Exercises 3.

Exercise 3.1. Let X be a Banach space and $u_k \in X$ such that $\sum_{k=1}^{\infty} \|u_k\| < \infty$. Show that the vectors

$$v_N := \sum_{k=1}^{N} u_k \in X$$

form a Cauchy sequence $(v_N)_{N=1}^{\infty}$, thus converging to

$$v = \lim_{N\to\infty} v_N =: \sum_{k=1}^{\infty} u_k \in X.$$

Moreover, when Y is another Banach space and $A \in \mathscr{B}(X,Y)$, show that here

$$A \sum_{k=1}^{\infty} u_k = \sum_{k=1}^{\infty} Au_k.$$

Exercise 3.2. Show: $\mathscr{B}(X,Y)$ is a Banach space if Y is Banach.

Exercise 3.3. Let $A : \mathbb{K}^n \to \mathbb{K}^m$ be a linear mapping defined by

$$(Au)_j := \sum_{k=1}^{n} A_{jk} u_k.$$

Show that

$$\|A\|_{\ell^1\to\ell^1} = \max_{k\in\{1,\cdots,n\}} \sum_{j=1}^{m} |A_{jk}|,$$

$$\|A\|_{\ell^\infty\to\ell^\infty} = \max_{j\in\{1,\cdots,m\}} \sum_{k=1}^{n} |A_{jk}|,$$

$$\|A\|_{\ell^2\to\ell^2} \leq \left(\sum_{j=1}^{m} \sum_{k=1}^{n} |A_{jk}|^2 \right)^{1/2}.$$

Exercise 3.4. Let $A \in \mathscr{B}(X)$. Assume that A is *power bounded*: this means that there is a constant $c > 0$ such that $\|A^k u\| \leq c\|u\|$ for all $u \in X$ and $k \geq 1$. Define $\|u\|_\star := \sup_{k\geq 0} \|A^k u\|$.

1. Show that $\|\cdot\|_\star$ is a norm.
2. Prove that the norms $\|\cdot\|$ and $\|\cdot\|_\star$ are equivalent: there are constants $a, b > 0$ such that $a\|u\| \leq \|u\|_\star \leq b\|u\|$ for all $u \in X$.
3. Let $\|A\|_\star := \sup_{u\in X:\|u\|_\star\leq 1} \|Au\|_\star$. Show that $\|A\|_\star \leq 1$.