Aalto University

Fractals and Kleinian groups

Shapes in Action 16th Oct 2020

Program schedule for Oct 16th

13:15 Some principles of Fractal geometry

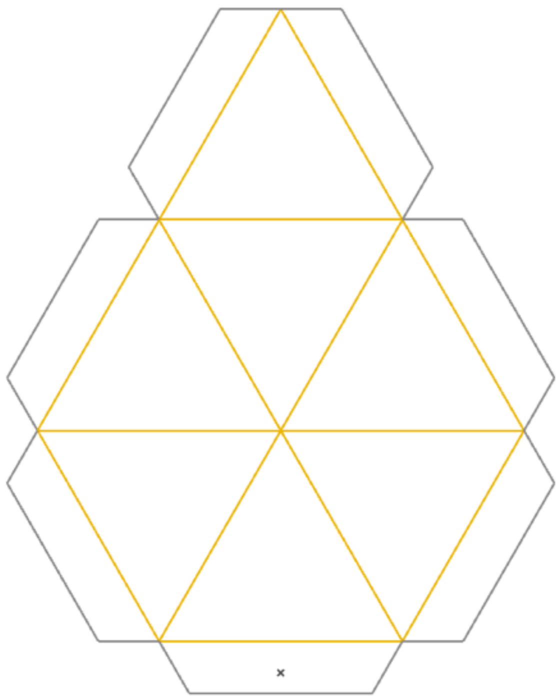
14:00 Break

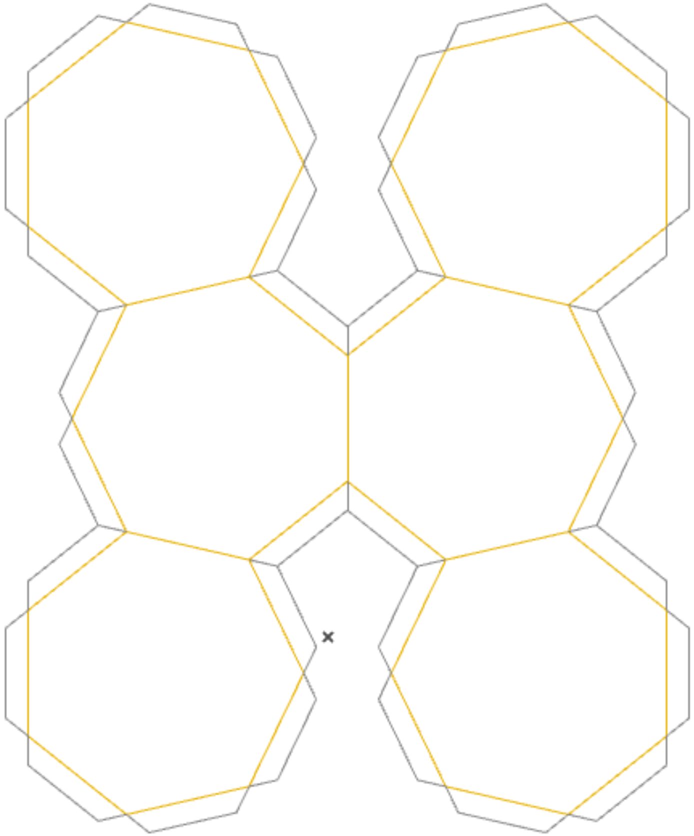
14:15 Kleinian groups and orbifolds

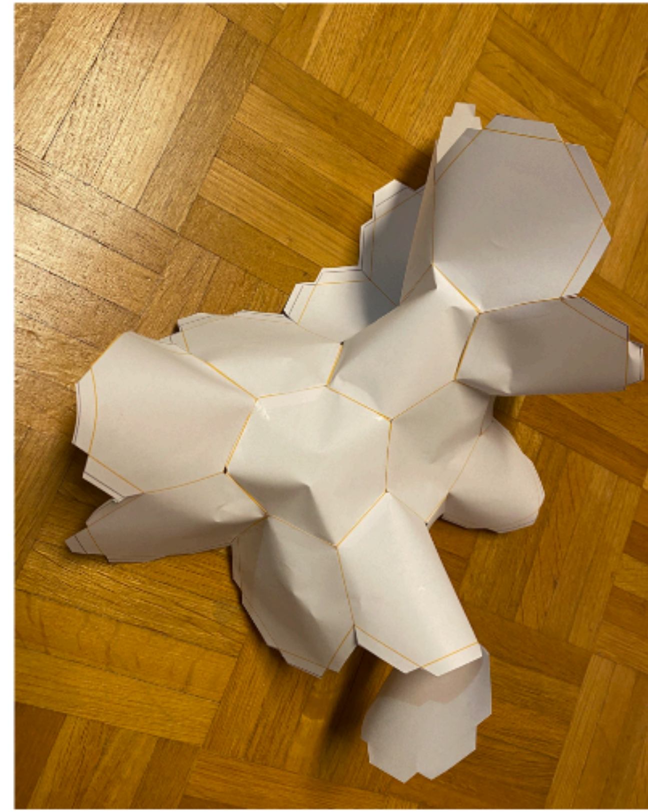
15:00 Break

15:15 Schottky groups and orbifolds

Heptagonal tiling by Severi Virolainen







Daina Taimina and a hyperbolic soccer ball

A workshop at Heureka 2017

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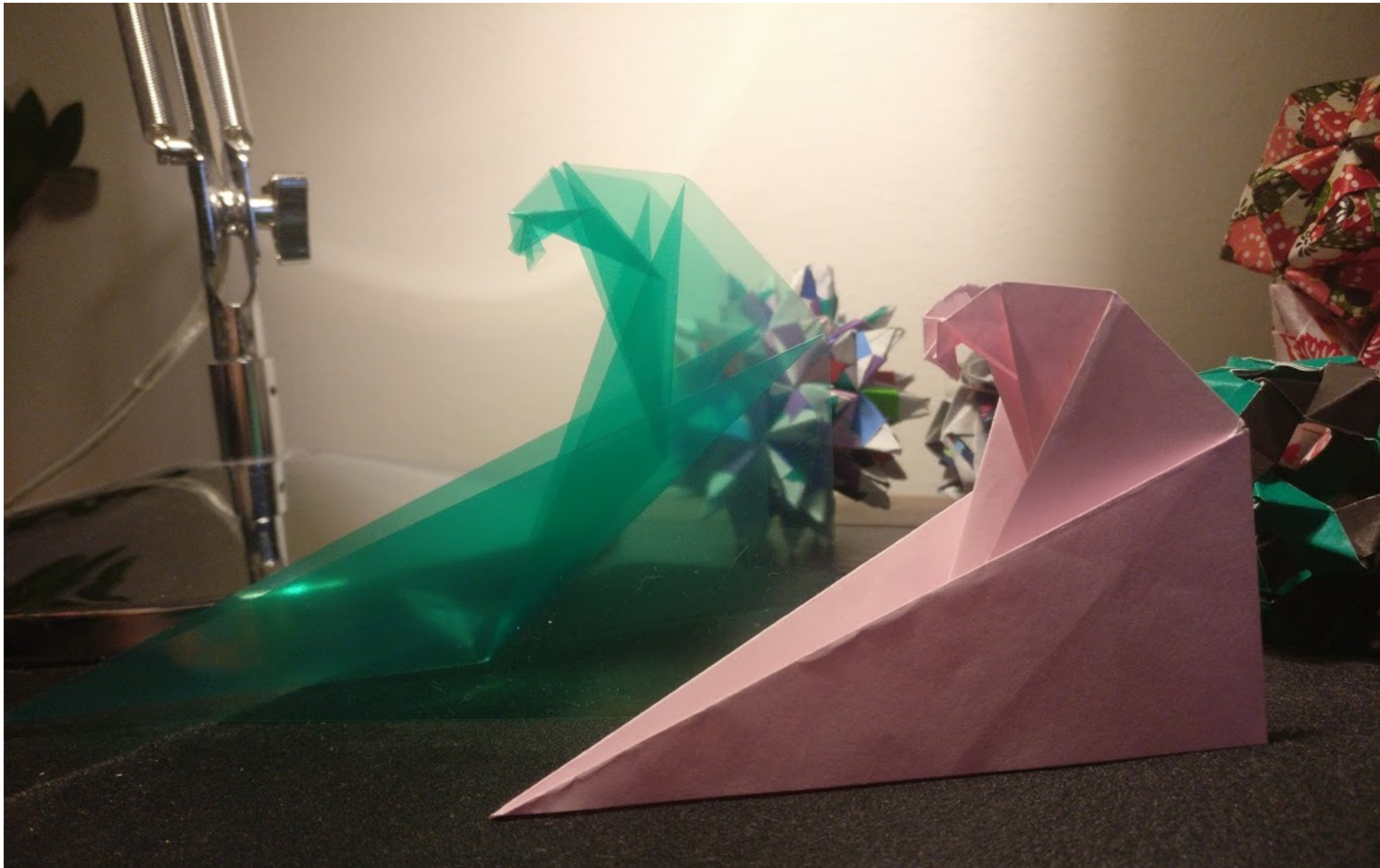
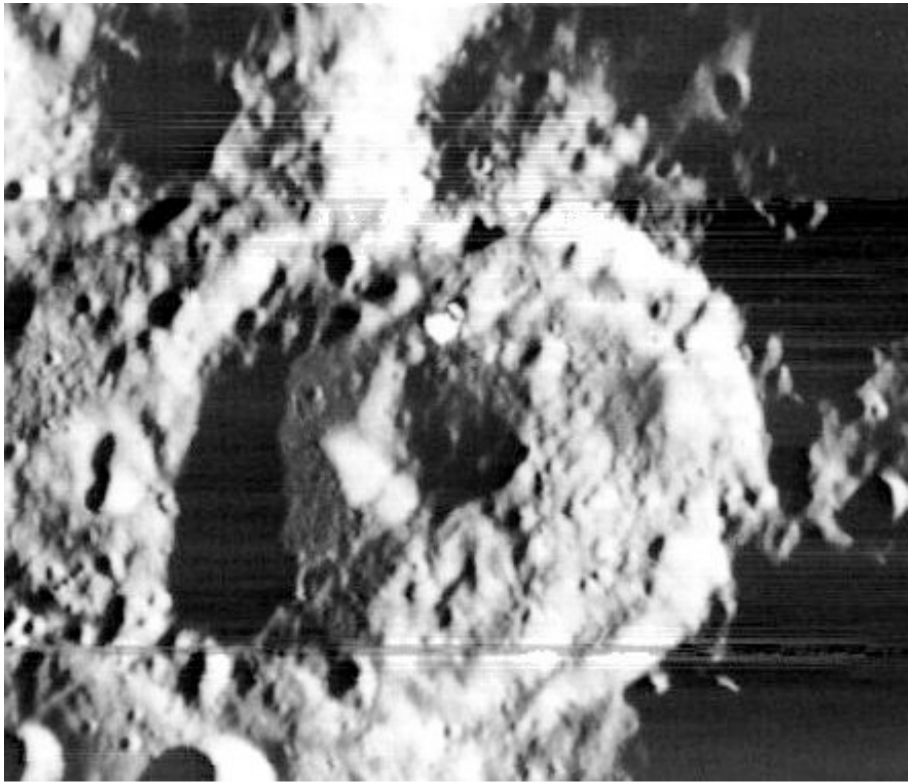
Design Factory 2017

