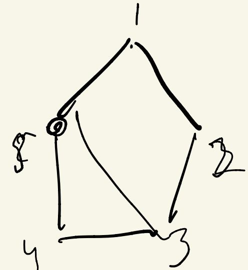
$\frac{1}{n} \times \frac{1}{n}$

0	0	1	1	1
0	0	1	1	0
1	1	0	0	0
1	1	0	0	0
1	0	0	0	0



symmetric $n \times n$ table $\rightsquigarrow$ n vertex graph
of zeroes & ones

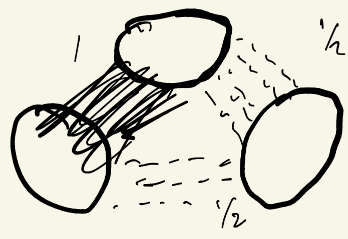
$\Updownarrow$

fcns $[0,1]^2 \to \{0,1\}$
constant on subsquares
of size $\frac{1}{n} \times \frac{1}{n}$.

SRL

ε-Approximated in L_1 by piecewise constant
functions $[0,1] \to [0,1]$
on boxes of size $\frac{1}{m} \times \frac{1}{m}$ $(m = m(\varepsilon))$

0	1/2	1
1/2	0	1/2
1	1/2	0



$0 \ll m \ll n$

Used SRL to prove Erdős-Stone:

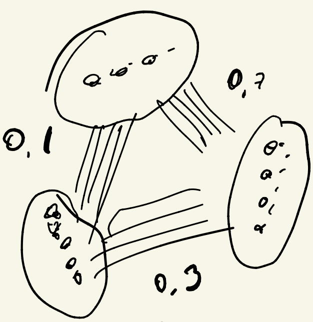
For every ε, s: If $|G| = n$ large enough and G has $\boxed{t_{r-1}(n)} + \varepsilon n^2$ edges, then G contains a $T_r(s)$

Idea:

If G has $> t_{r-1}(n)$ edges, then it must have a K_r subgraph.

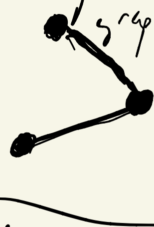
If G has $> t_{r-1}(n) + \varepsilon n^2$ edges, then it has a <u>blown up</u> K_r

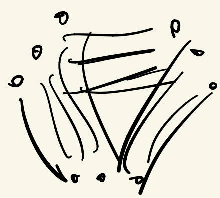
---



R

d = 0.2 regularity graph





Regularity graph has a K_r.

Lmi R_s might not be a subgraph of G, but low degree subgraphs of R_s are.

Blown up Regularity graph R_s

has a

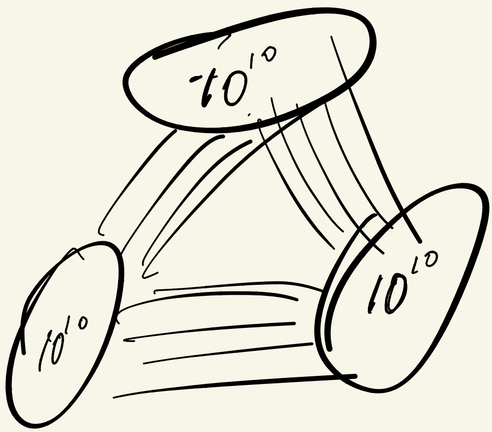
blown up complete subgraph $(K_r)_s = T_r(rs)$

bounded degree

$|G| \geq n_0 = 10^{10^{10^{10^{10}}}}$ $\quad s = 10^{60}$ $\quad \varepsilon = 10^{-4}$ $\quad r = 3$

$\|G\| = \frac{n^2}{4} + \frac{n^2}{10^4}$, then it has a blown up triangle

#edges



Ramsey Theory

"Every large enough graph (in terms of $|V|$ - no need to increase $|E|$) contains either H or its complement has a L"

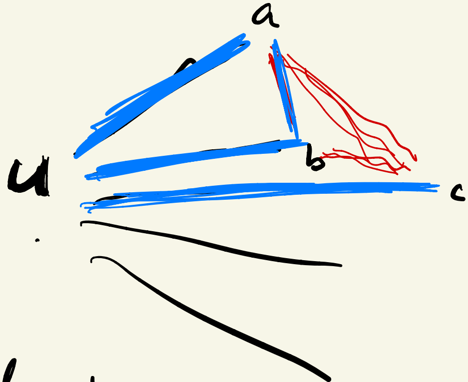
"Complete disorder is impossible"

Prop In a group of 6, there are always 3 mutual non-friends or 3 mutual friends.

In any graph G with $|G|=6$, either $K_3 \subseteq G$ or $K_3 \subseteq \overline{G}$

Proof: NTS: If I colour the edges of K_6 blue & red, there is either a blue or a red triangle.

Fix a vertex

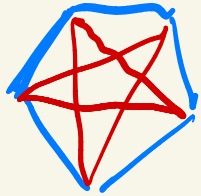


Without l.o.g., assume u has ≥ 3 blue edges going out, to a, b, c.

If a, b, c not a red triangle, say $a - b$ is blue edge, then

$u - a$, b blue triangle.

This (6) is best possible



2-coloured K_5 with no monochrom. triangles.

Def: Let $s, t \in \mathbb{N}$.
$R(s,t)$ smallest N s.t every red-blue colouring of K_n contains a blue K_s or a red K_t.

