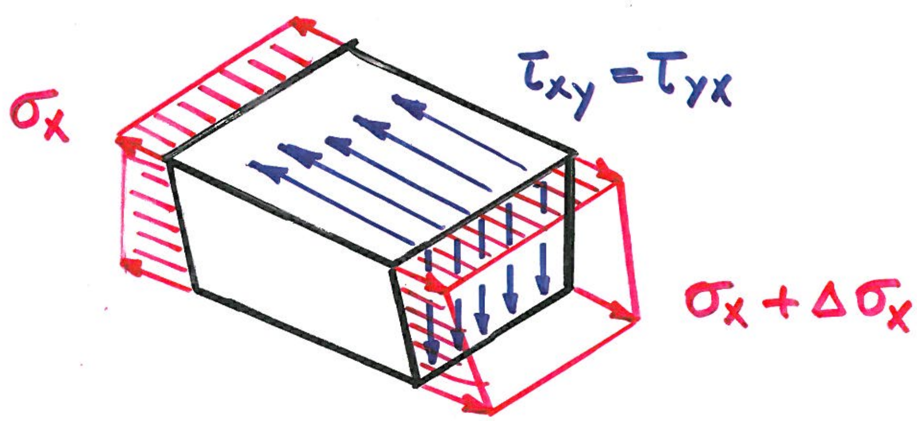




(E vakio)

Leikkausjännitykset $\tau$

# LEIKKAUSJÄNNITYSTEN JAKAANTUMINEN



Kirjoitetaan sauvan akselin suuntainen tasapainoehto:

$$\int_{\Delta A} (\cancel{\sigma_x} + \Delta\sigma_x)\, dA - \cancel{\int_{\Delta A} \sigma_x\, dA} - \int_{\Delta S} \tau_{xy}\, dS = 0$$

Tästä saadaan edelleen

$$\int_{\Delta A} \frac{\Delta M_z}{I_z} y\, dA = \tau_{xy} \cdot \Delta x \cdot b(y)$$

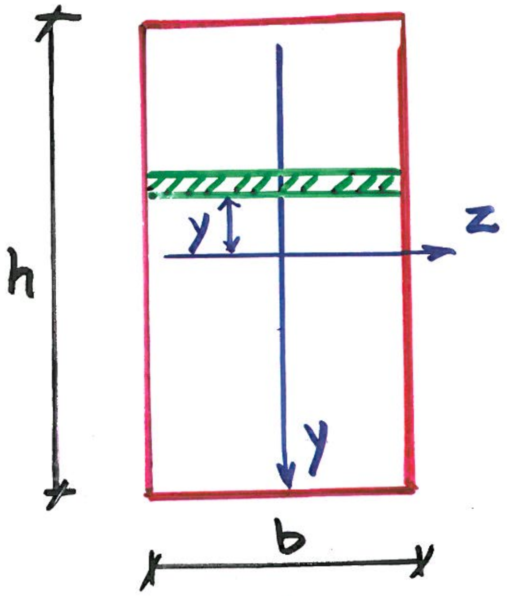
ja vielä

$$\frac{\Delta M_z}{I_z} \int_{\Delta A} y\, dA = \boxed{\frac{\Delta M_z}{I_z} S_z(y) = \tau_{xy} \Delta x\, b(y)}$$

$\Rightarrow$

$$\boxed{\tau_{xy} = \frac{\Delta M_z}{\Delta x} \frac{S_z(y)}{I_z\, b(y)} = \frac{Q(x)\, S_z(y)}{I_z\, b(y)}}$$

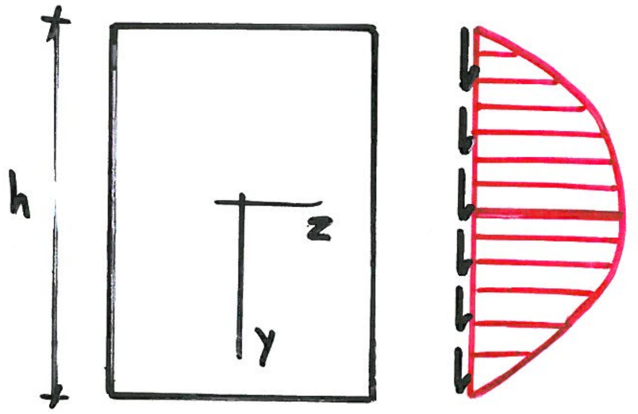
Esimerkki: **Suorakaidepoikkileikkaus**




$$S_z = \int_{\Delta A} y \, dA = \int_{y}^{-h/2} \int_{-b/2}^{b/2} y \, dz \, dy =$$

$$= b \Big/_{y}^{-h/2} \left( \frac{y^2}{2} \right) = \underline{\underline{\frac{bh^2}{8} \left( 1 - \left( \frac{y}{h/2} \right)^2 \right)}}$$

$$\underline{\tau_{xy}} = \frac{Q \, \frac{bh^2}{8} \left( 1 - \left( \frac{2y}{h} \right)^2 \right)}{\frac{1}{12} bh^3 \cdot b} = \underline{\frac{3}{2} \frac{Q}{A} \left( 1 - \frac{4y^2}{h^2} \right)}$$



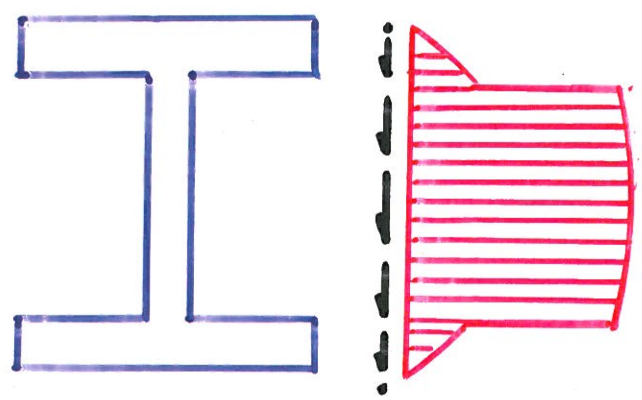

$\tau_{max} = \frac{3}{2}\frac{Q}{A}$

## Tarkastus:

$$\int_A \tau_{xy}\, dA = \frac{3}{2}\frac{Q}{A}\cdot b\int_{-h/2}^{h/2}\left(1-\frac{4y^2}{h^2}\right)dy$$

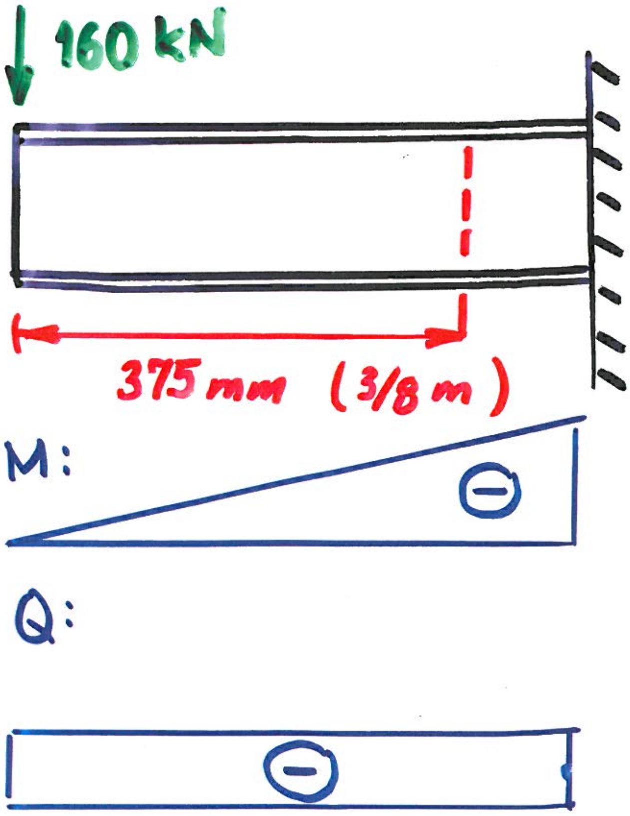
$$= \frac{3}{2}\frac{Q}{h}\Big/_{-h/2}^{h/2}\left(y-\frac{4}{3}\frac{y^3}{h^2}\right) = \underline{Q}$$

