



Aalto University  
School of Science

# Lecture 6: Electromagnetic waves and more

# Today's menu

- Basic EM waves in plasmas:
  - Transverse EM wave in the absence of a background B-field
  - Ordinary wave (O-wave)
  - Extraordinary wave (X-wave)
  - L-wave
  - R-wave
- Cut-offs & Resonances
- MHD waves:
  - Shear-Alfvén waves
  - Slow and fast magnetosonic waves (& compressional Alfvén wave)

# Electromagnetic waves in vacuum

# Allow time-dependent $E$ and $B$ fields → need Maxwell's equations

Maxwell's equations *in SI units* ...

$$\nabla \cdot \mathbf{E} = \rho / \epsilon_0$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t}$$

In vacuum:  $\rho = 0, \mathbf{j} = 0$

# Wave equation in vacuum

Take curl of Faraday's law →

$$\nabla \times (\nabla \times \mathbf{E}) = -\frac{\partial}{\partial t} \nabla \times \mathbf{B} = -\frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

$$\nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

→ Basic wave equation:  $\nabla^2 \mathbf{E} - \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$ .

Plane wave solution →  $k^2 \mathbf{E} - \frac{\omega^2}{c^2} \mathbf{E} = 0$  →  $\omega^2 = c^2 k^2$

**Dispersion relation !**

# Wave equation in plasmas

In plasmas,

- $\rho \approx 0$  can be assumed by quasineutrality (or simply make the choice to look at  $\mathbf{k} \cdot \mathbf{E} = 0$  since EM waves usually transverse)
- $\mathbf{j} = 0$  is a very bad assumption.

→ wave equation in plasmas:  $\nabla^2 \mathbf{E} = \mu_0 \frac{\partial \mathbf{j}}{\partial t} + \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2}$ .

FT & linearize →  $(\omega^2 - c^2 k^2) \mathbf{E}_1 = -i\omega c^2 \mu_0 \mathbf{j}_1$

EM waves are fast → ions immobile → current solely from electrons

# EM waves w/ no background B field

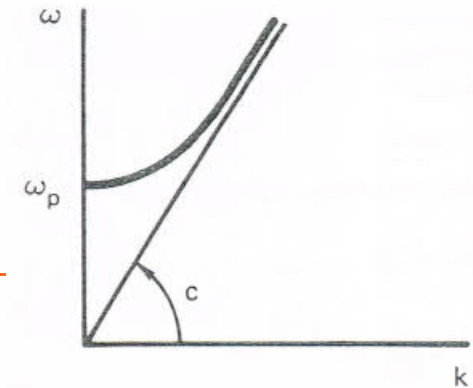
No guiding B field  $\rightarrow$  electrons are free to move:  $m \frac{\partial v}{\partial t} = -eE$

FT & linearize  $\rightarrow j_1 = -en_0 v_1 = -en_0 \left( \frac{-eE_1}{-i\omega m} \right) = i \frac{e^2 n_0}{m\omega} E_1$

$\rightarrow (\omega^2 - c^2 k^2) E_1 = -i\omega c^2 \mu_0 j_1 = \cancel{\omega} c^2 \mu_0 \frac{e^2 n_0}{\cancel{m\omega}} E_1 = \frac{e^2 n_0}{m\epsilon_0} E_1 = \omega_p^2 E_1$

$\rightarrow$  Dispersion relation for transverse EM waves propagating in plasmas in the absence of DC magnetic field:

