Taloustieteen matemaattiset menetelmät
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Problem Set 4: Solutions

1. **Solution**

   1. $f(x_1, x_2) = 3x_1x_2 - x_1^3 - x_2^3$. The FOCs reduce to the equations $x_2 = x_1^2$ and $x_1 = x_2^2$, which solve for $x_1 = x_2 = 0$ and $x_1 = x_2 = 1$. Thus the critical points are $(0, 0)$ and $(1, 1)$. The Hessian is
   $$H = D^2 f(x_1, x_2) = \begin{pmatrix} -6x_1 & 3 \\ 3 & -6x_2 \end{pmatrix}.$$
   At $(1, 1)$, the Hessian is negative definite. Therefore, $(1, 1)$ is a local maximizer. However, this is not a global maximizer. Notice that $f(1, 1) = 1 < 5 = f(-1, -1)$.

      At $(0, 0)$, the Hessian is indefinite (the determinant is -9). Therefore, $(0, 0)$ is a saddle point.

      Thus, this function attains only a local maximum at $(1, 1)$.

   2. $f(x_1, x_2) = 3x_1e^{x_2} - x_1^3 - e^{3x_2}$. The FOCs reduce to $e^{x_2} - x_1^2 = 0$ and $x_1 - e^{2x_2} = 0$, which solve for $x_1 = 1$ and $x_2 = 0$. So the only critical point is $(1, 0)$. The Hessian is
   $$H = D^2 f(x_1, x_2) = \begin{pmatrix} -6x_1 & 3e^{x_2} \\ 3e^{x_2} & -9e^{3x_2} \end{pmatrix}.$$
   At $(1, 0)$, the Hessian is negative definite, so implying that $(1, 0)$ is a local maximizer. However, $(1, 0)$ is not a global maximizer. Indeed we have $f(1, 0) = 1 < 7.18 \approx f(-2, -2)$.

2. **Solution**

   1. $f(x_1, x_2, x_3) = x_1^2 + x_2^2 + 3x_3^2 - x_1x_2 + 2x_1x_3 + x_2x_3$. The only critical point is the origin $(0, 0, 0)$. The Hessian is
   $$H = D^2 f(x_1, x_2, x_3) = \begin{pmatrix} 2 & -1 & 2 \\ -1 & 2 & 1 \\ 2 & 1 & 6 \end{pmatrix}$$
   The leading principal minors are $|H_1| = 2$, $|H_2| = 3$ and $|H_3| = 4$. Hence the Hessian is positive definite at every point of the domain. This implies that f is a convex function and $(0, 0, 0)$ is the unique global (and local) minimizer.

2. The unique critical point is the origin $(0,0)$. The Hessian is

$$H = D^2 f(x_1, x_2) = \begin{pmatrix} 2(1+x_2)^3 & 6(1+x_2)^2 x_1 \\ 6(1+x_2)^2 x_1 & 2+6(1+x_2)x_1^2 \end{pmatrix}$$

At $(0,0)$, the Hessian is positive definite, so implying that $(0,0)$ is a local minimizer. However, it is not a global minimizer. Notice that $f(x_1,-2) = -x_1^2 + 4$ tends to $-\infty$ as x goes to $+\infty$.

3. **Solution**

1. $f(x,y) = -2x + y + x^2 - 2xy + y^2$. The Hessian matrix is

$$H = D^2 f(x,y) = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$$

The two first order principal minors are positive, the second order principal minor (that is, the determinant) is equal to zero (consequently, non-negative). Hence the Hessian is positive semidefinite and the function is convex.

2. $f(x,y,z) = 100 - 2x^2 - y^2 - 3z - xy - e^{x+y+z}$. This is the same function of exercise 4 in Problem Set 3. Let $u := x+y+z$. The Hessian is

$$H = D^2 f(x,y,z) = \begin{pmatrix} -4-e^u & -1-e^u & -e^u \\ -1-e^u & -2-e^u & -e^u \\ -e^u & -e^u & -e^u \end{pmatrix}$$

The leading principal minors are $|A_1| = -4 - e^u < 0$, $|A_2| = 7 + 4e^u > 0$ and $|A_3| = -7e^u < 0$. Hence the Hessian is negative definite and f is concave.

4. **Solution**

The profit function is $\pi(q_1,q_2) = p_1q_1 + p_2q_2 - (2q_1^2 + q_1q_2 + 2q^2)$. The FOCs are:

$$\nabla\pi(q_1,q_2) = \begin{bmatrix} p_1 - 4q_1 - q_2 \\ p_2 - 4q_2 - q_1 \end{bmatrix} = 0 \quad \Rightarrow \quad \begin{cases} p_1 - 4q_1 - q_2 = 0 & (1) \\ p_2 - 4q_2 - q_1 = 0 & (2) \end{cases}$$

Substitute $q_2 = p_1 - 4q_1$ to the equation (2):

$$p_2 - 4(p_1 - 4q_1) - q_1 = p_2 - 4p_1 + 15q_1 = 0,$$

which solves for $q_1^* = \frac{4p_1 - p_2}{15}$ and $q_2^* = p_1 - 4\frac{4p_1-p_2}{15} = \frac{4p_2-p_1}{15}$. The Hessian is

$$H(\pi) = \begin{bmatrix} -4 & -1 \\ -1 & -4 \end{bmatrix}.$$

The leading principal minors are $|H_1| = -4 < 0$ and $|H_2| = |H(\pi)| = 15 > 0$, so the Hessian is negative definite and $\pi(q_1,q_2)$ is concave. Thus, $(q_1^*, q_2^*) = (\frac{4p_1-p_2}{15}, \frac{4p_2-p_1}{15})$ is the (global and local) maximizer of $\pi(q_1,q_2)$.

5. **Solution**

The two principal minors -2 and -4 are negative, hence A cannot be positive semidefinite and, consequently, cannot be positive definite either.

In order for A to be negative definite, it must be the case that the leading principal minors are as follows i) $a < 0$, ii) $|A_2| > 0$, which is true iff (if and only if) $a < -2$, and iii) $|A_3| = |A| < 0$, which is true iff $8a + 16 + 2b^2 < 0$. In sum, A is negative definite iff $a < -2$ and $4a + 8 + b^2 < 0$.

In order for A to be negative semidefinite, we must have $a \leq 0$ (first order principal minor), $a \leq -2$ and $4a + b^2 \leq 0$ (second order principal minors), and $4a + 8 + b^2 \leq 0$ (third order principal minor). In sum, A is negative semidefinite if $a \leq -2$ and $4a + 8 + b^2 \leq 0$.

In all the remaining cases, the matrix is indefinite.