Taloustieteen matemaattiset menetelmät
31C01100
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Problem Set 3: Solutions

1. **Solution**

(a) 1. $G(x,y) = y^3 + y - x^3 = 0$, $(x_0, y_0) = (0,1)$. The implicit function theorem cannot be applied at $(x_0, y_0) = (0,1)$, because $G(x_0, y_0) = 2 \neq 0$.

2. $G(x,y) = x^2 + y + ln(xy) = 0$, $(x_0, y_0) = (1,1)$. The implicit function theorem cannot be applied a $(x_0, y_0) = (1,1)$, because $G(x_0, y_0) = 2 \neq 0$.

(b) 1. $G(x,y,z) = x^3+y^3+z^3-xyz-1 = 0$, $(x_0, y_0, z_0) = (0,0,1)$. $G(x_0, y_0, z_0) = 0$ and $\frac{\partial G}{\partial z}(0,0,1) = 3 \neq 0$, so z can be defined implicitly as a function of $g(x,y)$ in a neighbourhood of the given point (x_0, y_0, z_0). Thus

$$\frac{\partial g}{\partial x}(x_0, y_0) = -\frac{\frac{\partial G}{\partial x}(0,0,1)}{\frac{\partial G}{\partial z}(0,0,1)} = 0$$

and

$$\frac{\partial g}{\partial y}(x_0, y_0) = -\frac{\frac{\partial G}{\partial y}(0,0,1)}{\frac{\partial G}{\partial z}(0,0,1)} = 0.$$

2. $G(x,y,z) = e^z - z^2 - x^2 - y^2 = 0$, $(x_0, y_0, z_0) = (1,0,0)$. $G(x_0, y_0, z_0) = 0$ and $\frac{\partial G}{\partial z}(1,0,0) = 1 \neq 0$, so z can be defined implicitly as a function of $g(x,y)$ in a neighbourhood of the given point (x_0, y_0, z_0). Thus

$$\frac{\partial g}{\partial x}(x_0, y_0) = -\frac{\frac{\partial G}{\partial x}(1,0,0)}{\frac{\partial G}{\partial z}(1,0,0)} = 2$$

and

$$\frac{\partial g}{\partial y}(x_0, y_0) = -\frac{\frac{\partial G}{\partial y}(1,0,0)}{\frac{\partial G}{\partial z}(1,0,0)} = 0.$$

2. **Solution**

(a) We want to approximate the value of the function $f(x_1, x_2, x_3) = 10x_1^{1/3}x_2^{1/2}x_3^{1/6}$ around the point $x = (27, 16, 64)$.

1. First, calculate the value of the function at the given point $x = (27, 16, 64)$.

$$f(27, 16, 64) = 10 * 27^{1/3} * 16^{1/2} * 4^{1/6} = 240$$

Next, use total differential to approximate how much f changes around the given point, when x_1 increases to 27.1, x_2 decreases to 15.7, and x_3 remains the same (i.e. $dx_1 = 0.1$, $dx_2 = -0.3$, and $dx_3 = 0$).

$$f + df \approx 240 + \frac{\partial f(x_1, x_2, x_3)}{\partial x_1}dx_1 + \frac{\partial f(x_1, x_2, x_3)}{\partial x_2}dx_2 + \frac{\partial f(x_1, x_2, x_3)}{\partial x_3}dx_3$$

$$= 240 + \frac{10}{3} * \frac{x_2^{1/2}x_3^{1/6}}{x_1^{2/3}}dx_1 + 5 * \frac{x_1^{1/3}x_3^{1/6}}{x_2^{1/2}}dx_2 + \frac{5}{3} * \frac{x_1^{1/3}x_2^{1/2}}{x_3^{5/6}}dx_3$$

$$= 240 + \frac{10}{3} * \frac{16^{1/2}64^{1/6}}{27^{2/3}} * 0,1 + 5 * \frac{27^{1/3}64^{1/6}}{16^{1/2}} * (-0.3)$$

$$\approx 238.046$$

2. $f(27.1, 15.7, 64) = 10 * 27.1^{1/3} * 15.7^{1/2} * 64^{1/6} \approx 238.032$

3. Now $dx_1 = dx_2 = 0.2$ and $dx_3 = -0.4$, so x_1 increases to 27.2, x_2 increases to 16.2, and x_3 decreases to 63.6. The approximated value is now $f(27, 16, 64) + df \approx 241.843$ and the actual value of the function is $f(27.2, 16.2, 63.6) \approx 241.837$.