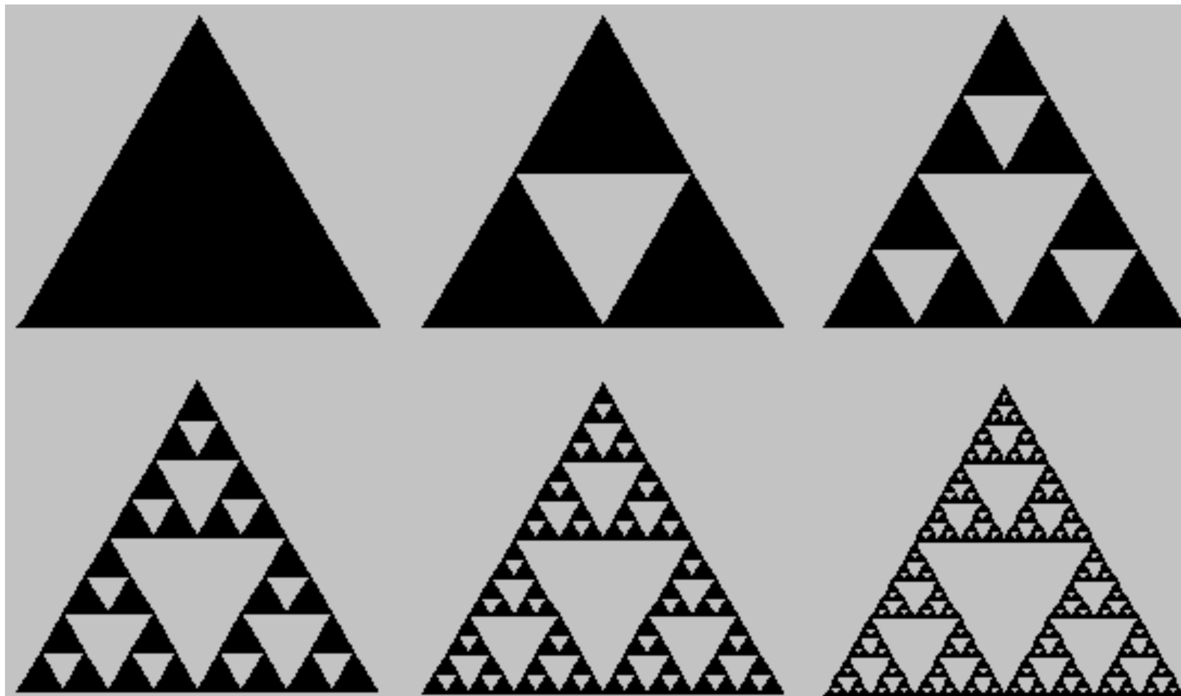
Self similar waves by Elias Seeve



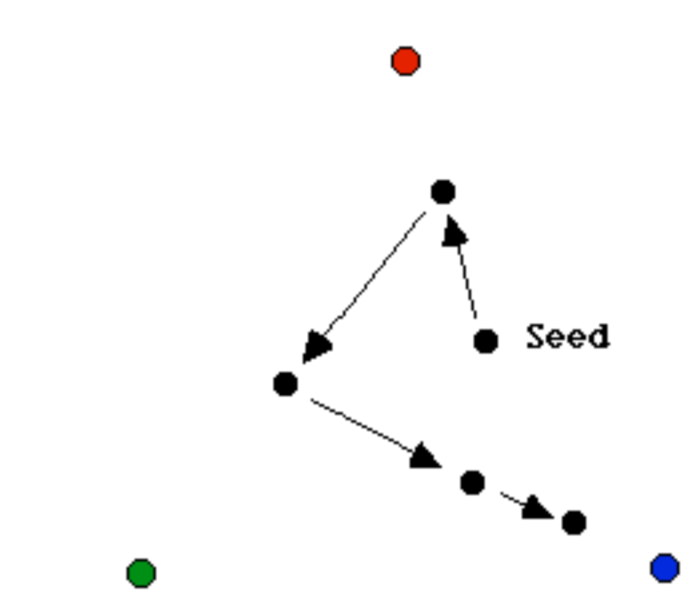
Wacław Franciszek Sierpiński 1882-1969

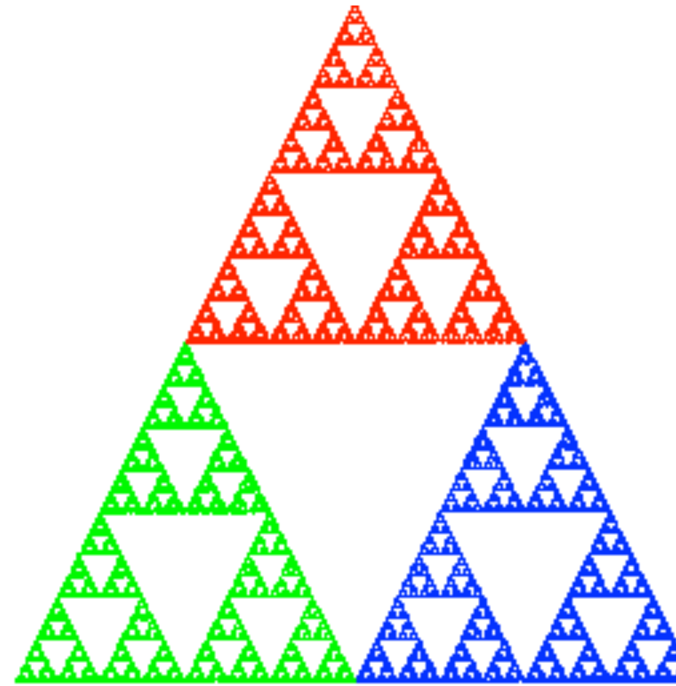


A self similar process in Sierpiński gasket (1916)



The Chaos Game (Barnsley)





Cathedral Anagni (Italy) 1104

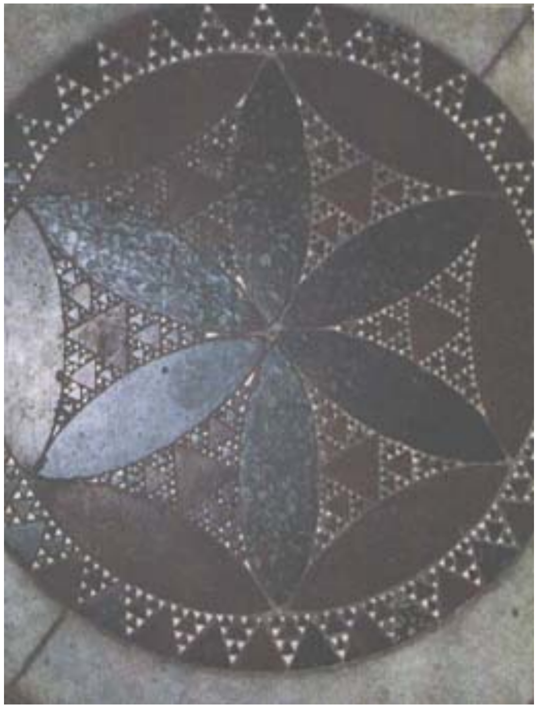
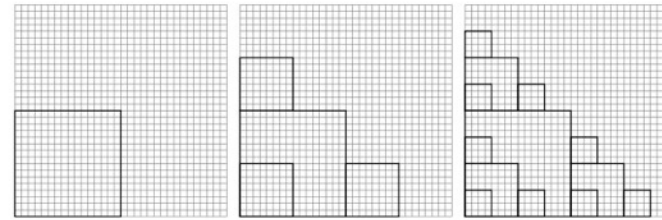
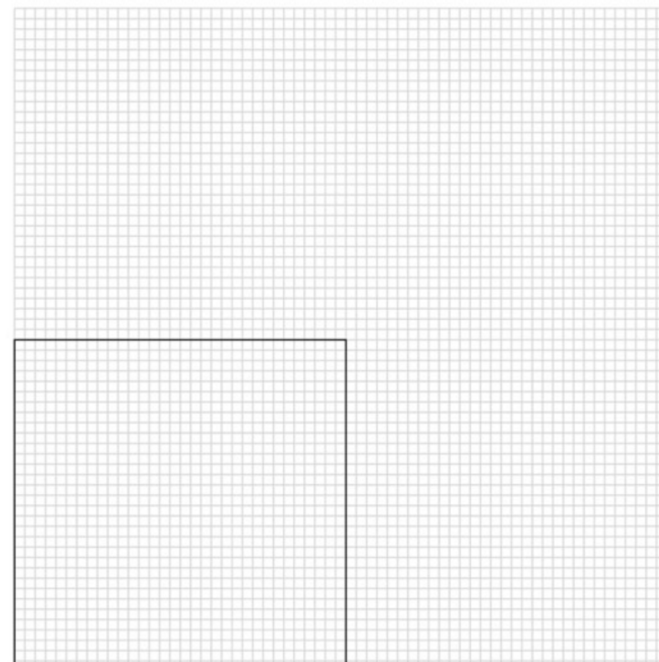
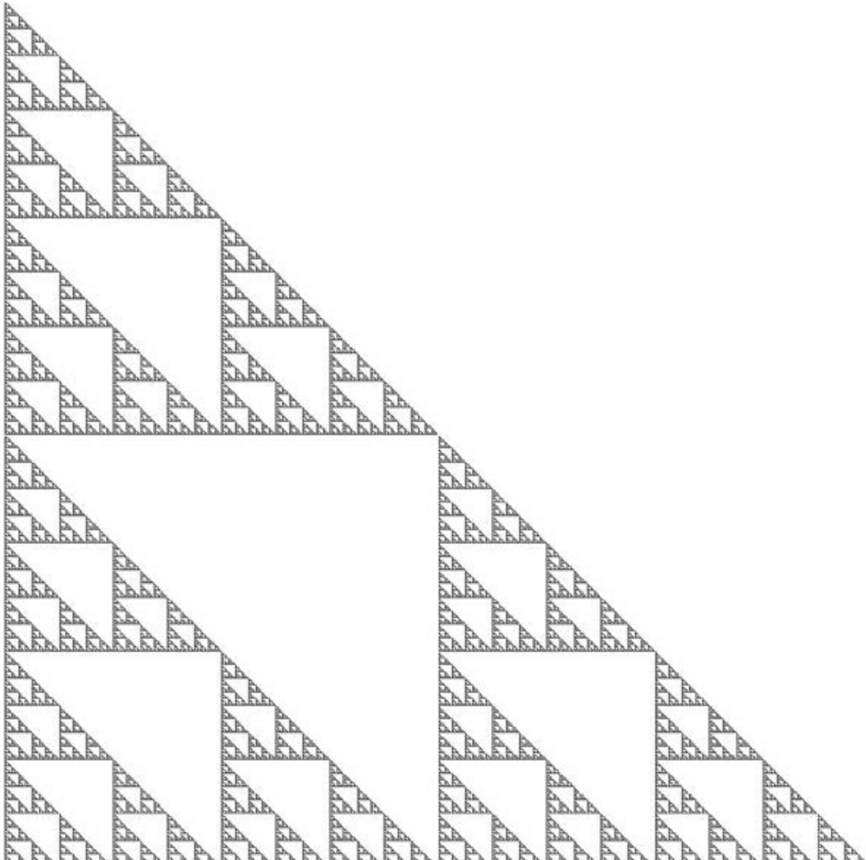