## I-poikkileikkaus



Hyvä approksimaatio $\tau_{keskim} = \frac{Q}{A_{uuma}}$

# Esimerkki:

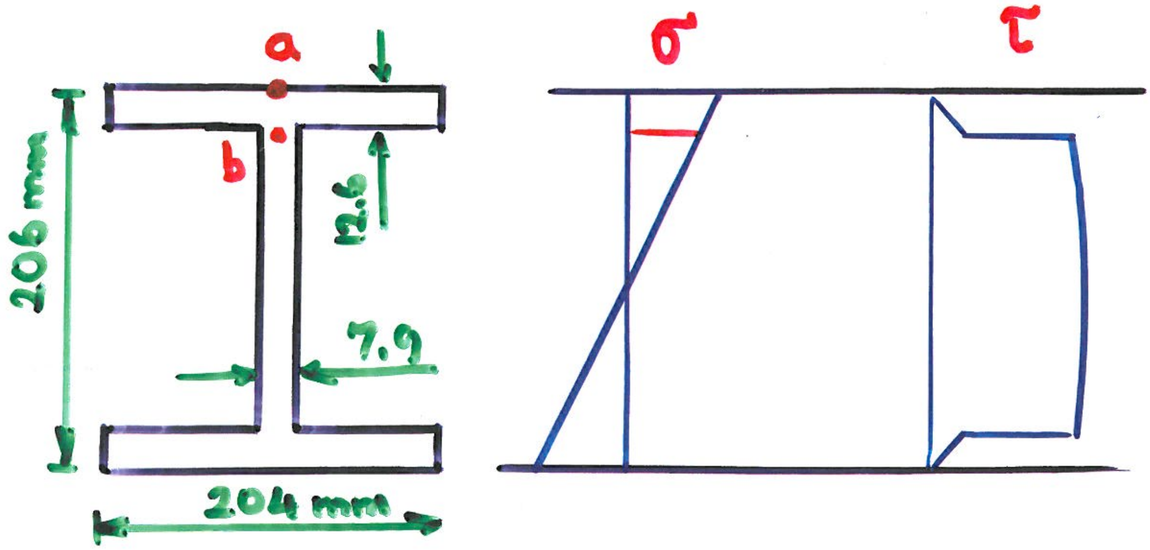



$I_z = 52.9 \cdot 10^{-6}\, m^4$

$M(3/8) = -60$ kNm

$Q(3/8) = -160$ kN

$\sigma_{sall} = 125$ MPa




$$\begin{cases} \sigma_a = \dfrac{-60\text{ kNm}}{52.9 \cdot 10^{-6}\, m^4} \cdot (-103\text{ mm}) = 116.7\text{ MPa} \\ \\ \tau_a = 0 \end{cases}$$

$$\sigma_b = \frac{-60\ kNm}{52.9 \cdot 10^{-6} m^4} \cdot 90.4\ mm = \underline{102.4\ MPa}$$

Staattinen momentti

$$S_b = (204 \cdot 12.6)(-96.7) = -249 \cdot 10^{-6} m^3$$

$$\tau_b = \frac{(-160\ kN)(-249 \cdot 10^{-6} m^3)}{52.9 \cdot 10^{-6} m^4\ \ 0.0079\ m} = \underline{95.3\ MPa}$$

Pääjännitykset

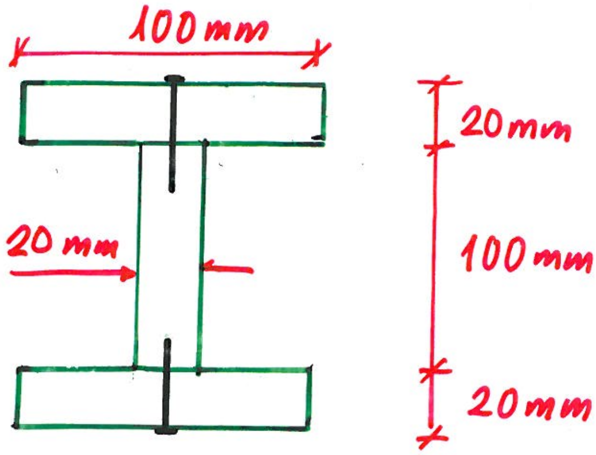
$$\sigma_{max} = \frac{\sigma_b}{2} + \sqrt{\left(\frac{\sigma_b}{2}\right)^2 + \tau_b^2} = \underline{159.4\ MPa}$$

$$\underline{\sigma_{max} > \sigma_{sall}}$$

Pääjännitys pisteessä b 36 % suurempi kuin pisteessä a.

Jos uloke pidempi kuin 874 mm, suurin normaalijännitys pisteessä a.

Esimerkki:




$Q = 500\,N$

Naulojen väli 25 mm

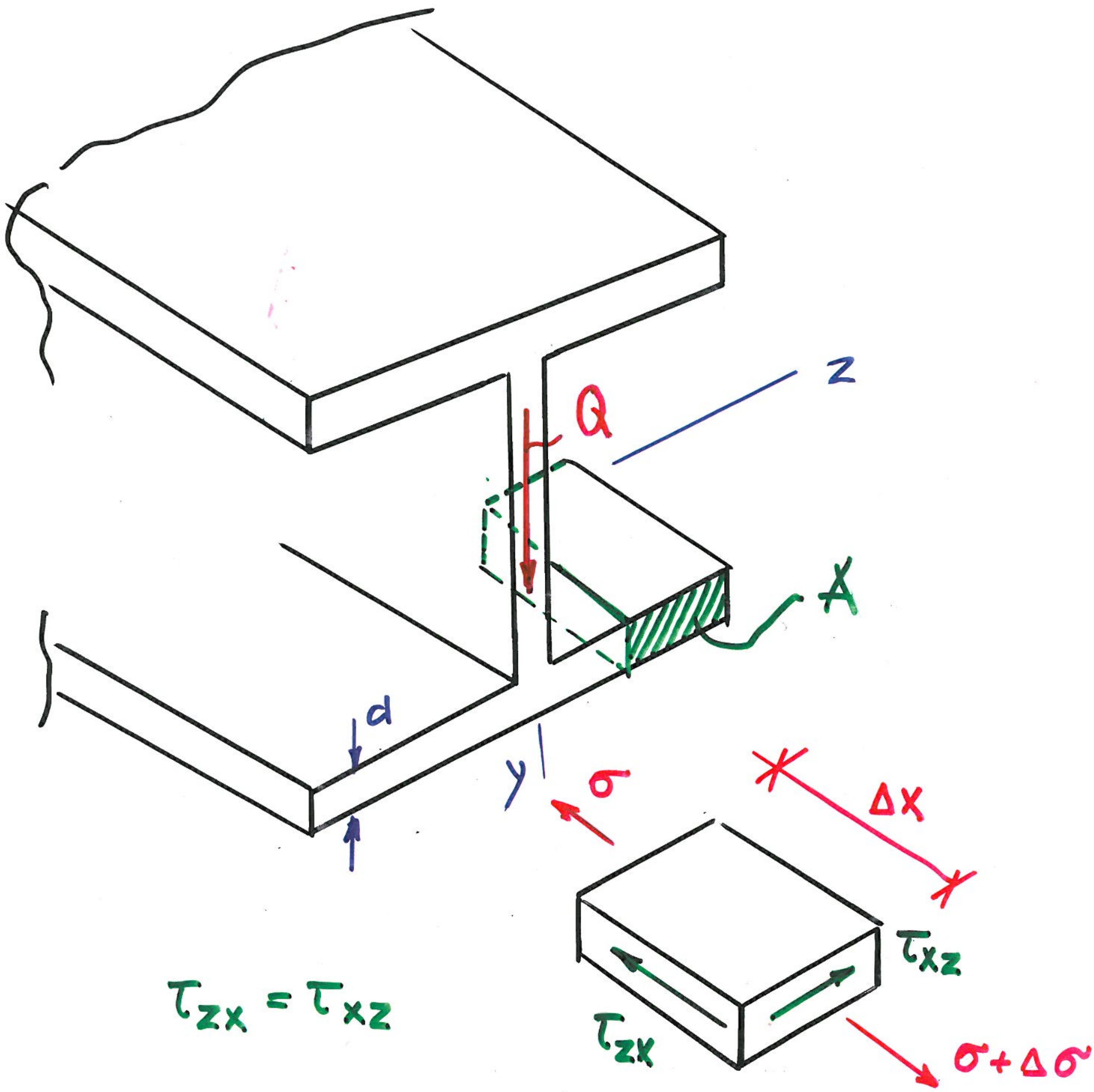
$$S = 100 \cdot 20 \cdot 60 = 0{,}12 \cdot 10^6\,mm^3$$

