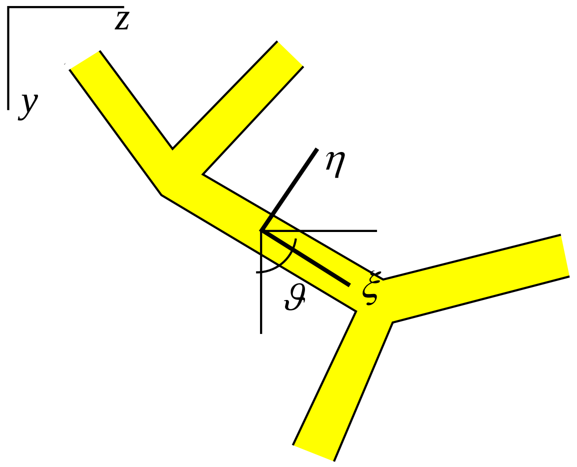
Slides by prof J. Paavola, 2018

Vlasov’s theory for thin-walled beams

Fundamental ideas

Kinematics



Rigid body rotation

$$\mathbf{u} = \boldsymbol{\theta} \times (\mathbf{r} - \hat{\mathbf{r}}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \theta_s & 0 & 0 \\ 0 & y - \hat{y} & z - \hat{z} \end{vmatrix} = -\left(z - \hat{z}\right)\theta_x \mathbf{j} + \left(y - \hat{y}\right)\theta_x \mathbf{k}$$

+ warping

Warping of the cross-section

$$\mathbf{u}(x, y, z) = \omega(y, z)\phi(x)\mathbf{i} - \left(z - \hat{z}\right)\theta_x(x)\mathbf{j} + \left(y - \hat{y}\right)\theta_x(x)\mathbf{k}$$

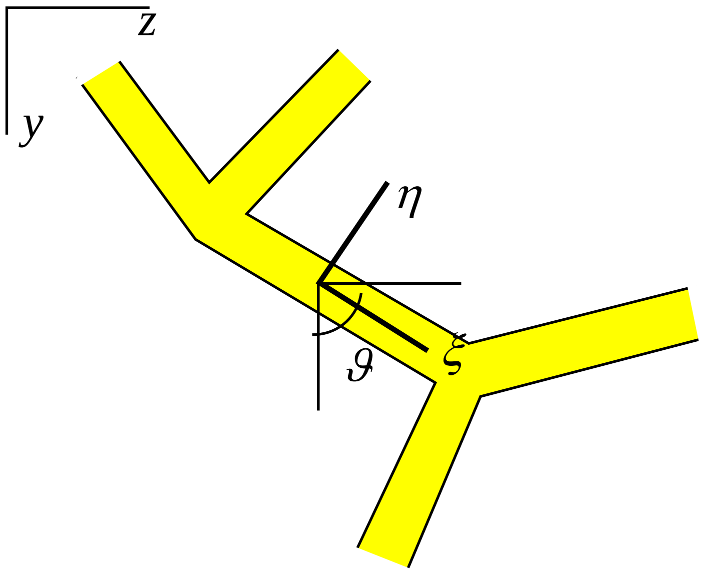
$(\hat{y}, \hat{z})$ shear centre

Global strain components

$$\varepsilon_x = \frac{\partial \mathbf{u}}{\partial x} \bullet \mathbf{i} = \omega \frac{\mathrm{d}\phi}{\mathrm{d}x}$$

$$\gamma_{yx} = \frac{\partial \mathbf{u}}{\partial x} \bullet \mathbf{j} + \frac{\partial \mathbf{u}}{\partial y} \bullet \mathbf{i} = -(z - \hat{z}) \frac{\mathrm{d}\theta_x}{\mathrm{d}x} + \frac{\partial \omega}{\partial y} \phi$$

$$\gamma_{zs} = \frac{\partial \mathbf{u}}{\partial x} \bullet \mathbf{k} + \frac{\partial \mathbf{u}}{\partial z} \bullet \mathbf{i} = (y - \hat{y}) \frac{\mathrm{d}\theta_x}{\mathrm{d}x} + \frac{\partial \omega}{\partial z} \phi$$



Coordinate transformation

$$y = y_\circ + \xi \cos \vartheta - \eta \sin \vartheta$$

$$z = z_\circ + \xi \sin \vartheta + \eta \cos \vartheta$$

$$\mathbf{e}_\xi = \cos \vartheta \mathbf{j} + \sin \vartheta \mathbf{k}$$

$$\mathbf{e}_\eta = -\sin \vartheta \mathbf{j} + \cos \vartheta \mathbf{k}$$

The chain rule for differentiation

$$\frac{\partial}{\partial \xi} = \frac{\partial}{\partial y}\frac{\partial y}{\partial \xi} + \frac{\partial}{\partial z}\frac{\partial z}{\partial \eta} = \cos\vartheta \frac{\partial}{\partial y} + \sin\vartheta \frac{\partial}{\partial z}$$

$$\frac{\partial}{\partial \eta} = \frac{\partial}{\partial y}\frac{\partial y}{\partial \eta} + \frac{\partial}{\partial z}\frac{\partial z}{\partial \eta} = -\sin\vartheta \frac{\partial}{\partial y} + \cos\vartheta \frac{\partial}{\partial z}$$