Fig. 6 SS. Giovanni e Paolo (13th century), Rome

Santa Maria in Cosmedin, Rome



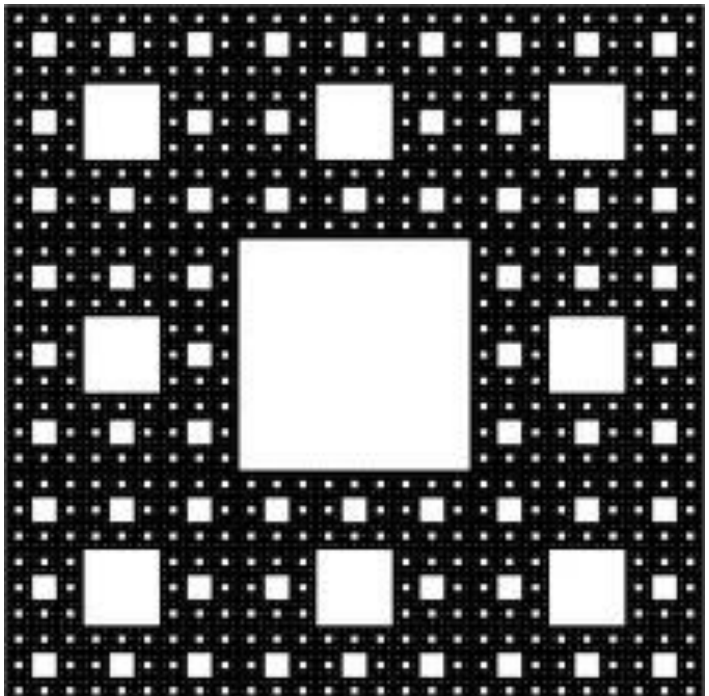


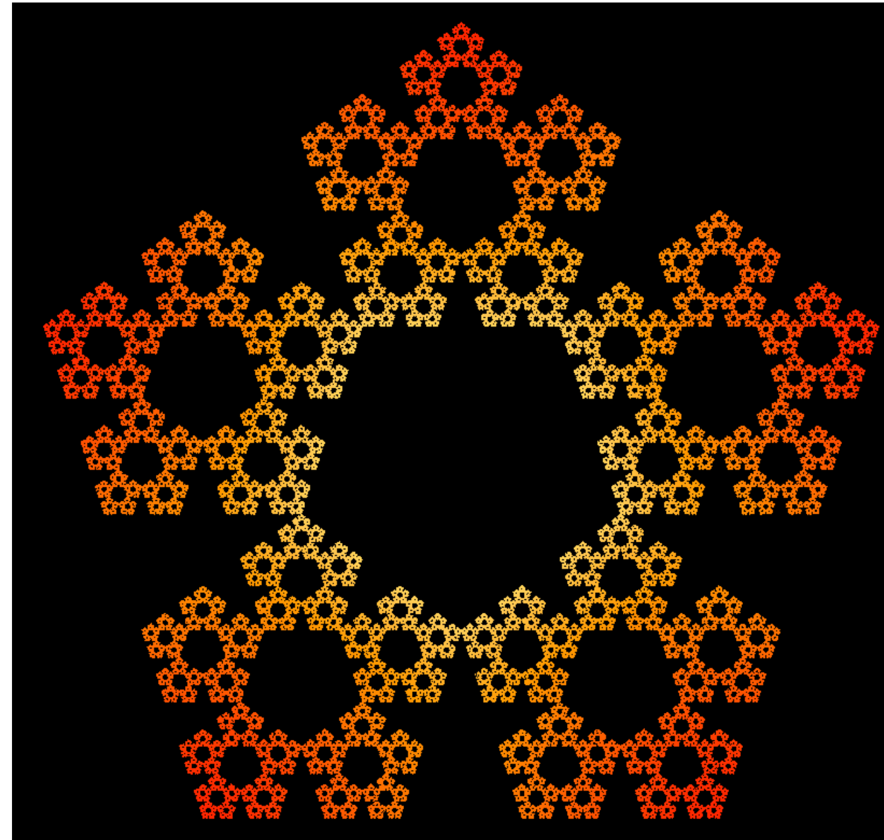


Escher's studies of Sierpinski gasket-type patterns

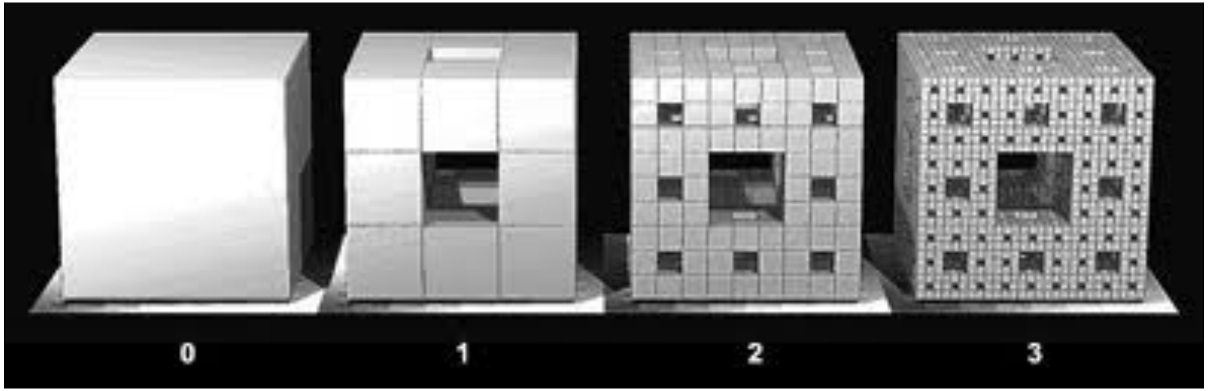
On twelfth-century pulpit of Ravello Cathedral, 1923