$$I = \frac{1}{12}\,100^3 \cdot 20 + 2\left(\frac{1}{12} \cdot 100 \cdot 20^3 + 100 \cdot 20 \cdot 60^2\right) = 16.2 \cdot 10^6\,mm^4$$

$$q = \tau t = \frac{QS}{I} = \frac{500\,N \cdot 0{,}12 \cdot \cancel{10^6}\,mm^3}{16{,}2 \cdot \cancel{10^6}\,mm^4} = 3{,}704\,\frac{N}{mm}$$

Voima yhdelle naulalle

$$F = 3.704\,\frac{N}{mm} \cdot 25\,mm = \underline{\underline{92.6\,N}}$$




$$\tau_{zx} = \tau_{xz}$$

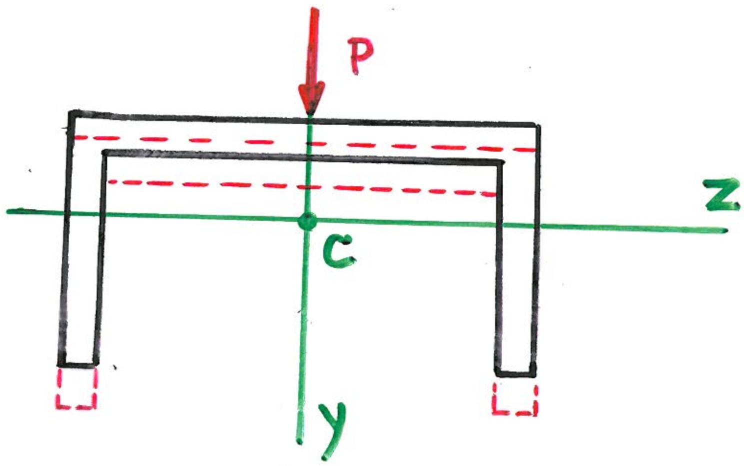
$$\tau = \frac{Q\,S_A}{I\,d}$$

$$\tau_{zx} \cdot d \cdot \Delta x = \int_A \Delta\sigma \, dA = \int_A \frac{\Delta M}{I}\, y \, dA$$

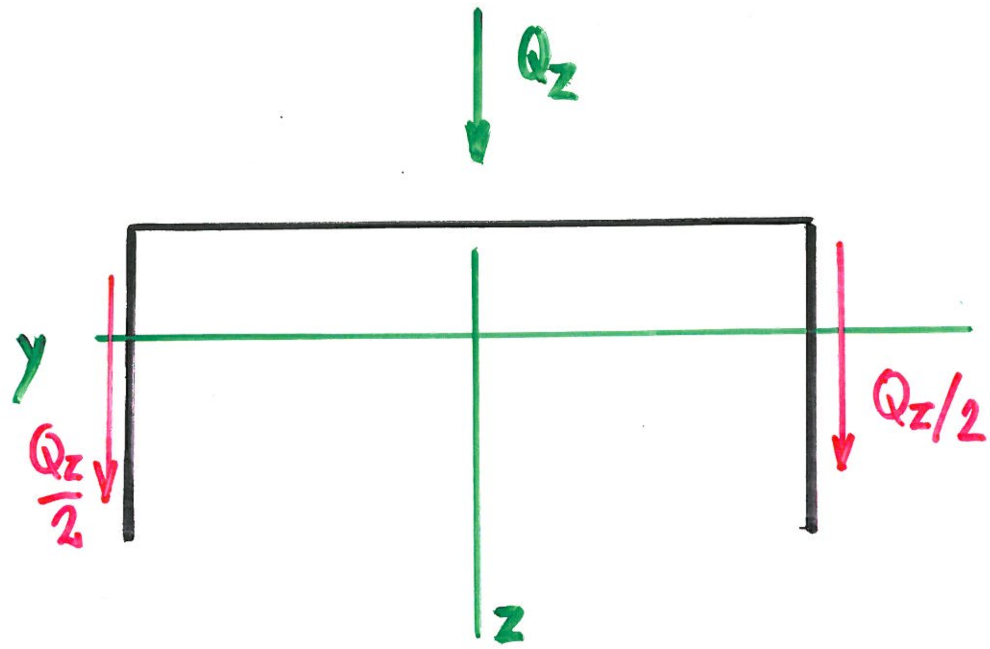
$$= \frac{\Delta M\, S_A}{I} \Rightarrow \underline{\underline{\tau_{zx} = \frac{Q\,S_A}{I\,d}}}$$

# OHUTSEINÄMÄISTEN SAUVOJEN EPÄSYMMETRINEN TAIVUTUS

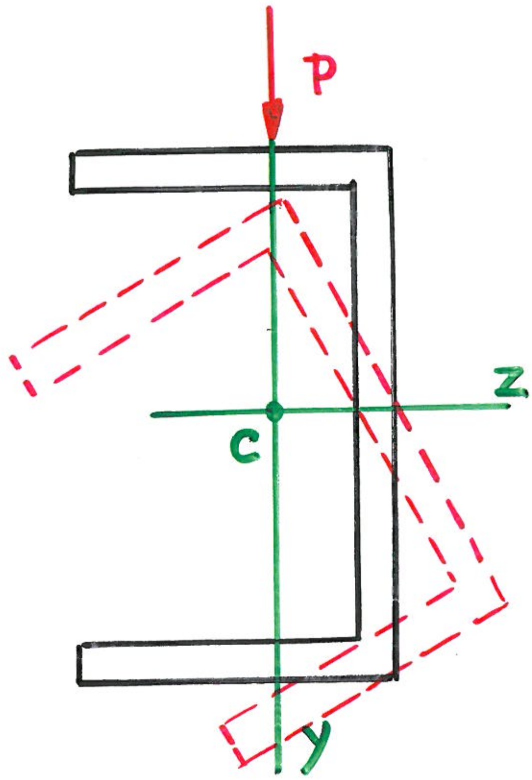
## LEIKKAUSKESKIÖ




Palkki taipuu y-koordinaattiakselin suunnassa




Vääntökeskiö sijaitsee symmetriatasossa.