Local shear strain components

$$\gamma_{\xi x} = \frac{\partial \mathbf{u}}{\partial x} \bullet \mathbf{e}_{\xi} + \frac{\partial \mathbf{u}}{\partial \xi} \bullet \mathbf{i}$$

$$\gamma_{\eta s} = \frac{\partial \mathbf{u}}{\partial x} \bullet \mathbf{e}_{\eta} + \frac{\partial \mathbf{u}}{\partial \eta} \bullet \mathbf{i}$$

$$\gamma_{\xi x} = \frac{\partial \mathbf{u}}{\partial \xi} \cdot \mathbf{i} + \frac{\partial \mathbf{u}}{\partial x} \cdot \mathbf{e}_\xi = \left(\frac{\partial \mathbf{u}}{\partial y} \frac{\partial y}{\partial \xi} + \frac{\partial \mathbf{u}}{\partial z} \frac{\partial z}{\partial \xi} \right) \cdot \mathbf{i} + \frac{\partial \mathbf{u}}{\partial x} \cdot (\cos \vartheta \mathbf{j} + \sin \vartheta \mathbf{k})$$

$$= \left(\frac{\partial \mathbf{u}}{\partial y} \cos \vartheta + \frac{\partial \mathbf{u}}{\partial z} \sin \vartheta \right) \cdot \mathbf{i} + \frac{\partial \mathbf{u}}{\partial x} \cdot (\cos \vartheta \mathbf{j} + \sin \vartheta \mathbf{k}) =$$

$$= \left(\frac{\partial \mathbf{u}}{\partial y} \cdot \mathbf{i} + \frac{\partial \mathbf{u}}{\partial x} \cdot \mathbf{j} \right) \cos \vartheta + \left(\frac{\partial \mathbf{u}}{\partial z} \cdot \mathbf{i} + \frac{\partial \mathbf{u}}{\partial x} \cdot \mathbf{k} \right) \sin \vartheta = \gamma_{yx} \cos \vartheta + \gamma_{zx} \sin \vartheta$$

and correspondingly

$$\gamma_{\eta x} = -\left(\frac{\partial \mathbf{u}}{\partial y} \cdot \mathbf{i} + \frac{\partial \mathbf{u}}{\partial x} \cdot \mathbf{j}\right) \sin \vartheta + \left(\frac{\partial \mathbf{u}}{\partial z} \cdot \mathbf{i} + \frac{\partial \mathbf{u}}{\partial x} \cdot \mathbf{k}\right) \cos \vartheta$$

$$= -\gamma_{yx} \sin \vartheta + \gamma_{zx} \cos \vartheta$$

By inserting we get the kinematics

$$\gamma_{\xi x} = \frac{\partial \omega}{\partial \xi}\phi - \left[(z_{\circ} - \hat{z})\cos\vartheta - (y_{\circ} - \hat{y})\sin\vartheta + \eta\right]\frac{\mathrm{d}\theta_x}{\mathrm{d}x}$$

$$\gamma_{\eta x} = \frac{\partial \omega}{\partial \eta}\phi + \left[(z_{\circ} - \hat{z})\sin\vartheta + (y_{\circ} - \hat{y})\cos\vartheta + \xi\right]\frac{\mathrm{d}\theta_x}{\mathrm{d}x}$$

Vlasov's assumptions

$$\gamma_{\eta x}(\xi,\eta) \equiv 0 \quad \Rightarrow \quad \phi(x) = -\frac{\mathrm{d}\theta_x}{\mathrm{d}x}(x)$$

$$\frac{\partial \omega}{\partial \eta} = (z_{\circ} - \hat{z})\sin\vartheta + (y_{\circ} - \hat{y})\cos\vartheta + \xi$$

$$\omega = \left[(z_{\circ} - \hat{z})\sin\vartheta + (y_{\circ} - \hat{y})\cos\vartheta\right]\eta + \xi\eta + g(\xi)$$

$$\gamma_{\xi x}(\xi,0) = 0 \qquad g(\xi) = \left[-(z_{\circ} - \hat{z})\cos\vartheta + (y_{\circ} - \hat{y})\sin\vartheta\right]\xi + u_{\circ}$$

$$\omega = (y - \hat{y})(z - z_o) - (z - \hat{z})(y - y_o) + \eta\left[(y - y_o)\cos\vartheta + (z - z_o)\sin\vartheta\right] + u_o$$

This will often be given in the form $(\eta = 0)$

$$\mathrm{d}\tilde{\omega} = -(z_o - \hat{z})\mathrm{d}y + (y_o - \hat{y})\mathrm{d}z$$

$$\mathrm{d}\tilde{\boldsymbol{\omega}} = \pm h_S(\boldsymbol{\xi})\mathrm{d}\boldsymbol{\xi}$$