(b) Both x and y increase by 1%, so $dx = 2 * 0.01 = 0.02$ and $dy = 3 * 0.01 = 0.03$. Thus

$$df = \frac{\partial f}{\partial x}(2, 3)dx + \frac{\partial f}{\partial y}(2, 3)dy = 4 \times 0.02 + 2 \times 0.03 = 0.14.$$

3. **Solution**

(a)

$$MRS^v(x_0, y_0) = -\frac{\frac{\partial v}{\partial x}(x_0, y_0)}{\frac{\partial v}{\partial y}(x_0, y_0)} = -\frac{\frac{\partial f}{\partial x}(u(x_0, y_0))}{\frac{\partial f}{\partial y}(u(x_0, y_0))}$$

(b)

$$MRS^v(x_0, y_0) = -\frac{\frac{\partial f}{\partial x}(u(x_0, y_0))}{\frac{\partial f}{\partial y}(u(x_0, y_0))} = -\frac{\frac{\partial f}{\partial u}(x_0, y_0)\frac{\partial u}{\partial x}(x_0, y_0)}{\frac{\partial f}{\partial u}(x_0, y_0)\frac{\partial u}{\partial y}(x_0, y_0)}$$

$$= -\frac{\frac{\partial u}{\partial x}(x_0, y_0)}{\frac{\partial u}{\partial y}(x_0, y_0)} = MRS^u(x_0, y_0),$$

(c) For every $t > 0$,

$$MRS(tx_0, ty_0) = \frac{\frac{\partial u}{\partial x}(tx_0, ty_0)}{\frac{\partial u}{\partial y}(tx_0, ty_0)} = \frac{t^{k-1}\frac{\partial u}{\partial x}(x_0, y_0)}{t^{k-1}\frac{\partial u}{\partial y}(x_0, y_0)} = MRS(x_0, y_0),$$

where the second equality follows directly from the proposition at p. 17 of the slides from Lecture 7.

4. **Solution**

(a) $f(x, y, z) = x^2 + 5y^2 + 4xy - 2yz$, so the Hessian matrix is

$$H = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial z \partial x} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} & \frac{\partial^2 f}{\partial z \partial y} \\ \frac{\partial^2 f}{\partial x \partial z} & \frac{\partial^2 f}{\partial y \partial z} & \frac{\partial^2 f}{\partial z^2} \end{pmatrix} = \begin{pmatrix} 2 & 4 & 0 \\ 4 & 10 & -2 \\ 0 & -2 & 0 \end{pmatrix}.$$

(b) $g(x, y, z) = 100 - 2x^2 - y^2 - 3z - xy - e^{x+y+z}$. Define $u := x + y + z$. The Hessian matrix is

$$H = \begin{pmatrix} -4 - e^u & -1 - e^u & -e^u \\ -1 - e^u & -2 - e^u & -e^u \\ -e^u & -e^u & -e^u \end{pmatrix}.$$

5. **Solution**

Let $t > 1$ be an arbitrary scale factor. For each production function, the following is true.

(a) $f(x_1, x_2) = ax_1 + bx_2$

$$\begin{aligned} f(tx_1, tx_2) &= atx_1 + btx_2 \\ &= t(ax_1 + bx_2) \\ &= t\,f(x_1, x_2)\,. \end{aligned}$$

Returns to scale are constant.

(b) $f(x_1, x_2) = ax_1^c + bx_2^c$

$$\begin{aligned} f(tx_1, tx_2) &= at^c x_1^c + bt^c x_2^c \\ &= t^c(ax_1^c + bx_2^c) \\ &= t^c\,f(x_1, x_2)\,. \end{aligned}$$

Returns to scale are constant if $c = 1$, increasing if $c > 1$, and decreasing if $c < 1$.

(c) $f(x_1, x_2) = \min\{ax_1, bx_2\}$

$$\begin{aligned} f(tx_1, tx_2) &= \min\{atx_1, btx_2\} \\ &= t\,\min\{ax_1, bx_2\} \\ &= t\,f(x_1, x_2)\,. \end{aligned}$$

Returns to scale are constant.

(d) $f(x_1, x_2) = \max\{ax_1, bx_2\}$

$$\begin{aligned} f(tx_1, tx_2) &= \max\{atx_1, btx_2\} \\ &= t\,\max\{ax_1, bx_2\} \\ &= t\,f(x_1, x_2)\,. \end{aligned}$$

Returns to scale are constant.

(e) $f(x_1, x_2) = x_1^a x_2^b$

$$\begin{aligned} f(tx_1, tx_2) &= t^a x_1^a t^b x_2^b \\ &= t^{a+b} x_1^a x_2^b \\ &= t^{a+b}\,f(x_1, x_2)\,. \end{aligned}$$

Returns to scale are constant if $(a + b) = 1$, increasing if $(a + b) > 1$, and decreasing if $(a + b) < 1$.

(f) $f(x_1, x_2) = \frac{1}{\frac{1}{x_1} + \frac{1}{x_2}}$

$$\begin{aligned} f(tx_1, tx_2) &= \frac{1}{\frac{1}{tx_1} + \frac{1}{tx_2}} \\ &= \frac{1}{\frac{1}{t}\left(\frac{1}{x_1} + \frac{1}{x_2}\right)} \\ &= t\,\frac{1}{\frac{1}{x_1} + \frac{1}{x_2}} \\ &= t\,f(x_1, x_2). \end{aligned}$$

Returns to scale are constant.