Palkki taipuu ja kiertyy.

Miksi ?

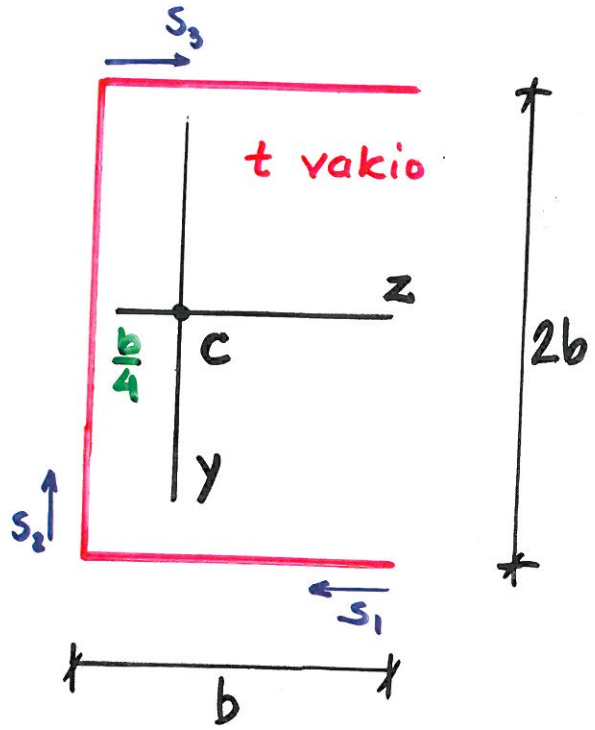
Missä kuorman tulisi sijaita, jotta kiertymää ei esiintyisi ?

Leikkauskeskiössä !

# Esimerkki:

$$I_z = \frac{1}{12}t(2b)^3 + 2tb^3$$
$$= \frac{8}{3}tb^3$$

$$I_y = \frac{5}{12}tb^3$$

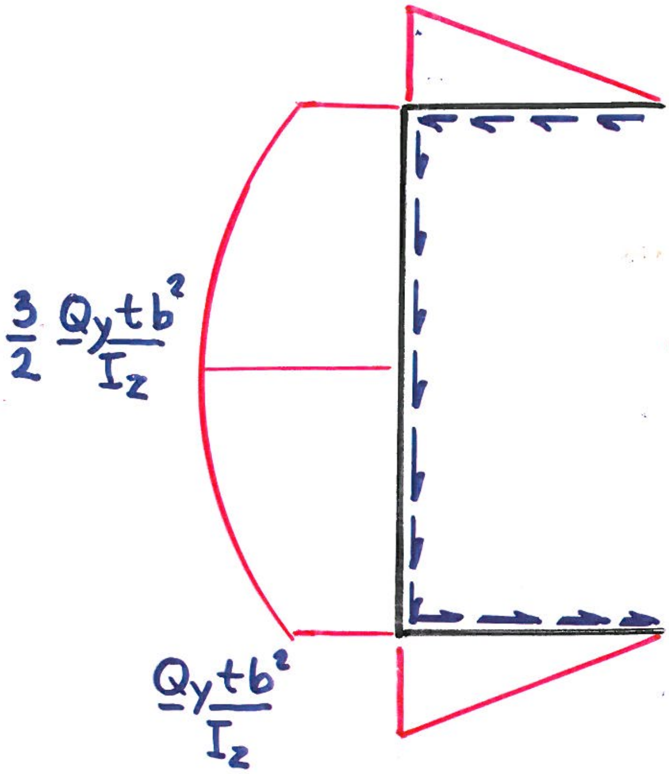



## Staattinen momentti:

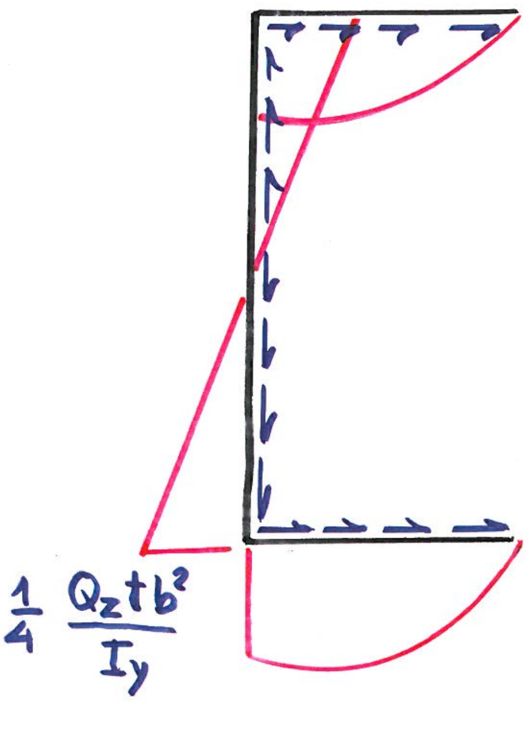
Alalaippa:
$$\begin{cases} S_z = s_1 t b \\ S_y = s_1 t\left(\frac{3}{4}b - \frac{s_1}{2}\right) \end{cases}$$

Uuma:
$$\begin{cases} S_z = tb^2 + s_2 t\left(b - \frac{s_2}{2}\right) \\ S_y = \frac{1}{4}tb^2 + s_2 t\left(-\frac{1}{4}b\right) \end{cases}$$

Ylälaippa:
$$\begin{cases} S_z = tb^2 + s_3 t(-b) \\ S_y = -\frac{1}{4}tb^2 + s_3 t\left(\frac{s_3}{2} - \frac{b}{4}\right) \end{cases}$$




$$\tau = \frac{Q_y S_z}{I_z t}$$




$$\tau = \frac{Q_z S_y}{I_y t}$$

Integroidaan voimaresultantit yli poikkileikkausalan osa kerrallaan.

## Ylälaippa

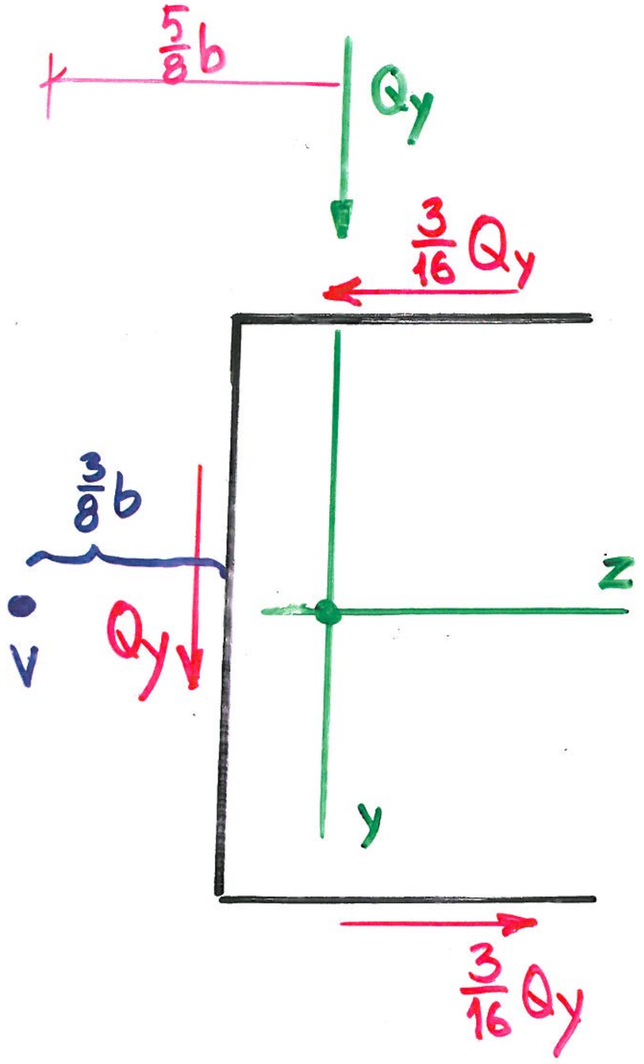
$$\begin{cases} R_1 = \int_0^b \tau t \, ds_3 = \dfrac{Q_y}{I_z} \int_0^b S_z(s_3) \, ds_3 = \dfrac{3}{16} Q_y \\ T_1 = \int_0^b \tau t \, ds_3 = \dfrac{Q_z}{I_y} \int_0^b S_y(s_3) \, ds_3 = \dfrac{1}{2} Q_z \end{cases}$$

