ELEC-E7130
Examples on parameter estimation

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1 Discrete uniform distribution $U^{\mathrm{d}}(1, N)$, one unknown parameter N

Let $U^{\mathrm{d}}(1, N)$ denote the discrete uniform distribution, for which $N \in \{1, 2, \ldots\}$ and the point probabilities are given by

$$p(i) = \mathrm{P}\{X = i\} = \frac{1}{N}, \quad i \in \{1, \ldots, N\}.$$

It follows that

$$\begin{aligned}
\mathrm{E}(X) &= \sum_{i=1}^{N} \mathrm{P}\{X = i\} i = \frac{1}{N} \sum_{i=1}^{N} i = \frac{1}{N} \frac{N(N+1)}{2} = \frac{N+1}{2}, \\
\mathrm{E}(X^2) &= \sum_{i=1}^{N} \mathrm{P}\{X = i\} i^2 = \frac{1}{N} \sum_{i=1}^{N} i^2 = \frac{1}{N} \frac{N(N+1)(2N+1)}{6} \\
&= \frac{(N+1)(2N+1)}{6}, \\
\mathrm{V}(X) &= \mathrm{E}(X^2) - (\mathrm{E}(X))^2 = \frac{(N+1)(2N+1)}{6} - \left(\frac{N+1}{2}\right)^2 \\
&= \frac{(N+1)(N-1)}{12} = \frac{N^2 - 1}{12}.
\end{aligned}$$

1.1 Estimation of N: Method of Moments (MoM)

Consider an IID sample $(x_1, \ldots, x_n)$ of size n from distribution $U^{\mathrm{d}}(1, N)$. The first (theoretical) moment equals

$$\mu_1 = \mathrm{E}(X) = \frac{N+1}{2},$$

and the corresponding sample moment is the sample mean $\bar{x}_n$:

$$\hat{\mu}_1 = \bar{x}_n = \frac{1}{n}\sum_{i=1}^{n} x_i.$$

Let $\hat{N}$ denote the estimator of the unknown parameter N.

MoM estimation: By solving the requirement (for the first moment) that

$$\hat{\mu}_1 = \frac{\hat{N}+1}{2},$$

we get the estimator

$$\hat{N}^{\mathrm{MoM}} = 2\hat{\mu}_1 - 1 = 2\bar{x}_n - 1. \tag{1}$$

1.2 Estimation of N: Maximum Likelihood (ML)

Consider again an IID sample $(x_1, \ldots, x_n)$ of size n from distribution $U^{\mathrm{d}}(1, N)$. Let m_n denote the maximum value of this sample,

$$m_n = \max\{x_1, \ldots, x_n\}.$$

The likelihood function for this discrete distribution with unknown parameter N equals

$$\begin{aligned} L(x_1, \ldots, x_n; N) &= P\{X_1 = x_1, \ldots, X_n = x_n; N\} \\ &= P\{X_1 = x_1; N\} \cdots P\{X_n = x_n; N\} \\ &= \begin{cases} \left(\frac{1}{N}\right)^n, & \text{if } N \geq m_n; \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Let $\hat{N}$ denote the estimator of the unknown parameter N.

ML estimation: Since

$$\max_{N\in\{1,2,\ldots\}} L(x_1,\ldots,x_n;N) = \max\left\{0, \left(\frac{1}{m_n}\right)^n, \left(\frac{1}{m_n+1}\right)^n, \ldots\right\} = \left(\frac{1}{m_n}\right)^n,$$

we get the estimator

$$\hat{N}^{\mathrm{ML}} = \arg\max_N L(x_1,\ldots,x_n;N) = m_n. \tag{2}$$

1.3 Example

Assume that your IID sample of size $n=5$ (from distribution $U^{\mathrm{d}}(1,N)$ with unknown N) is as follows:

$$(x_1,\ldots,x_n) = (5,2,7,5,3).$$

Now, the sample mean is $\bar{x}_n = \frac{22}{5} = 4.4$, and the sample maximum equals $m_n = 7$.

MoM estimate: By (1), we get

$$\hat{N}^{\mathrm{MoM}} = 2\bar{x}_n - 1 = 7.8$$

ML estimate: By (2), we get

$$\hat{N}^{\mathrm{ML}} = m_n = 7.0$$

Two following two estimates are borrowed from the *German tank problem* discussed in the lecture slides 41-42.[1]

Bayesian estimate:

$$\hat{N}^{\mathrm{Bayes}} = \frac{(m_n-1)(n-1)}{n-2} = \frac{24}{3} = 8.0$$

Minimum-variance unbiased estimate:

$$\hat{N}^{\mathrm{MinV}} = \frac{m_n(n+1)}{n} - 1 = \frac{37}{5} = 7.4$$

[1]Note, however, that there is a slight difference in our estimation problem when compared to the German tank problem. Can you identify the difference?

