Exercises 4.

Exercise 4.1. Use Baire's Theorem to prove that an algebraic basis of an infinite-dimensional Banach space must be uncountable.

Exercise 4.2. Prove the following easy converse to Zabreiko's Lemma:
Bounded seminorms on Banach spaces are countably subadditive.

Exercise 4.3. Prove the following corollary to Uniform Boundedness Principle:
Let operators $A_k \in \mathscr{B}(X, Y)$ satisfy for all $u \in X$

$$\lim_{k\to\infty} \|A_k u - Au\|_Y = 0.$$

Then $A = (u \mapsto Au) : X \to Y$ is a bounded linear operator.
(Above, we say that $A_k \to A$ in *strongly*
– this does not necessarily imply norm convergence $\|A_k - A\|_Y \to 0$.

Exercise 4.4. Use the Closed Graph Theorem to directly prove the Open Mapping Theorem (without using Zabreiko's Lemma).