## Uuma

$$\begin{cases} R_2 = \int_0^{2b} \tau t \, ds_2 = \dfrac{Q_y}{I_z} \int_0^{2b} S_z(s_2) \, ds_2 = Q_y \\ T_2 = \int_0^{2b} \tau t \, ds_2 = \dfrac{Q_z}{I_y} \int_0^{2b} S_y(s_2) \, ds_2 = 0 \end{cases}$$

## Alalaippa

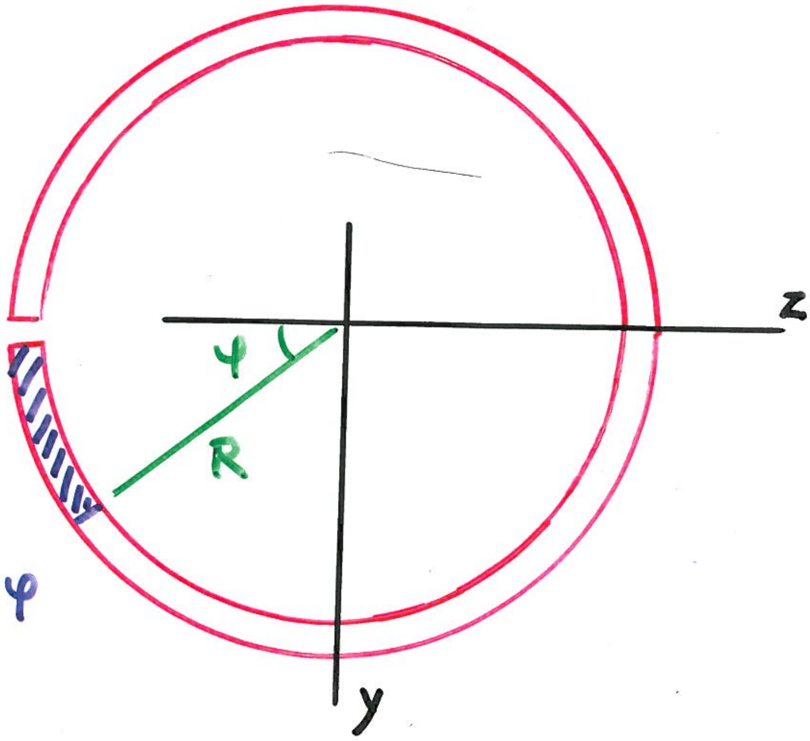
$$\begin{cases} R_3 = \int_0^b \tau t \, ds_1 = \dfrac{Q_y}{I_z} \int_0^b S_z(s_1) \, ds_1 = \dfrac{3}{16} Q_y \\ T_3 = \int_0^b \tau t \, ds_1 = \dfrac{Q_z}{I_y} \int_0^b S_y(s_1) \, ds_1 = \dfrac{1}{2} Q_z \end{cases}$$




$$\frac{3}{16}\cancel{Q_y}\cdot b + \frac{3}{16}\cancel{Q_y}\cdot b + \cancel{Q_y}\cdot e = 0$$

$$\underline{\underline{e = -\frac{3}{8}b}}$$

# Avoin ympyräpoikkileikkaus




$dA = Rt\,d\varphi$

$$\underline{S_z} = \int_{\Delta A} y\,dA = \int_0^{\varphi} R\sin\varphi\; Rt\,d\varphi =$$

$$= -\Big/_0^{\varphi} R^2 t\cos\varphi = \underline{\underline{R^2 t\,(1-\cos\varphi)}}$$

$$\Rightarrow \quad \tau = \frac{Q_y}{I_z}\cdot\frac{S_z}{t} = \frac{R^2 \cancel{t}\, Q_y}{I_z \cancel{t}}(1-\cos\varphi)$$

$$\tau_{max} = \frac{2R^2 Q_y}{I_z}$$

Tarkastetaan tasapainoehdot

$$I_z = \int_A y^2 dA = R^3 t \int_0^{2\pi} \sin^2\varphi \, d\varphi = \pi R^3 t$$

Voimatasapaino

$$Q_y = \int_A -\tau \cos\varphi \, dA = \frac{Q_y R^3 t}{I_z} \int_0^{2\pi} (-\cos\varphi + \cos^2\varphi) \, d\varphi$$

$$= \frac{Q_y R^3 t}{I_z} \Big/_0^{2\pi} \left(-\sin\varphi + \frac{\varphi}{2} + \frac{1}{4}\sin 2\varphi\right)$$

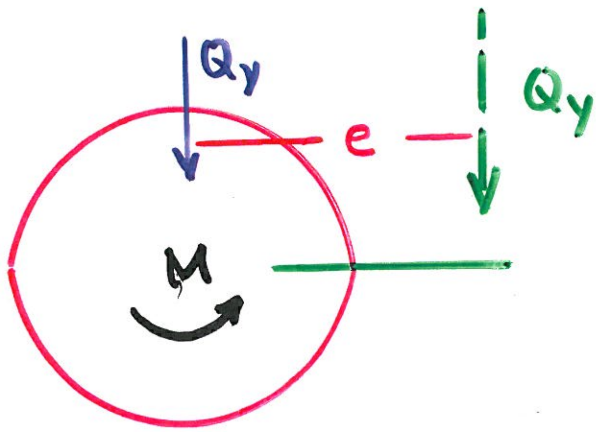
$$= \frac{Q_y \pi R^3 t}{I_z} = \underline{\underline{Q_y}} \quad \checkmark$$

Momenttitasapaino

$$M = \int_A \tau R \, dA = \frac{R^4 t Q_y}{I_z} \int_0^{2\pi} (1 - \cos\varphi) \, d\varphi$$

$$= \frac{R^4 t Q_y}{I_z} \Big/_0^{2\pi} \varphi - \sin\varphi = \frac{2\pi R^4 t Q_y}{I_z} = \underline{\underline{2 R Q_y}}$$