2 Continuous uniform distribution $U^{\mathrm{C}}(0,b)$, one unknown parameter b

Let $U^{\mathrm{c}}(0,b)$ denote the continuous uniform distribution, for which $b>0$ and the probability density function equals

$$f(x)=\begin{cases}\dfrac{1}{b}, & x\in[0,b];\\ 0, & \text{otherwise.}\end{cases}$$

It follows that

$$\begin{aligned}\mathrm{E}(X)&=\int_0^b f(x)\,x\,dx=\frac{1}{b}\int_0^b x\,dx=\frac{b}{2},\\ \mathrm{E}(X^2)&=\int_0^b f(x)\,x^2\,dx=\frac{1}{b}\int_0^b x^2\,dx=\frac{b^2}{3},\\ \mathrm{V}(X)&=\mathrm{E}(X^2)-(\mathrm{E}(X))^2=\frac{b^2}{3}-\left(\frac{b}{2}\right)^2=\frac{b^2}{12}.\end{aligned}$$

2.1 Estimation of b: Method of Moments (MoM)

Consider an IID sample $(x_1,\ldots,x_n)$ of size n from distribution $U^{\mathrm{c}}(0,b)$. The first (theoretical) moment equals

$$\mu_1=\mathrm{E}(X)=\frac{b}{2},$$

and the corresponding sample moment is the sample mean $\bar{x}_n$:

$$\hat{\mu}_1=\bar{x}_n=\frac{1}{n}\sum_{i=1}^{n}x_i.$$

Let $\hat{b}$ denote the estimator of the unknown parameter b.

MoM estimation: By solving the requirement (for the first moment) that

$$\hat{\mu}_1=\frac{\hat{b}}{2},$$

we get the estimator

$$\hat{b}^{\mathrm{MoM}}=2\hat{\mu}_1=2\bar{x}_n. \tag{3}$$

2.2 Estimation of b: Maximum Likelihood (ML)

Consider again an IID sample $(x_1, \ldots, x_n)$ of size n from distribution $U^c(0, b)$. Let m_n denote the maximum value of this sample,

$$m_n = \max\{x_1, \ldots, x_n\}.$$

The likelihood function for this continuous distribution with unknown parameter b equals

$$\begin{aligned} L(x_1, \ldots, x_n; b) &= f(x_1; b) \cdots f(x_n; b) \\ &= \begin{cases} \left(\frac{1}{b}\right)^n, & \text{if } b \geq m_n; \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Let $\hat{b}$ denote the estimator of the unknown parameter N.

ML estimation: Since

$$\max_{b>0} L(x_1, \ldots, x_n; b) = \max_{b \geq m_n} \left(\frac{1}{b}\right)^n = \left(\frac{1}{m_n}\right)^n,$$

we get the estimator

$$\hat{b}^{\mathrm{ML}} = \arg\max_b L(x_1, \ldots, x_n; b) = m_n. \tag{4}$$

2.3 Example

Assume that your IID sample of size $n = 5$ (from distribution $U^c(0, b)$ with unknown b) is as follows:

$$(x_1, \ldots, x_n) = (30, 20, 90, 100, 60).$$

Now, the sample mean is $\bar{x}_n = \frac{300}{5} = 60.0$, and the sample maximum equals $m_n = 100$.

MoM estimate: By (3), we get

$$\hat{b}^{\mathrm{MoM}} = 2\bar{x}_n = 120.0$$

ML estimate: By (4), we get

$$\hat{b}^{\mathrm{ML}} = m_n = 100.0$$

3 Continuous uniform distribution $U^{\mathrm{c}}(a,b)$, two unknown parameters a and b

Let $U^{\mathrm{c}}(a,b)$ denote the continuous uniform distribution, for which $a<b$ and the probability density function equals

$$f(x)=\begin{cases}\dfrac{1}{b-a}, & x\in[a,b];\\ 0, & \text{otherwise.}\end{cases}$$