Ex:

$$R(3,3) = 6$$

$$R(s,t) = R(t,s)$$

$$R(s,2) = s$$

(either there is a red edge, or all edges are bleue)

Lm: $R(s,t)$ finite for all s,t, namely:

$$R(s,t) \le \binom{s+t-2}{s-1}$$

Proof: Enough to show $R(s,t) \le \boxed{R(s-1,t) + R(s,t-1)}$

$(R(2,s) = s,\ R(s,2) = s)$

Let G be a 2-coloured edge $K_{\underbrace{R(s-1,t)}_{N_1} + \underbrace{R(s,t-1)}_{N_2}}$. Fix $u \in G$.

u has $N_1 + N_2 - 1$ n'bours, so by pigeonhole principle, either N_1 blue edges out, or N_2 red edges out.

Assume wlog N_1 blue edges. Let the set of n'bours of u with blue edges be A.

Inside A is a blue K_{s-1} or a red K_t.

If A has a red K_t, then G — " —

If A has a blue K_{s-1}, then this together w/ u forms a blue K_s in G.

$R(3,3) = 6$

$R(4,4) = 18$

$R(5,5) = ?$

$R(6,6) = ????$

$$R(s,s) \leq \binom{2s-2}{s-1} \approx c\,\frac{2^{2s-2}}{\sqrt{s}}$$

"essentially" best upper bound, since 50 years.

<u>Def</u>: $R(s_1 \ldots s_k)$ smallest N s.t. every k-colouring of K_N has a K_{s_i} with colour i for some $i = 1 \ldots k$.

<u>Thm</u>: $$\underbrace{R(s_1, \ldots, s_k)}_{\substack{k\text{-coloured} \\ \text{Ramsey number}}} \leq R\underbrace{\left(R(s_1, s_2), s_3, \ldots s_k\right)}_{\substack{(k-1)\text{-coloured} \\ \text{Ramsey number}}}$$

(in particular, all Ramsey numbers are finite.)

Note (base case) $R(s,t) = R(R(s,t))$

Proof: NTS: In a k-colouring of K_N, $N = R(R(s_1, s_2), s_3 \ldots s_k)$

there is a c_i-coloured s_i-clique for some i.

I know there is either

c_k-coloured K_{s_k}

or c_{k-1}-coloured $K_{s_{k-1}}$

$\vdots$

or c_3-coloured K_{s_3}

or $\boxed{\cancel{c_1}\, c_2\text{-coloured } K_{R(s_1, s_2)}}$

in this case, either

c_1-coloured K_{s_1}

or c_2-coloured K_{s_2}.

$\blacksquare$

Obs: $R(s,t) \leq N$ means that any graph G on N nodes has $\alpha(G) \geq t$ or $\omega(G) \geq s$.

Thm: (Erdős, 1947)

$$R(s,s) > \underline{\sqrt{2}^{\,s}}$$

Pf: NTS exists graph G on $n \approx \sqrt{2}^{\,s}$ nodes with $\alpha(G) < s$, $\omega(G) < s$.

Consider all $2^{\binom{n}{2}}$ graphs on n nodes.

A <u>given</u> set of s nodes is

- a clique in $2^{\binom{n}{2} - \binom{s}{2}}$ of these graphs
- a indep set in $2^{\binom{n}{2} - \binom{s}{2}}$ of these graphs

So # graphs where no s-tuple is clique or indep

$$\geq 2^{\binom{n}{2}} - \binom{n}{s} \cdot 2 \cdot 2^{\binom{n}{2} - \binom{s}{2}} = A$$

Done if we can show $A > 0$.

NTS

$$2^{\binom{n}{2}} \overset{?}{>} \binom{n}{s} \cdot 2 \cdot 2^{\binom{n}{2} - \binom{s}{2}}$$

$$1 \overset{?}{>} \binom{n}{s} \cdot 2^{1 - \binom{s}{2}}$$

this holds for $n = \sqrt{2}^{s}$

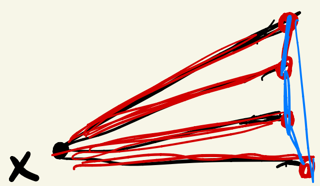
$\square$

$\underline{\text{Thm}}$: $R_k(3) = \underbrace{R(3, \ldots, 3)}_{k \text{ colours}} \leq \lfloor ek! \rfloor + 1$

$\underline{\text{Pf}}$: By induction. $k=2$

$R(3,3) = 6 = \lfloor e \cdot 2 \rfloor + 1$ ✓

Suppose x is a vertex of a k-coloured graph on $N = k(R_{k-1}(3) - 1) + 1$ nodes



There is a (red) colour of which x has $R_{k-1}(3)$ outgoing edges (by pigeonhole)

If a red edge within the set of red n'bours of x then red K_3.

Otherwise red n'hood of x has $k-1$ - colours, so monochrom Δ.

$$R_k(3) \leq k\left(R_{k-1}(3) - 1\right) + 1$$

$$R_2(3) = 6.$$

The number sequence defined by

$$a_2 = 6$$

$$a_k = k\, a_{k-1} - k + 1$$

is $a_k = \lfloor e\, k! \rfloor + 1$.

(simple calculation using

$$e = \sum_{j=0}^{\infty} \frac{1}{j!}$$

) $\blacksquare$