# Leikkauskeskiön paikan määrittäminen

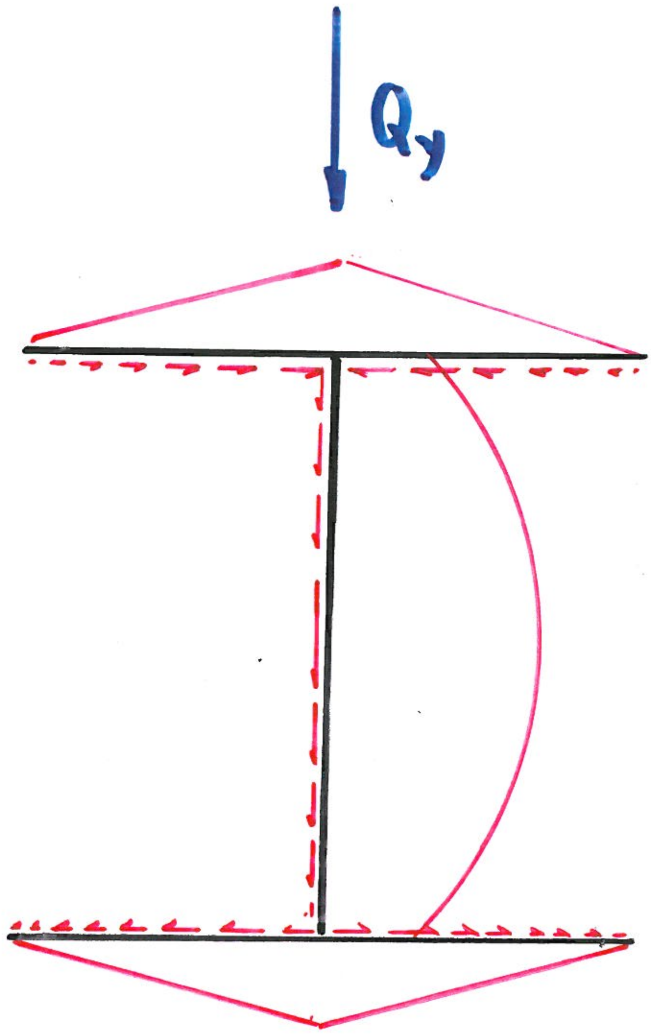


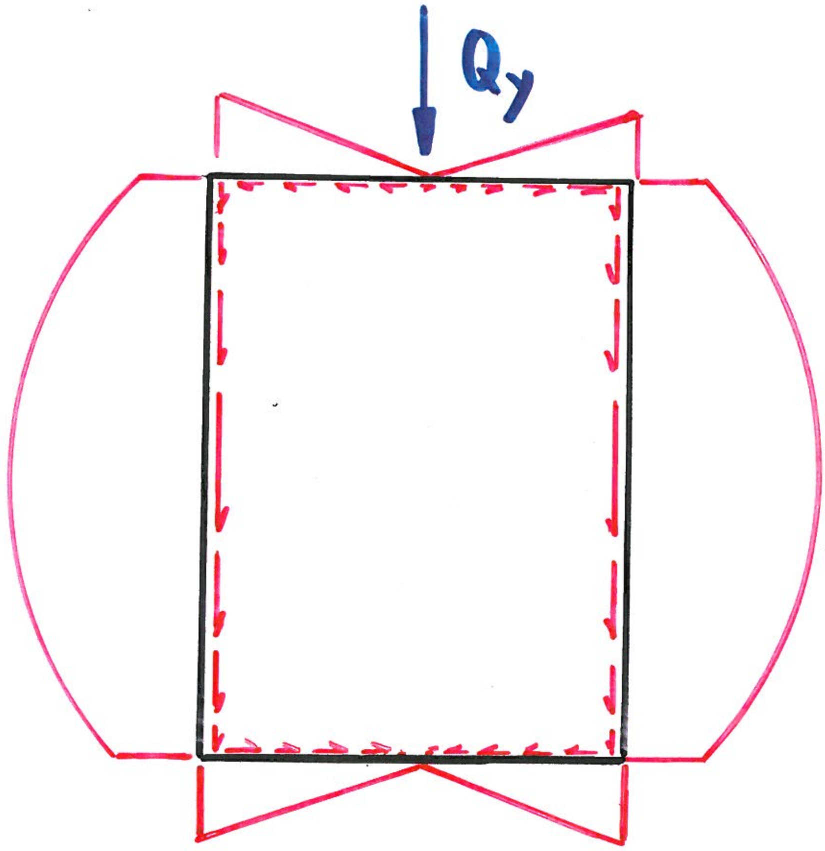

$$M = 2RQ_y$$

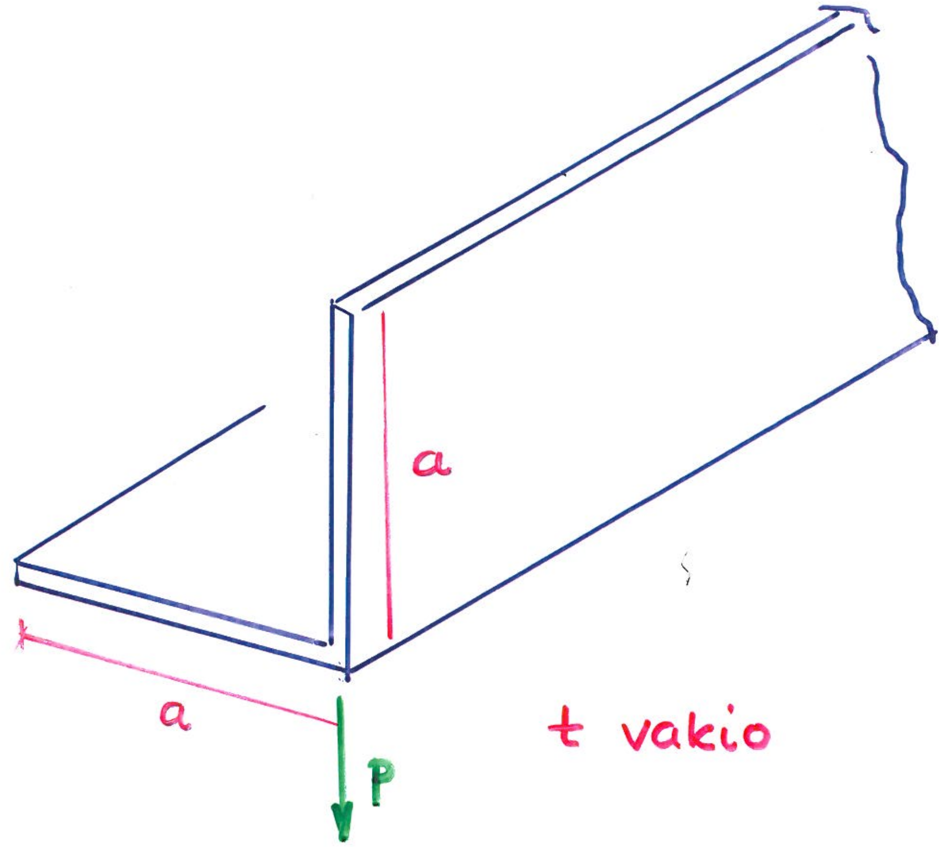
Ehto leikkauskeskiölle

$$0 = M - Q_y \cdot e \iff 2RQ_y - Q_y e = 0$$

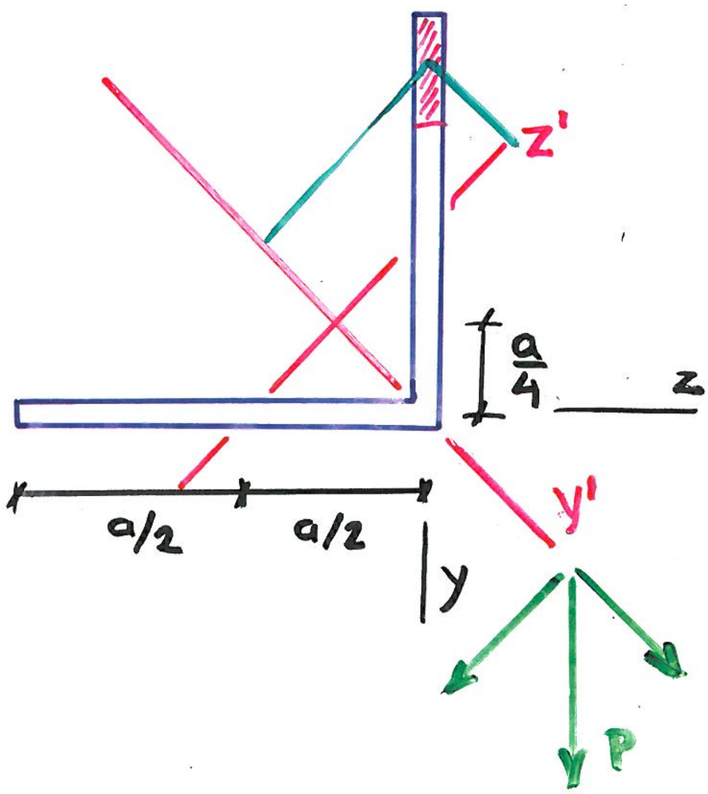
$$\Rightarrow \quad \underline{\underline{e = 2R}}$$


Qy


Qy




Pääjäyhyyssuunnat (symmetria)



$$I_{y'} = 2\left[\frac{1}{3}\left(\frac{t}{\cos \pi/4}\right)\left(a \cdot \cos\frac{\pi}{4}\right)^3\right] = \frac{1}{3} t a^3$$

$$I_{z'} = 2\left[\frac{1}{12}\left(\frac{t}{\cos \pi/4}\right)\left(a \cdot \cos\frac{\pi}{4}\right)^3\right] = \frac{1}{12} t a^3$$

Kuormana $V_{y'}$

$$S_{z'} = t(a+y)\left(\frac{a}{2} - \left(\frac{a}{2} + \frac{y}{2}\right)\right)\cos\frac{\pi}{4}$$

$$= \underline{-\frac{1}{2} t y (a+y) \cos\frac{\pi}{4}}$$