Sierpiński Carpet and generalizations

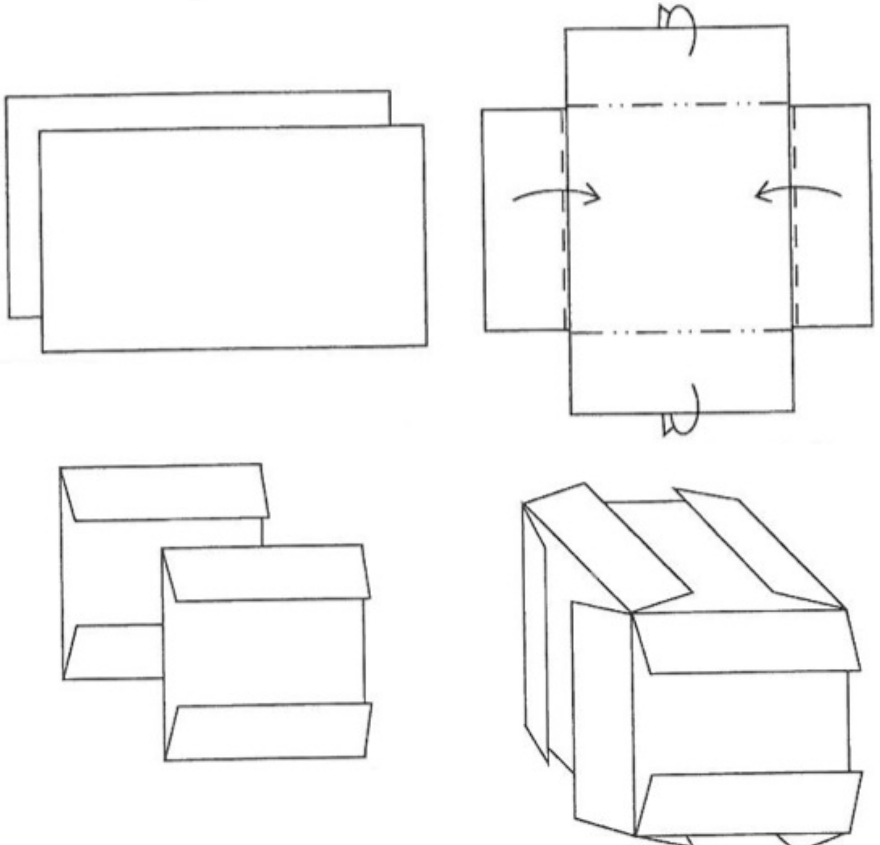


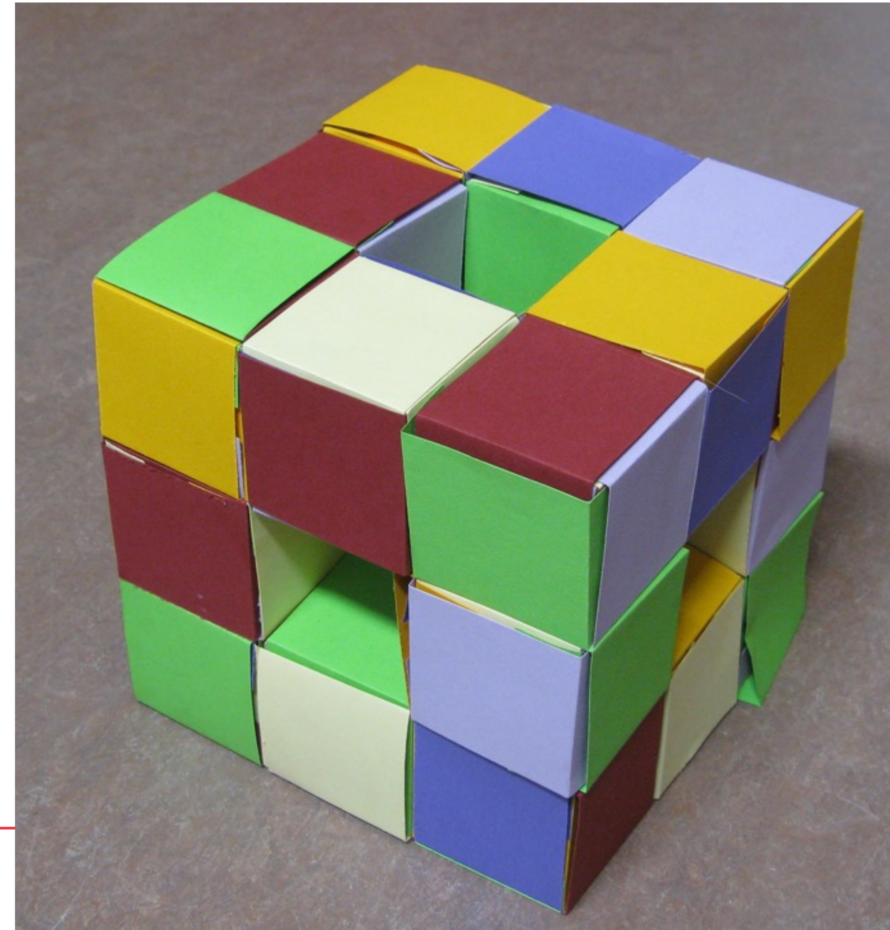


Karl Menger 1902-1985 and his sponge 1926



Menger sponge via business card origami





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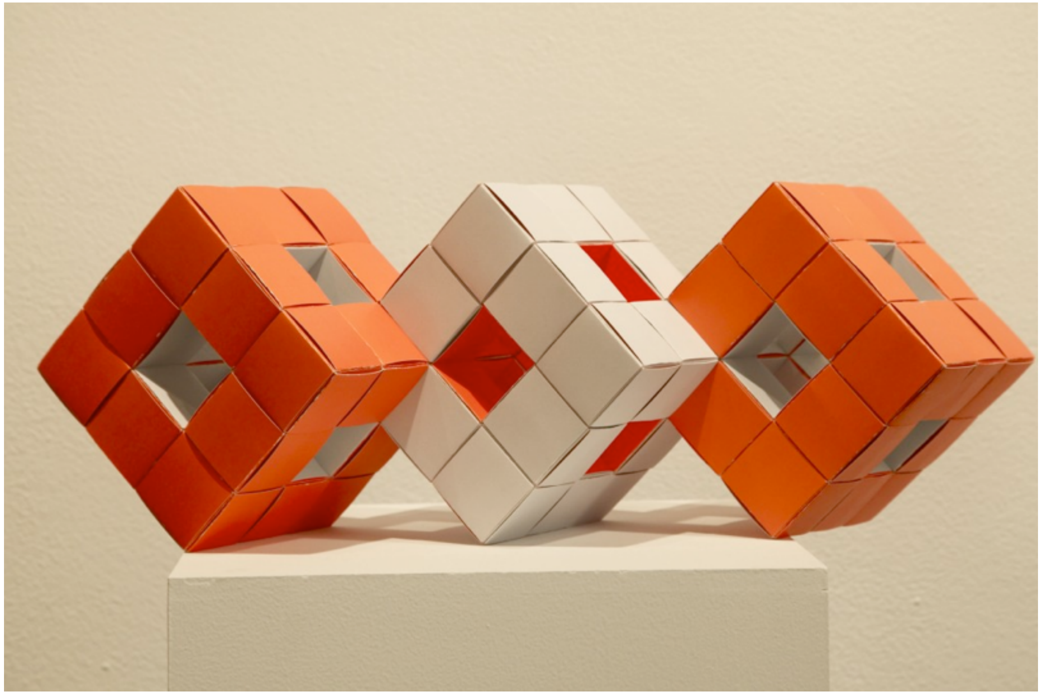
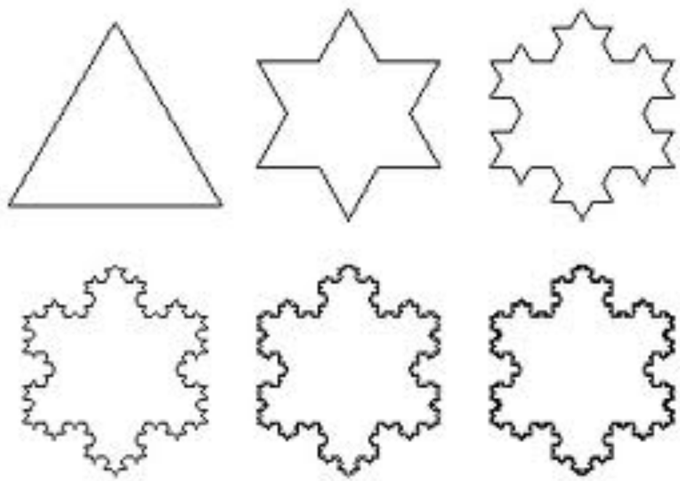


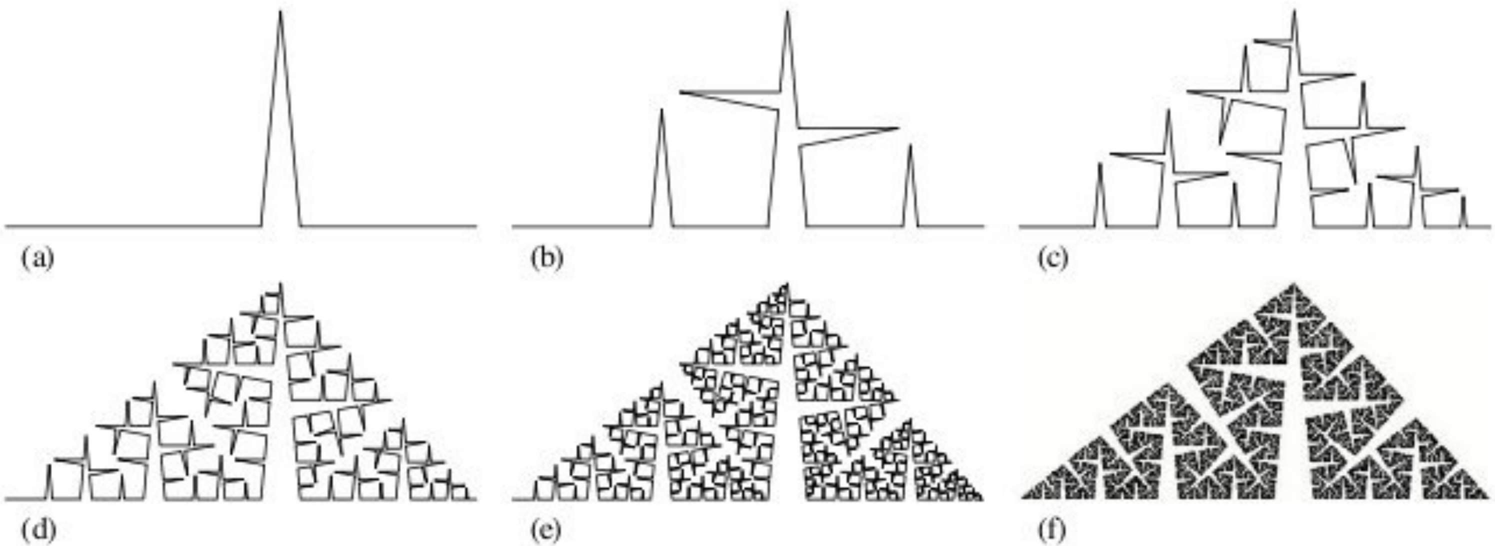
Three interlinked Level One Menger Sponges, by Margaret Wertheim.

