Exercises 5.

Exercise 5.1. Prove complex scalar Hahn–Banach Theorem.

Exercise 5.2. Show: if $v \in X \setminus Z$ (where $Z \subset X$ is a closed subspace) then there exists $\Phi \in X'$ such that $\Phi(v) \neq 0 = \Phi(u)$ for all $u \in Z$.
(Hint: Let $Z_\varphi := \{\lambda v - u : \ \lambda \in \mathbb{K},\ u \in Z\}$ and $\varphi(\lambda v - u) := \lambda$.)

Exercise 5.3. Let X, Y be Banach spaces. Show that for $A \in \mathscr{B}(X, Y)$ there is unique $A' \in \mathscr{B}(Y', X')$ so that

$$\langle Av, \psi \rangle = \langle v, A'\psi \rangle$$

for every $v \in X$ and $\psi \in Y'$. Moreover, show:
(a) $\|A'\| = \|A\|$.
(b) $(BA)' = A'B'$ if $B \in \mathscr{B}(Y, Z)$.
(c) $(A^{-1})' = (A')^{-1}$ if A is invertible.

Exercise 5.4. Let X, Y be Banach spaces and $A \in \mathscr{B}(X, Y)$. Show that

$$\begin{aligned}
\mathrm{Ran}(A)^\perp &= \mathrm{Ker}(A'),\\
\overline{\mathrm{Ran}(A)} &= {}^\perp\mathrm{Ker}(A'),\\
{}^\perp\mathrm{Ran}(A') &= \mathrm{Ker}(A),\\
\overline{\mathrm{Ran}(A')} &= \mathrm{Ker}(A)^\perp.
\end{aligned}$$