$$\Rightarrow \tau_1 = \frac{V_{y'} S_{z'}}{I_{z'} t} = \frac{P\cos\frac{\pi}{4}\left[-\frac{1}{2} t y (a+y)\cos\frac{\pi}{4}\right]}{\frac{1}{12} t^2 a^3}$$

$$= \frac{-3Py(a+y)}{a^3 t}$$

Kuormana $V_{z'}$

$$S_{y'} = t(a+y)\left(a - \left(\frac{a}{2} + \frac{y}{2}\right)\right)\cos\frac{\pi}{4}$$

$$= \underline{\frac{1}{2}(a^2 - y^2)\cos\frac{\pi}{4} \cdot t}$$

$\Rightarrow$

$$\tau_2 = \frac{V_{z'} S_{y'}}{I_{y'} t} = \frac{P\cos\frac{\pi}{4} \cdot t\frac{1}{2}(a^2-y^2)\cos\frac{\pi}{4}}{\frac{1}{3}t^2a^3}$$

$$= \frac{3P(a^2-y^2)}{4a^3t}$$

Pystyasennossa olevassa laipassa leikkausjännitys on

$$\underline{\underline{\tau = \tau_1 + \tau_2}}$$

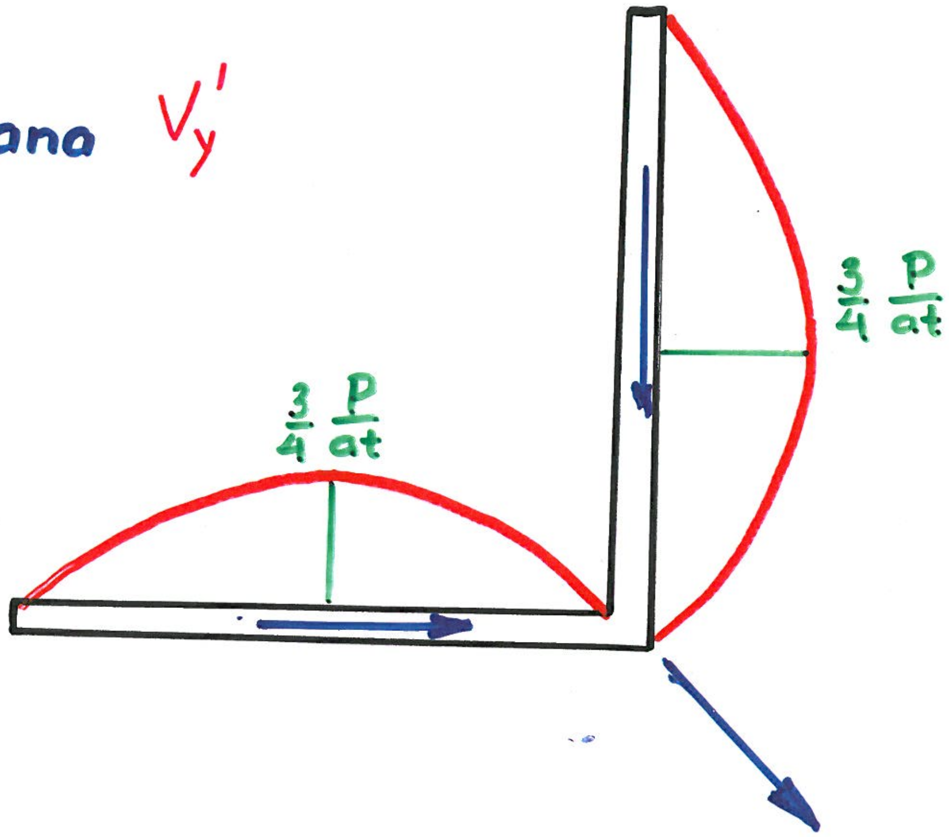
Vaaka-asennossa olevassa laipassa

$$\underline{\underline{\tau = \tau_2 - \tau_1}}$$

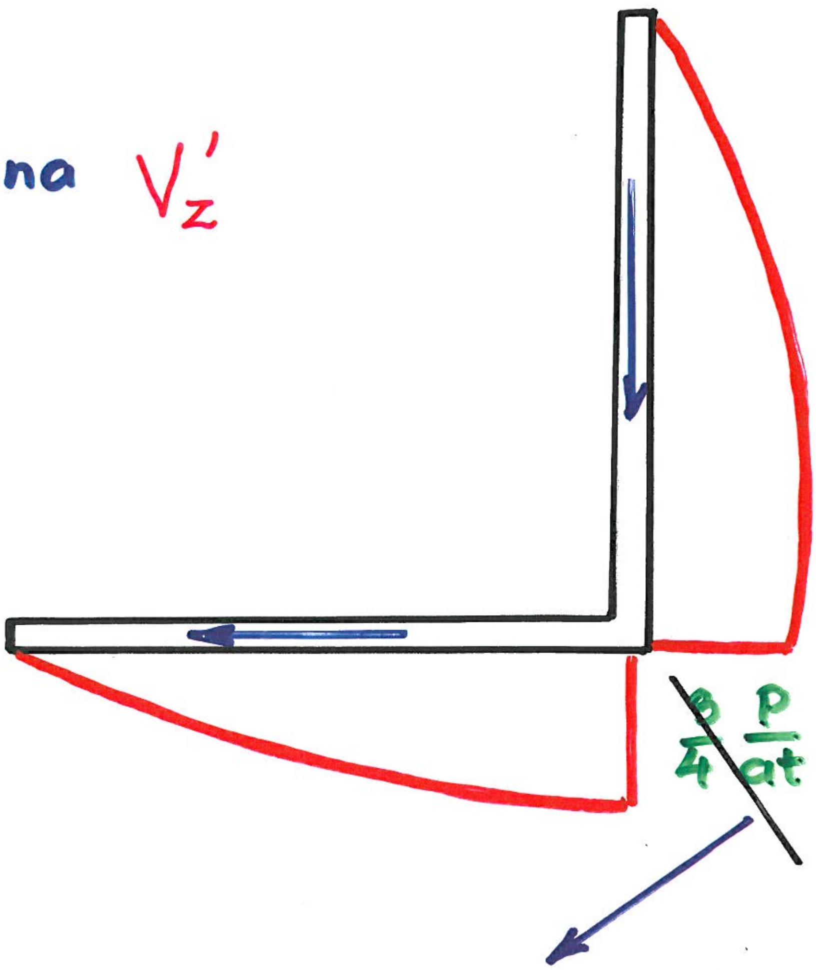
$$\tau = \frac{3P}{a^3t}\left[\frac{1}{4}(a^2-y^2) \mp y(a+y)\right]$$