Jeannine Mosely
66048 business cards

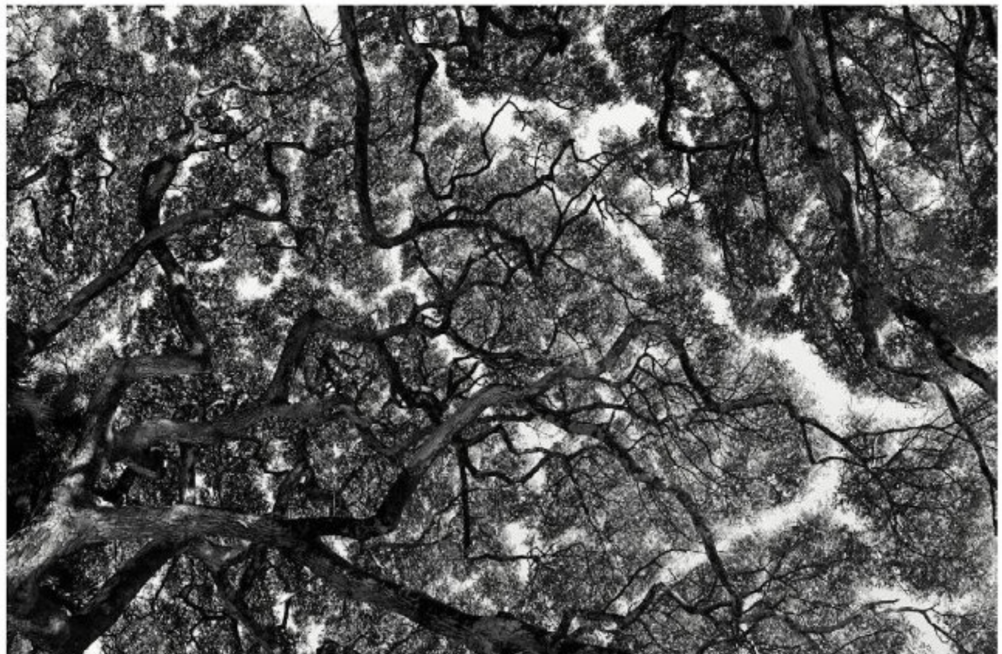
Niels Fabian Helge von Koch (1870-1924) and his snowflake (1904)



Evolution à la Mandelbrot



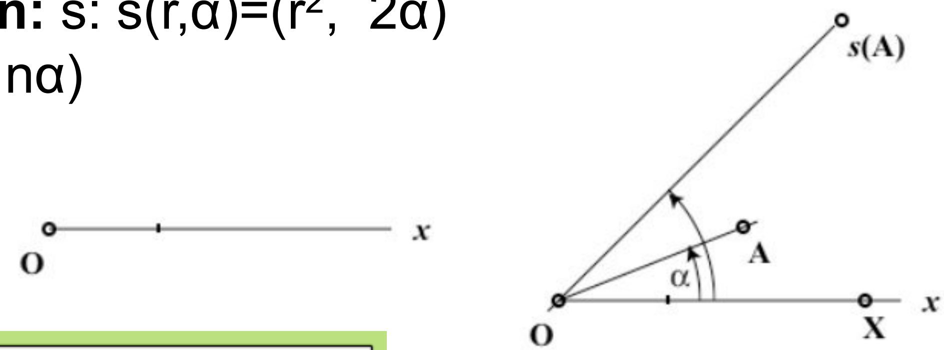
Canopy, by Craig Harris 2008

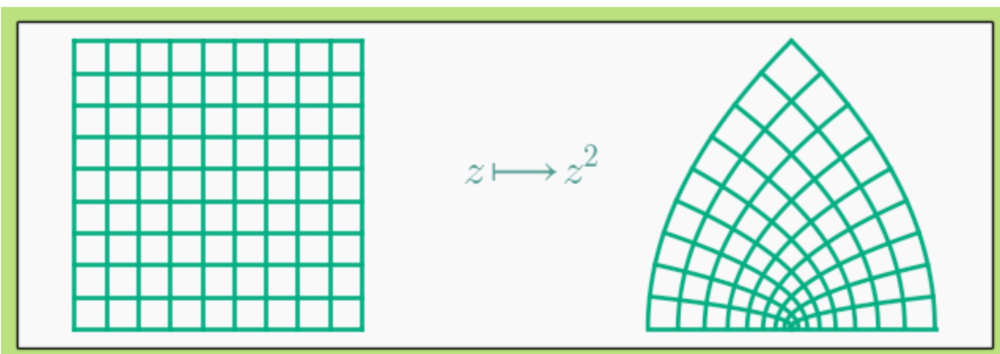


Gaston Maurice Julia 1893-1978

Iteration of planar rational functions

Squaring transformation: s: s(r,α)=(r^2, 2α)
Power n: pn: pn(r,α)=(r^n, nα)

