Exercises 6.

Banach algebras

Exercise 6.1. Let $\mathscr{A}$ be a Banach algebra. The *commutant* of a set $S \subset \mathscr{A}$ is

$$\mathscr{C}(S) := \{A \in \mathscr{A} : AB = BA \text{ for all } B \in S\}. \tag{6.1}$$

Prove the following claims:
a) $\mathscr{C}(S)$ is a Banach (sub)algebra.
b) $S \subset \mathscr{C}(\mathscr{C}(S))$.
c) If $AB = BA$ for all $A, B \in S$, then $\mathscr{B} := \mathscr{C}(\mathscr{C}(S))$ is a commutative algebra such that $\sigma_{\mathscr{B}}(C) = \sigma_{\mathscr{A}}(C)$ for all $C \in \mathscr{B}$.

Exercise 6.2. In a Banach algebra $\mathscr{A}$, element $A \in \mathscr{A}$ is a *topological zero divisor* if there exists a sequence $(B_n)_{n=1}^{\infty} \subset \mathscr{A}$ such that $\|B_n\| = 1$ for all n and

$$\lim_{n\to\infty} A\,B_n = 0 = \lim_{n\to\infty} B_n\,A.$$

(a) Show that if $(A_n)_{n=1}^{\infty} \subset G_{\mathscr{A}}$ satisfies $A_n \to A \in \partial G_{\mathscr{A}}$ then $\|A_n^{-1}\| \to \infty$.
(b) Show that the boundary points of $G_{\mathscr{A}}$ are topological zero divisors.

Exercise 6.3. Let $\mathscr{A}$ be a Banach algebra.
(a) Assume that 0 is the only topological zero divisor. Show that $\mathscr{A} \cong \mathbb{C}$ isometrically. (Hint: modify the Gelfand–Mazur proof.)
(b) Assume that there exists constant $k < \infty$ such that

$$\|A\|\,\|B\| \le k\,\|AB\|$$

for every $A, B \in \mathscr{A}$. Show that $\mathscr{A} \cong \mathbb{C}$ isometrically. (Hint: Apply (a).)

Exercise 6.4. Let $\mathscr{A}$ be a Banach algebra, $A, B \in \mathscr{A}$. Prove the following claims:
(a) $\rho(AB) = \rho(BA)$.
(b) If $A \in \mathscr{A}$ is *nilpotent* (i.e. $A^k = 0$ for some $k \in \mathbb{N}$) then $\sigma(A) = \{0\}$.
(c) If $AB = BA$ then $\rho(AB) \le \rho(A)\,\rho(B)$.