$$\omega^2 = \omega_p^2 + c^2 k^2$$



# Observations on the dispersion relation

1.  $v_{ph}^2 = \frac{\omega^2}{k^2} = c^2 + \frac{\omega_p^2}{k^2} > c^2$

2.  $v_{gr} = \frac{d\omega}{dk} = \frac{c^2}{v_{ph}} < c$

3. At large  $k$  (small  $\lambda$ )  $\rightarrow$  ordinary light waves,  $\omega = ck$

4. There is a *cut-off frequency* for waves to propagate ...

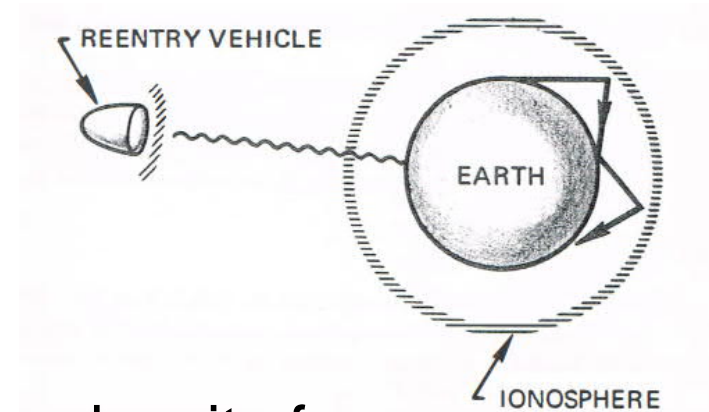
$$\omega < \omega_{cut-off} = \omega_p \rightarrow ck = \sqrt{\omega^2 - \omega_p^2} = i \sqrt{\omega_p^2 - \omega^2}$$

$\rightarrow e^{ikx} = e^{-Im(k)x} = e^{-\frac{x}{\delta}}$ , where  $1/\delta \equiv \frac{1}{c} \sqrt{\omega_p^2 - \omega^2}$ ,

$\rightarrow$  an exponentially attenuated wave with skin depth  $\delta$



# Radio communication



Remember  $\omega_p^2 = \frac{e^2 n_0}{m \epsilon_0}$

- For a given frequency  $\omega$ , there is a maximum density for plasmas through which the wave can propagate.
- This is the basis of short-wavelength radio communication
- To communicate with a satellite, the wave frequency has to be chosen sufficiently high to penetrate all atmospheric layers
- Space vehicle entering the atmosphere will suffer a communication black-out due to the shock wave in front of it

# Electromagnetic waves with $B_0 \neq 0$

$k \perp B_0$

# Transverse waves propagating perpendicular to $B_0$ : *ordinary wave*

Transverse wave:  $k \perp E$

Propagation perpendicular to magnetic field:  $k \perp B_0$

#1. Take  $E_1 \parallel B_0$

Then the magnetic field does not constrain the electron motion and the math of  $B_0 = 0$  case applies

$$\omega^2 = \omega_p^2 + c^2 k^2$$

This is called an *ordinary wave* – or just O-wave between friends.

# Finding the *extraordinary* wave

#2. Take  $\mathbf{E}_1 \perp \mathbf{B}_0$

Now electron motion *is* constrained by  $\mathbf{B}$ .

Take x-axis so that  $\mathbf{k} = k\hat{\mathbf{x}}$  and  $\mathbf{E}_1 = E_x\hat{\mathbf{x}} + E_y\hat{\mathbf{y}}$

- It is known that in this case a longitudinal component will arise  $\rightarrow E_x \neq 0$ .

Electron EoM, linearized and FT'd:

$$-im\omega\mathbf{v}_1 = -e(\mathbf{E}_1 + \mathbf{v}_1 \times \mathbf{B}_0)$$

$$v_x = \frac{-ie}{m\omega} (E_x + v_y B_0)$$

$$v_x = \frac{e}{m\omega} \left[ -iE_x - \frac{\Omega_e}{\omega} E_y \right] / \left( 1 - \frac{\Omega_e^2}{\omega^2} \right)$$

$$v_y = \frac{-ie}{m\omega} (E_y - v_x B_0)$$

$$v_y = \frac{e}{m\omega} \left[ -iE_y + \frac{\Omega_e}{\omega} E_x \right] / \left( 1 - \frac{\Omega_e^2}{\omega^2} \right)$$