It follows that

$$\begin{aligned}
\mathrm{E}(X) &= \int_a^b f(x)\,x\,dx=\frac{1}{b-a}\int_a^b x\,dx=\frac{a+b}{2},\\
\mathrm{E}(X^2) &= \int_0^b f(x)\,x^2\,dx=\frac{1}{b-a}\int_a^b x^2\,dx=\frac{a^2+ab+b^2}{3},\\
\mathrm{V}(X) &= \mathrm{E}(X^2)-(\mathrm{E}(X))^2=\frac{a^2+ab+b^2}{3}-\left(\frac{a+b}{2}\right)^2=\frac{(b-a)^2}{12}.
\end{aligned}$$

3.1 Estimation of a and b: Method of Moments (MoM)

Consider an IID sample $(x_1,\ldots,x_n)$ of size n from distribution $U^{\mathrm{c}}(a,b)$. The first two (theoretical) moments equal

$$\mu_1=\mathrm{E}(X)=\frac{a+b}{2},\quad \mu_2=\mathrm{E}(X^2)=\frac{a^2+ab+b^2}{3}.$$

and the corresponding sample moments are given by

$$\hat\mu_1=\frac{1}{n}\sum_{i=1}^n x_i,\quad \hat\mu_2=\frac{1}{n}\sum_{i=1}^n x_i^2.$$

Let $\hat a$ and $\hat b$ denote the estimators of the two unknown parameters a and b, respectively.

MoM estimation: By solving the requirements (for the first two moments) that

$$\hat\mu_1=\frac{\hat a+\hat b}{2},\quad \hat\mu_2=\frac{\hat a^2+\hat a\hat b+\hat b^2}{3},$$

we get the estimators

$$\hat a^{\mathrm{MoM}}=\hat\mu_1-\sqrt{3(\hat\mu_2-\hat\mu_1^2)},\quad \hat b^{\mathrm{MoM}}=\hat\mu_1+\sqrt{3(\hat\mu_2-\hat\mu_1^2)}. \tag{5}$$

3.2 Estimation of a and b: Maximum Likelihood (ML)

Consider again an IID sample $(x_1, \ldots, x_n)$ of size n from distribution $U^{\mathrm{c}}(a, b)$. Let u_n and v_n denote the minimum and the maximum value of this sample, respecively, so that

$$u_n = \min\{x_1, \ldots, x_n\}, \quad v_n = \max\{x_1, \ldots, x_n\}.$$

The likelihood function for this continuous distribution with unknown parameters a and b equals

$$\begin{aligned} L(x_1, \ldots, x_n; a, b) &= f(x_1; a, b) \cdots f(x_n; a, b) \\ &= \begin{cases} \left(\frac{1}{b-a}\right)^n, & \text{if } a \leq u_n \text{ and } b \geq v_n; \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Let $\hat{a}$ and $\hat{b}$ denote the estimators of the unknown parameters a and b, respectively.

ML estimation: Since

$$\max_{a<b} L(x_1, \ldots, x_n; a, b) = \max_{a \leq u_n, b \geq v_n} \left(\frac{1}{b-a}\right)^n = \left(\frac{1}{v_n - u_n}\right)^n,$$

we get the following pair of estimators:

$$(\hat{a}^{\mathrm{ML}}, \hat{b}^{\mathrm{ML}}) = \arg\max_{(a,b)} L(x_1, \ldots, x_n; a, b) = (u_n, v_n). \tag{6}$$

3.3 Example

Assume that your IID sample of size $n = 5$ (from distribution $U^{\mathrm{c}}(a, b)$ with unknown a and b) is as follows:

$$(x_1, \ldots, x_n) = (30, 20, 90, 100, 60).$$

Now, the first two sample moments are

$$\hat{\mu}_1 = \frac{300}{5} = 60.0, \quad \hat{\mu}_2 = \frac{23000}{5} = 4600.0$$

In addition, the sample mimimum u_n and the sample maximum v_n equal

$$u_n = 20.0, \quad v_n = 100.0$$

MoM estimate: By (5), we get

$$\hat{a}^{\text{MoM}} = \hat{\mu}_1 - \sqrt{3(\hat{\mu}_2 - \hat{\mu}_1^2)} = 60 - \sqrt{3000} = 5.2,$$
$$\hat{b}^{\text{MoM}} = \hat{\mu}_1 + \sqrt{3(\hat{\mu}_2 - \hat{\mu}_1^2)} = 60 + \sqrt{3000} = 114.8$$

ML estimate: By (6), we get

$$\hat{a}^{\text{ML}} = u_n = 20.0, \quad \hat{b}^{\text{ML}} = v_n = 100.0$$