$$= \underline{\underline{\frac{3P}{a^3t}(a+y)\left(\frac{1}{4}(a-y) \mp y\right)}}$$

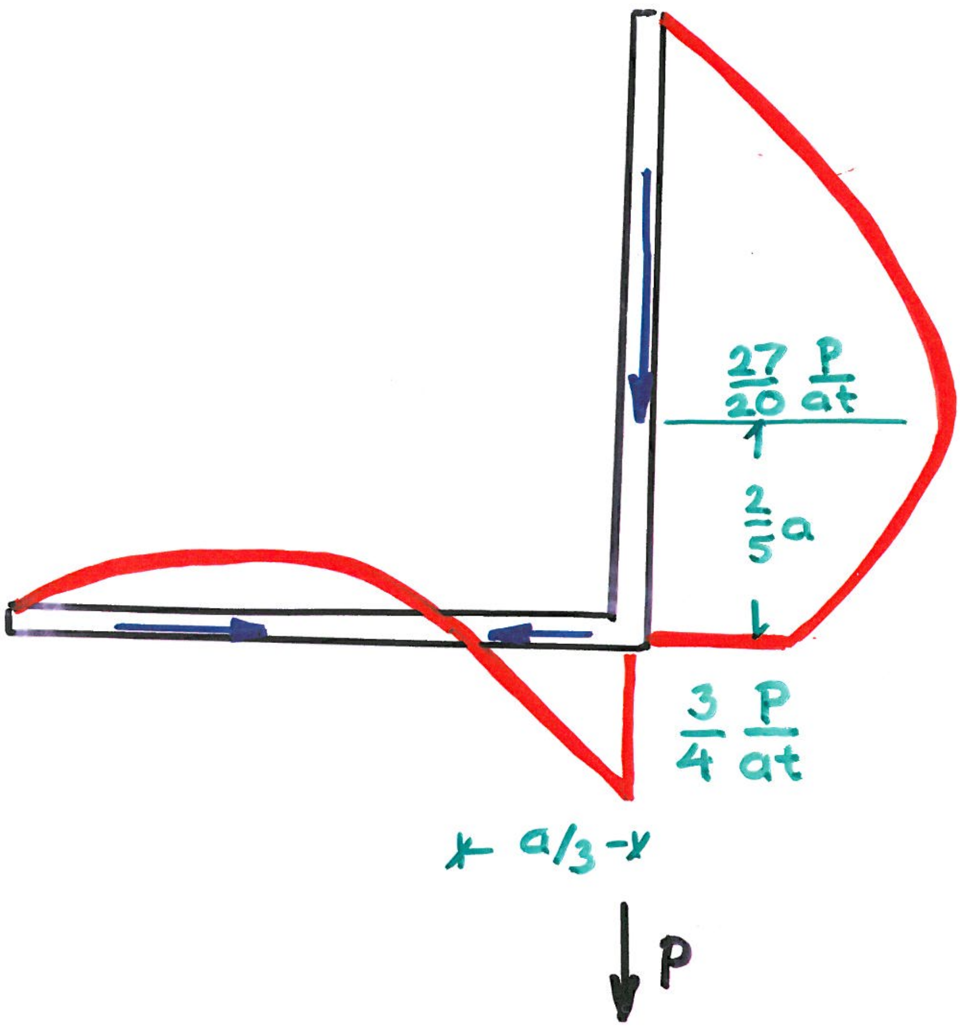
Kuormana $V_y'$



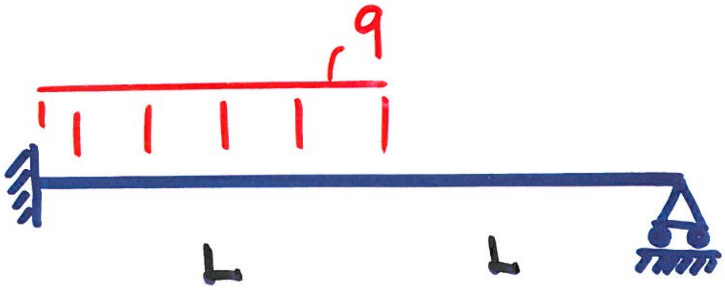

Kuormana $V_z'$



Leikkausjännitysjakauma:



## Esimerkki




Kuormitus $q(x) = q\,(1 - \langle x - L\rangle^0)$

Diffyhtälö $EIv'''' = q(x)$

Ratkaisu $v(x) = C_1 x^3 + C_2 x^2 + C_3 x + C_4 + \frac{q}{EI}\left(\frac{x^4}{24} - \frac{\langle x-L\rangle^4}{24}\right)$

Reunaehdot $v(0) = 0 \Rightarrow C_4 = 0$

$v'(0) = 0 \Rightarrow C_3 = 0$

$v(2L) = 0 \Rightarrow 8L^3 C_1 + 4L^2 C_2 + \frac{q}{24EI}(16L^4 - L^4) = 0$

$v''(2L) = 0 \Rightarrow 12LC_1 + 2C_2 + \frac{q}{2EI}(4L^2 - L^2) = 0$

$\Rightarrow \quad C_1 = -\frac{19}{128}\frac{qL}{EI}$

$C_2 = \frac{18}{128}\frac{qL^2}{EI}$

$\Rightarrow$

$$v(x) = \left(-\frac{9}{16}x^3L + \frac{9}{8}x^2L^2 + \frac{x^4}{3} - \frac{\langle x-L\rangle^4}{3}\right)\frac{q}{8EI}$$

$$v'(x) = \left(-\frac{27}{16}x^2L + \frac{9}{4}xL^2 + \frac{4}{3}x^3 - \frac{4}{3}\langle x-L\rangle^3\right)\frac{q}{8EI}$$

$$M(x) = \left(\frac{27}{8}xL - \frac{9}{4}L^2 - 4x^2 + 4\langle x-L\rangle^2\right)\frac{q}{8}$$

$$Q(x) = \left(\frac{27}{8}L - 8x + 8\langle x-L\rangle\right)\frac{q}{8}$$

$$\Rightarrow \left.\begin{aligned} Q(0) &= \frac{27}{64}qL \\ Q(2L) &= -\frac{37}{64}qL \end{aligned}\right\} Q(0)-Q(2L) = qL$$

%.