# Dispersion relation for the X wave

Now careful with the wave equation:  $E_x \neq 0 \rightarrow \mathbf{k} \cdot \mathbf{E} \neq 0$

$$\rightarrow (\omega^2 - c^2 k^2) \mathbf{E}_1 + c^2 k E_x \mathbf{k} = -i\omega c^2 \mu_0 \mathbf{j}_1 = i\omega c^2 \mu_0 n_0 e \mathbf{v}_1$$

We already have  $\mathbf{v}_1 = \mathbf{v}_1(\mathbf{E}_1) \rightarrow$

A matrix equation: 
$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} E_x \\ E_y \end{bmatrix} = 0$$

$\rightarrow$  Use the determinant condition to find the non-trivial solution ...

HW  $\rightarrow$  
$$\frac{c^2 k^2}{\omega^2} = 1 - \frac{\omega_p^2}{\omega^2} \frac{\omega^2 - \omega_p^2}{\omega^2 - \omega_h^2},$$
 dispersion relation for the X-wave

# Cut-offs and resonances

We have just obtained our first *non-trivial* dispersion relation.

In particular, it can happen that

- $k \rightarrow 0$ , i.e.,  $\lambda \rightarrow \infty$ . This is the cut-off that we already got for  $\mathbf{B}_0 = 0$ . A cut-off corresponds to reflection of the EM wave
- $k \rightarrow \infty$ , i.e.,  $\lambda \rightarrow 0$ . This is called a resonance, and here the wave can be absorbed.

How do cut-offs and resonances look for the *X*-wave?

# Cut-offs and resonances of the X-wave

- Resonance:

- $k \rightarrow \infty$  when  $\omega \rightarrow \omega_h$

→ Resonance occurs at a point in the plasma where  $\omega^2 = \omega_h^2 = \omega_p^2 + \Omega_e^2$

But this dispersion relation we know: electrostatic 'waves' across  $B_0$  !

→ When an EM wave approaches a point in a plasma where  $\omega \rightarrow \omega_h$ , both  $v_{ph}$  and  $v_{gr}$  go to zero and the wave is converted into upper hybrid oscillations !

- Cut-off:

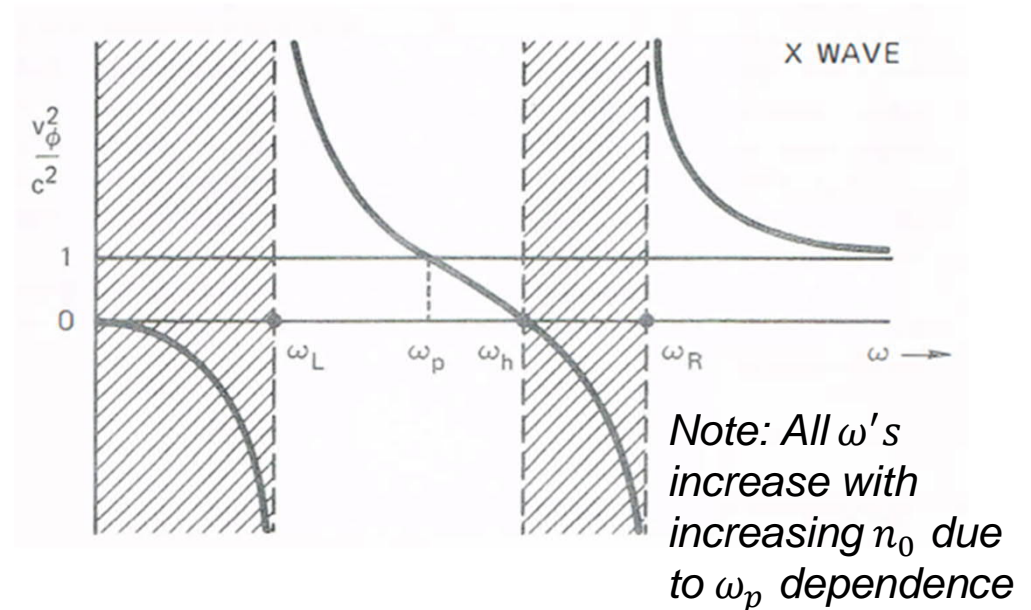
- $k \rightarrow 0$  when  $1 = \frac{\omega_p^2}{\omega^2} \frac{\omega^2 - \omega_p^2}{\omega^2 - \omega_h^2} = \frac{\omega_p^2}{\omega^2} \left[ 1 - \frac{\Omega_e^2}{\omega^2 - \omega_p^2} \right]^{-1}$  **HW** →  $\omega^2 \mp \Omega_e \omega - \omega_p^2 = 0$

