Exercises 2.

Exercise 2.1. Let X be a normed space, $C \subset X$ convex, and $V \subset X$ a finite-dimensional subspace. Show that V is closed and closure $\overline{C}$ is convex. Moreover, show that the interior of C is convex.

Exercise 2.2. Let $a, b \in \mathbb{R}$ with $a < b$. For $M = [a, b]$ and for $u \in C(M)$ define $\|u\| = \|u\|_{C(M)} := \sup_{x \in M} |u(x)|$, and let

$$\|u\|_1 := \int_a^b |u(x)|\,\mathrm{d}x.$$

Show that $C(M)$ is not complete with respect to the norm $u \mapsto \|u\|_1$.

Exercise 2.3. Let $X_p := (\mathbb{R}^2, \|\cdot\|_p)$, where $\|x\|_p^p = |x_1|^p + |x_2|^p$ for $1 \leq p < \infty$, and $\|x\|_\infty := \max\{|x_1|, |x_2|\}$. Draw the closed unit ball

$$\overline{B_p} := \overline{\mathbb{B}_{X_p}(0,1)} = \left\{x \in \mathbb{R}^2 : \|x\|_p \leq 1\right\}.$$

when $p = 1$, $p = 1 + \varepsilon$, $p = 2$, $p = 1/\varepsilon$, $p = \infty$, where $\varepsilon > 0$ is small.
Due to compactness, there is a closest point to the point $(1, 2)$ in $\overline{B_p}$.
How does the closest point change in the different cases? Is it unique?

Exercise 2.4. Show that

$$V := \{u \in \ell^p(J) : \ \{j \in J : \ u_j \neq 0\} \text{ finite}\}$$

is a dense subspace of $\ell^p(J)$ for $1 \leq p < \infty$.