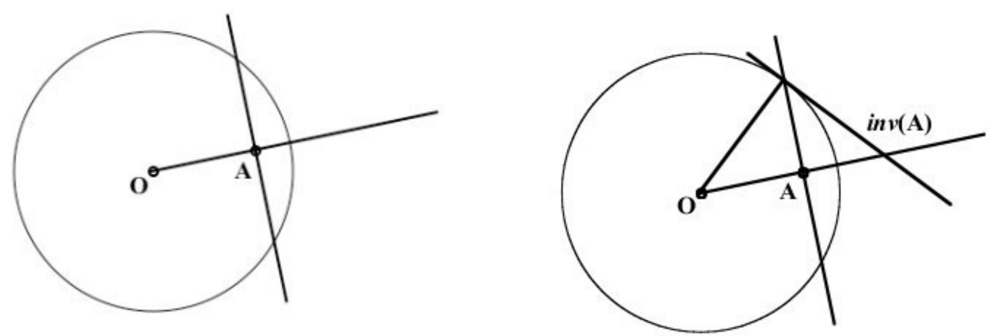




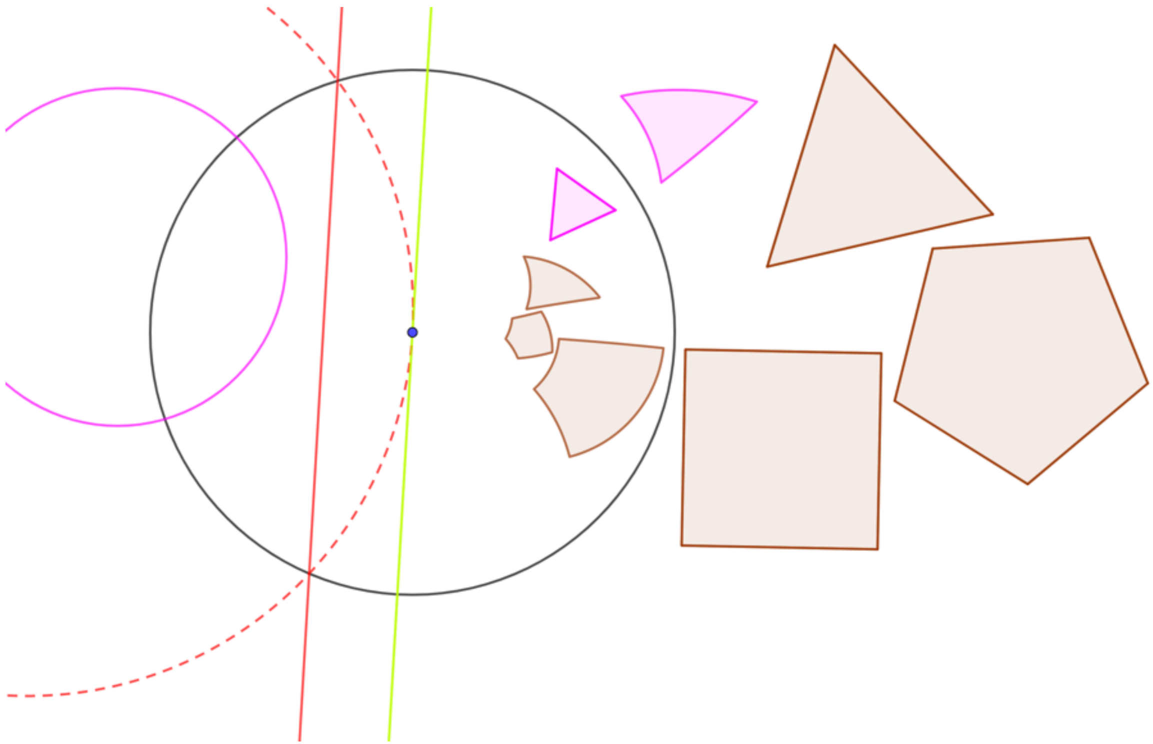
Preserves angles outside the origin !

… and (geometric) inversion (=reflection) in a circle

Planar rational maps are compositions of similarities, powers and inversions.