# Stop bands for X-wave

HW → 2 cut-off frequencies:

$$\omega_R = \frac{1}{2} \left[ \sqrt{\Omega_e^2 + 4\omega_p^2} + \Omega_e \right]$$

$$\omega_L = \frac{1}{2} \left[ \sqrt{\Omega_e^2 + 4\omega_p^2} - \Omega_e \right]$$



The resonance and cut-off frequencies divide the dispersion diagram into propagation and nonpropagation zones.

→ X-wave has two regions of propagation, separated by a 'stop band' where it cannot propagate.



# The new *dispersion diagram*

Note: the dispersion diagram for the  $X$ -wave was no longer of the type  $\omega = \omega(k)$ .

The reason is that we do not have simple enough functional dependence between  $\omega$  and  $k$ .

→ Plotting  $\frac{\omega}{ck} = \frac{v_{ph}}{c}$  as a function of  $\omega$  has proven to be enlightening.

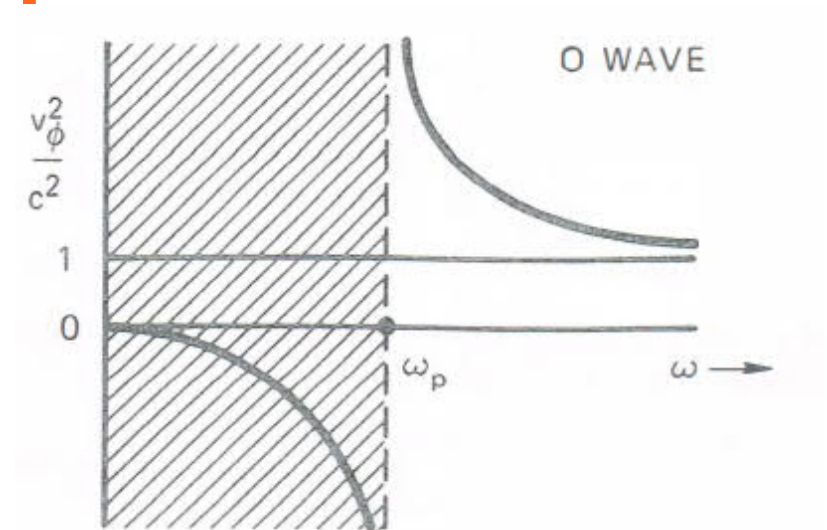
# Stop band for the O wave?

Simpler:  $\omega^2 = \omega_p^2 + c^2 k^2$

→  $\frac{c^2 k^2}{\omega^2} = 1 - \frac{\omega_p^2}{\omega^2}$

→ No resonances

→ One cut-off:  $k \rightarrow 0$  when  $\omega = \omega_p$  (as was discovered already 😊)



# Electromagnetic waves with $B_0 \neq 0$ $k \parallel B_0$

## How about waves parallel to $B_0$ ?

Now  $\mathbf{k} \parallel \mathbf{B}_0 \rightarrow \mathbf{k} = k\hat{\mathbf{z}}$ , and from electron motion we can expect  $\mathbf{E}_1 = E_x\hat{\mathbf{x}} + E_y\hat{\mathbf{y}} \rightarrow$  we can use the wave equation from X wave with the substitutions  $\mathbf{k} = k\hat{\mathbf{x}} \rightarrow k\hat{\mathbf{z}} \rightarrow$

$$(\omega^2 - c^2k^2)E_x = \alpha \left( E_x - \frac{i\Omega_e}{\omega} E_y \right), \quad \text{where } \alpha = \frac{\omega_p^2}{1 - \Omega_e^2/\omega^2}$$

$$(\omega^2 - c^2k^2)E_x = \alpha \left( E_x + \frac{i\Omega_e}{\omega} E_y \right)$$

Again we have a coupled set of equations  $\rightarrow$  use  $\det = 0 \rightarrow$

$$\omega^2 - c^2k^2 - \alpha = \pm \alpha \frac{\Omega_e}{\omega}$$

# Wave names by polarization

We then obtain two waves propagation along the B field:

- R-wave:  $\frac{c^2 k^2}{\omega^2} = 1 - \frac{\omega_p^2}{\omega^2} \left[ 1 - \frac{\Omega_e}{\omega} \right]^{-1}$
- L-wave:  $\frac{c^2 k^2}{\omega^2} = 1 - \frac{\omega_p^2}{\omega^2} \left[ 1 + \frac{\Omega_e}{\omega} \right]^{-1}$

Reason for names:

The  $E_1$  vector of the *R*-wave rotates clockwise in time as viewed in the direction of propagation → *right-hand circularly polarized wave*

Vice versa for the *L*-wave.

# Cut-offs and resonances for $L$ and $R$ ?

## Resonances:

- R-wave:  $k \rightarrow \infty @ \omega = \Omega_e$ , giving a resonance. Physics of the resonance: polarization allows the E field to be in sync with the electron gyration  $\rightarrow$  wave dumps its energy to electrons  $\rightarrow$  electron cyclotron resonance heating (ECRH)
- L-wave: no resonance found. (would exist if ion motion were included)

## Cut-offs:

- R-wave:  $k \rightarrow 0 @ \omega = \omega_R$ .
- L-wave:  $k \rightarrow 0 @ \omega = \omega_L$ .

The names of the frequencies make sense! 😊

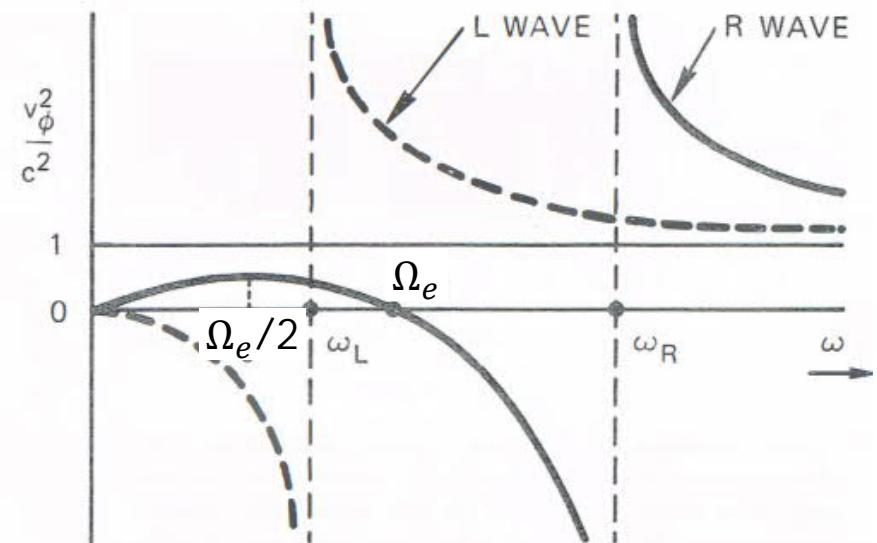
# Stop bands for *R* and *L* waves

## The *L*-wave

- Has a stop band at low-  $\omega \rightarrow$  behaves like an O-wave except with replacement  $\omega_p \rightarrow \omega_L$

## The *R*-wave

- Has a stop band between  $[\Omega_e, \omega_R]$
  - The low-frequency band,  $\omega < \Omega_e$ , has an interesting history and relevance
- $\rightarrow$  HW: Food for Thought