Some (amazing!) properties of reflection wrt a circle



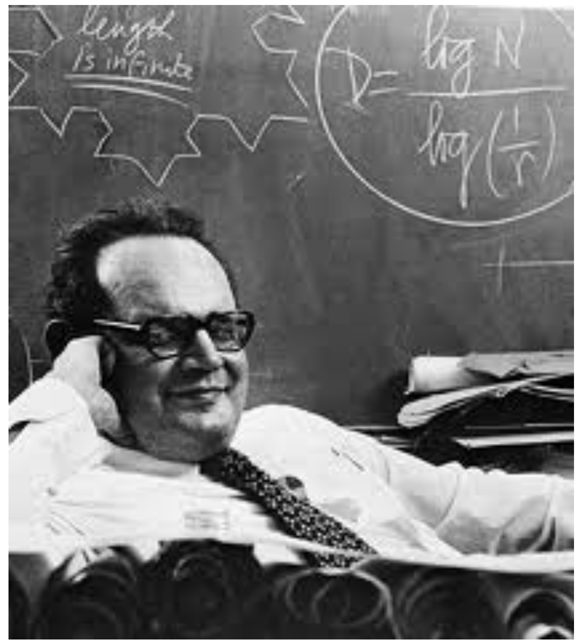
Pierre Joseph Louis Fatou 1878-1929

- ‘Fatou set’
- Holomorphic dynamics

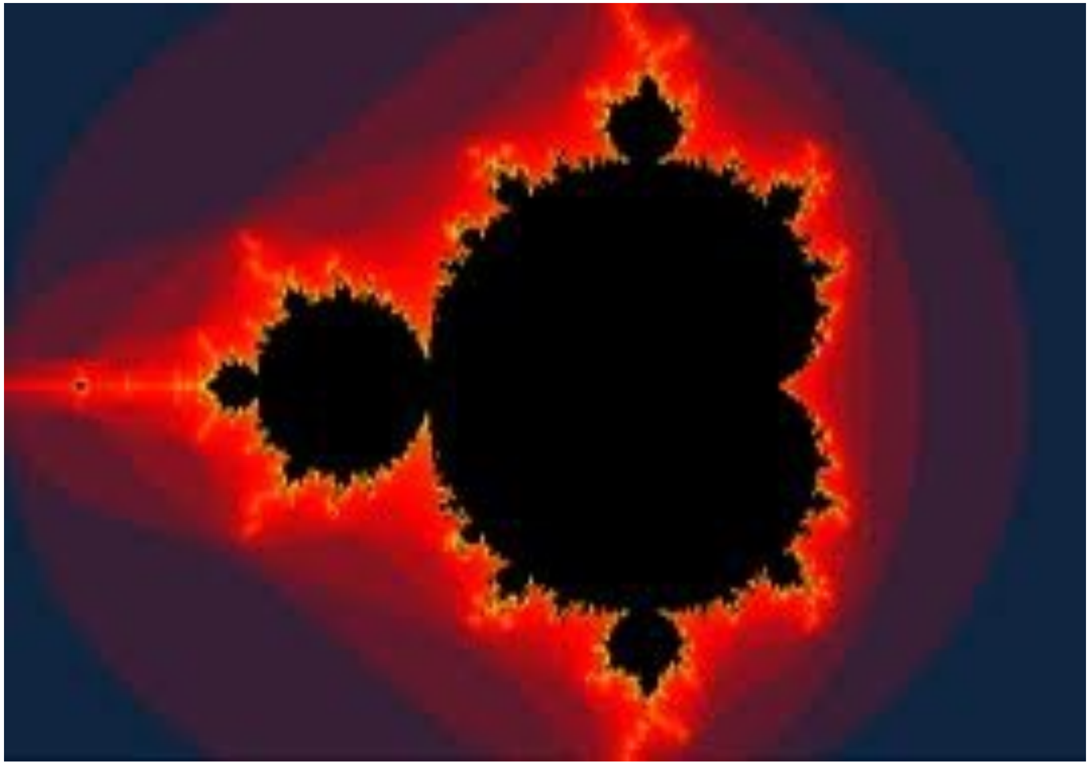
Benoit Mandelbrot 1924-2010

Mandelbrot coined (70's) the word 'fractal' to explain self similar objects

Fractus= fractured, broken