# Summary of EM waves in plasmas

Along the B field:

- Right-hand ( $R$ ) and left-hand ( $L$ ) circularly polarized waves

Across the B-field:

- plane-polarized ordinary ( $O$ ) wave and elliptically polarized extraordinary ( $X$ ) wave



# Magnetohydrodynamic waves

# What is different now?

- Until now we have always been aware that plasma consists of ions and electrons → we have made choices of which dynamics to include.
- In magnetohydrodynamics, the plasma is just a *fluid*  
→ in MHD, the waves are supported/carried by plasma fluid, where the ion and electron species have just as much to say as oxygen and hydrogen have in regular *hydrodynamics*.

We shall now apply our procedure to the MHD equations ...

# Linearized MHD equations

Do the linearization procedure for the MHD equations →

$$\begin{aligned}\frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \cdot \mathbf{V}_1 &= 0 \\ \rho_0 \frac{\partial \mathbf{V}_1}{\partial t} + \nabla p_1 - \frac{(\nabla \times \mathbf{B}_1) \times \mathbf{B}_0}{\mu_0} &= 0 \\ -\frac{\partial \mathbf{B}_1}{\partial t} + \nabla \times (\mathbf{V}_1 \times \mathbf{B}_0) &= 0 \\ \frac{\partial}{\partial t} \left( \frac{p_1}{p_0} - \frac{\gamma \rho_1}{\rho_0} \right) &= 0\end{aligned}$$

here  $\rho$  is the fluid density

## ”Plane wave” solution ...

Now take the harmonic approximation and assume that each perturbed quantity is a sum of plane waves w/ given  $\mathbf{k}$  and  $\omega$  :

$$\begin{aligned} -\omega\rho_1 + \rho_0\mathbf{k}\cdot\mathbf{V}_1 &= 0 \Rightarrow \rho_1 = \rho_0\frac{\mathbf{k}\cdot\mathbf{V}_1}{\omega} \\ \omega\mathbf{B}_1 + \mathbf{k}\times(\mathbf{V}_1\times\mathbf{B}_0) &= 0 \Rightarrow \mathbf{B}_1 = \frac{(\mathbf{k}\cdot\mathbf{V}_1)\mathbf{B}_0 - (\mathbf{k}\cdot\mathbf{B}_0)\mathbf{V}_1}{\omega} \\ -\omega\left(\frac{p_1}{p_0} - \frac{\gamma\rho_1}{\rho_0}\right) &= 0 \Rightarrow p_1 = \gamma p_0\frac{\mathbf{k}\cdot\mathbf{V}_1}{\omega} \\ -\omega\rho_0\mathbf{V}_1 + \mathbf{k}p_1 - \frac{(\mathbf{k}\times\mathbf{B}_1)\times\mathbf{B}_0}{\mu_0} &= 0 \end{aligned}$$

# 4 equations, 4 unknowns!

Substitute the expressions for  $\mathbf{B}_1$  and  $p_1 \rightarrow$  equation for  $\mathbf{V}_1$  :

$$\left[ \omega^2 - \frac{(\mathbf{k} \cdot \mathbf{B}_0)^2}{\mu_0 \rho_0} \right] \mathbf{V}_1 = \left\{ \left[ \frac{\gamma p_0}{\rho_0} + \frac{B_0^2}{\mu_0 \rho_0} \right] \mathbf{k} - \frac{(\mathbf{k} \cdot \mathbf{B}_0)}{\mu_0 \rho_0} \mathbf{B}_0 \right\} (\mathbf{k} \cdot \mathbf{V}_1) - \frac{(\mathbf{k} \cdot \mathbf{B}_0) (\mathbf{V}_1 \cdot \mathbf{B}_0)}{\mu_0 \rho_0} \mathbf{k}$$

# Get real(istic) in a very general way

Time to fix the space:

- Align the coordinate system so that  $\mathbf{B}_0 = B_0 \mathbf{z}$ ,  $\mathbf{k} = k_x \mathbf{x} + k_z \mathbf{z}$ ,
- Angle  $\theta$  defined to be the angle between  $\mathbf{k}$  and  $\mathbf{B}_0$

Write the linearized equation of motion in (x,y,z) components:

→ Matrix equation

$$\begin{pmatrix} \omega^2 - k^2 V_A^2 - k^2 V_S^2 \sin^2 \theta & 0 & -k^2 V_S^2 \sin \theta \cos \theta \\ 0 & \omega^2 - k^2 V_A^2 \cos^2 \theta & 0 \\ -k^2 V_S^2 \sin \theta \cos \theta & 0 & \omega^2 - k^2 V_S^2 \cos^2 \theta \end{pmatrix} \begin{pmatrix} V_{1x} \\ V_{1y} \\ V_{1z} \end{pmatrix} = 0,$$

Here  $V_A^2 = \frac{B_0^2}{\mu_0 \rho_0}$  is the so-called *Alfvén speed*, and  $V_S^2 = \frac{\gamma p_0}{\rho_0}$  is the *sound speed*

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# Non-trivial ( $\neq 0$ ) solutions only for $\det = 0$

→ Product of three terms (easiest by using either middle row or middle column)

→ 
$$(\omega^2 - k^2 V_A^2 \cos \theta) [\omega^4 - \omega^2 k^2 (V_A^2 + V_S^2) + k^4 V_A^2 V_S^2 \cos^2 \theta] = 0$$

→ 3rd order equation for  $\omega^2$

→ Different kinds of *Alfvén waves* ...

# Shear Alfvén wave

$$\omega^2 - k^2 V_A^2 \cos^2 \theta = 0$$

This corresponds to  $\mathbf{k} \cdot \mathbf{V}_1 = 0$

→ No density or pressure perturbation associated with the wave

Also  $\mathbf{V}_1 \cdot \mathbf{B}_0 = 0$

→ Motion only perpendicular to the magnetic field



# Compressional Alfvén wave

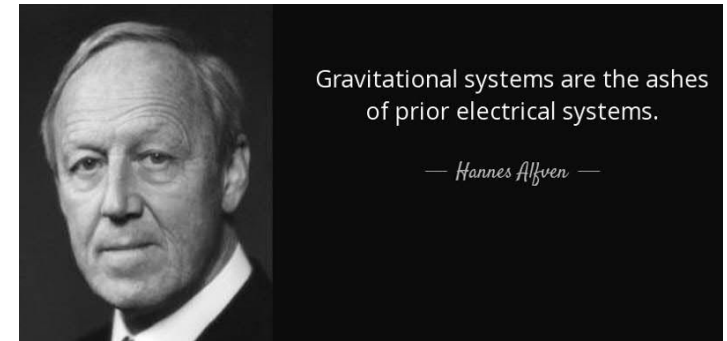
The other two roots:  $\omega = kV_{\pm}$ , where

$$V_{\pm} = \left\{ \frac{1}{2} \left[ V_A^2 + V_S^2 \pm \sqrt{(V_A^2 + V_S^2)^2 - 4V_A^2 V_S^2 \cos^2 \theta} \right] \right\}^{1/2}$$

These are the fast and slow *magnetosonic* waves – or *fast* and *slow* waves between friends.

- In the cold plasma limit ( $p \rightarrow 0$ ), the *fast* wave becomes the so-called *compressional Alfvén wave*:  $\omega = kV_A$  and slow wave dies.
- For  $V_A \ll V_S$  the *fast* wave becomes a sound wave, modified by the presence of the magnetic field:  $\omega = kV_S$

# Hannes Alfvén (1908 – 1995)



- developed the MHD theory
- was the first to discover these wave motions within MHD
- Nobel prize 1970

Alfvén waves can be used to diagnose the plasma (especially in space, but also in laboratory plasmas)