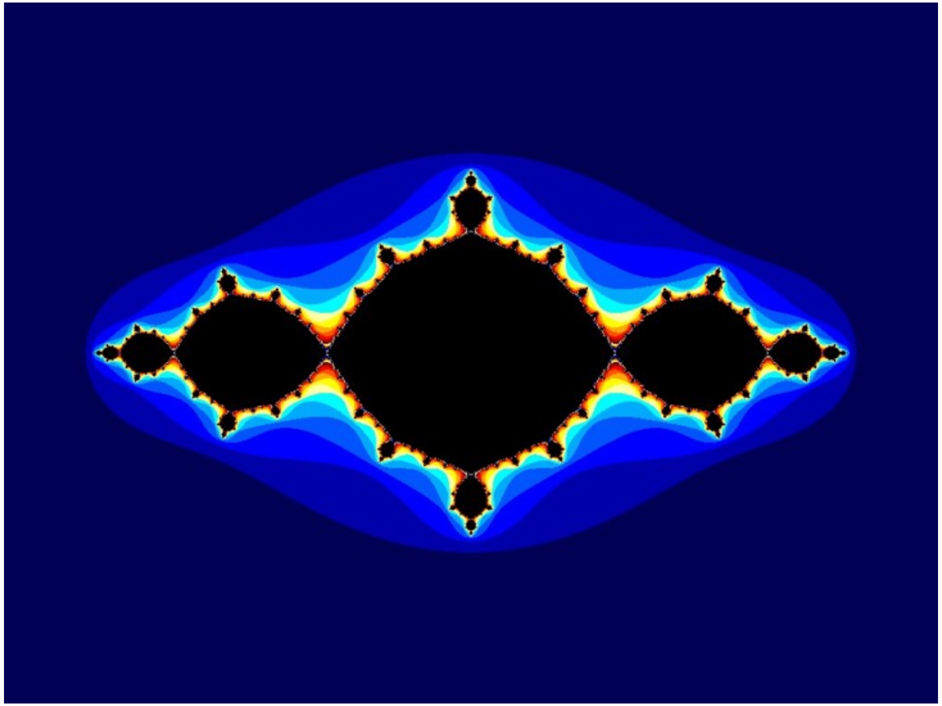
Mandelbrot set



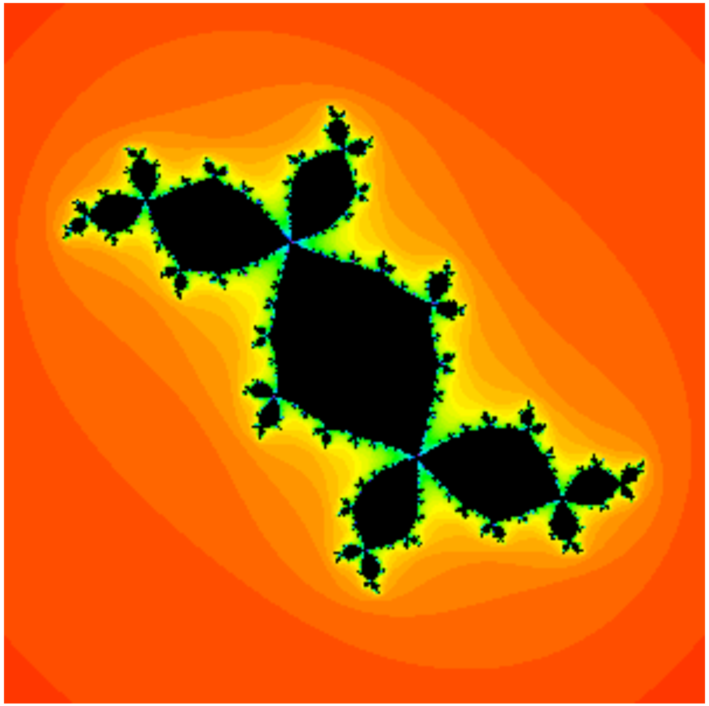
Parameter space for $C=(C_x,C_y)$ under f: $f(r,\alpha)=(r^2,2\alpha)+C$

Look at C=0 once more!

C=-1, Julia/Fatou set

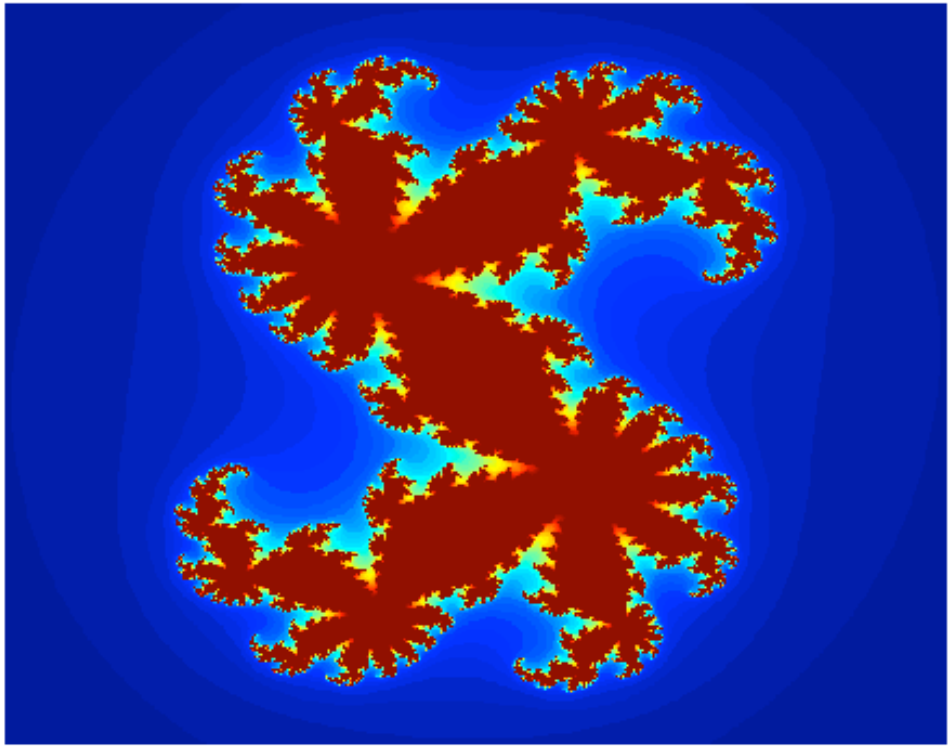


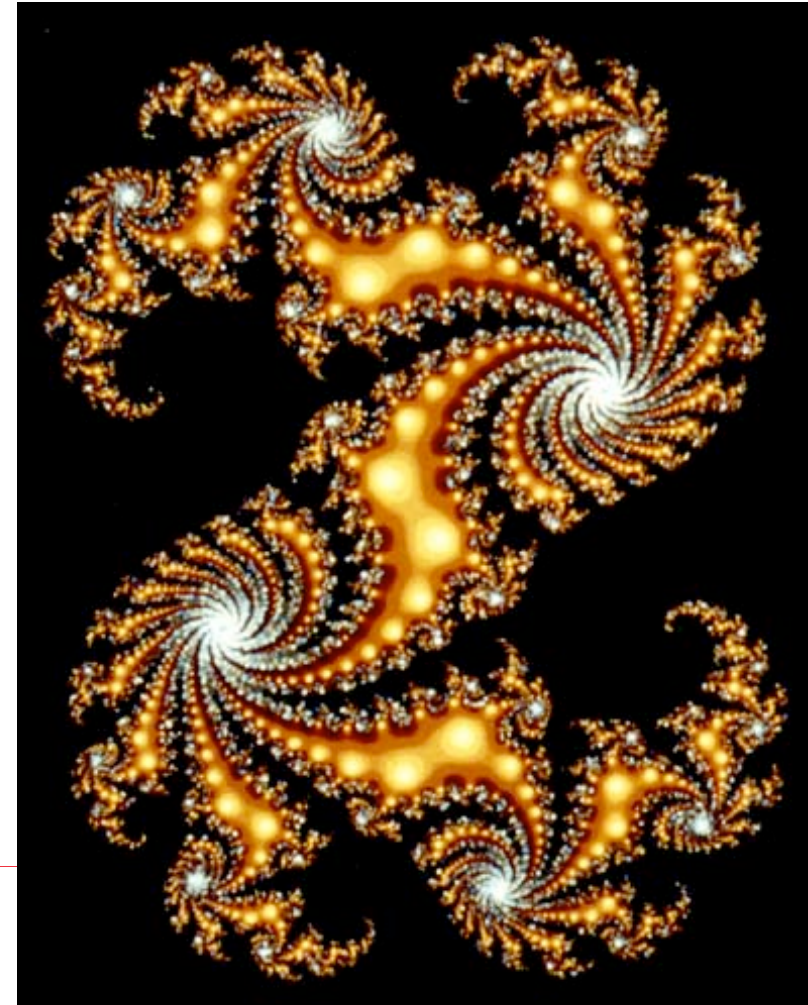
Douady's rabbit (Adrien Douady 1935-2006)



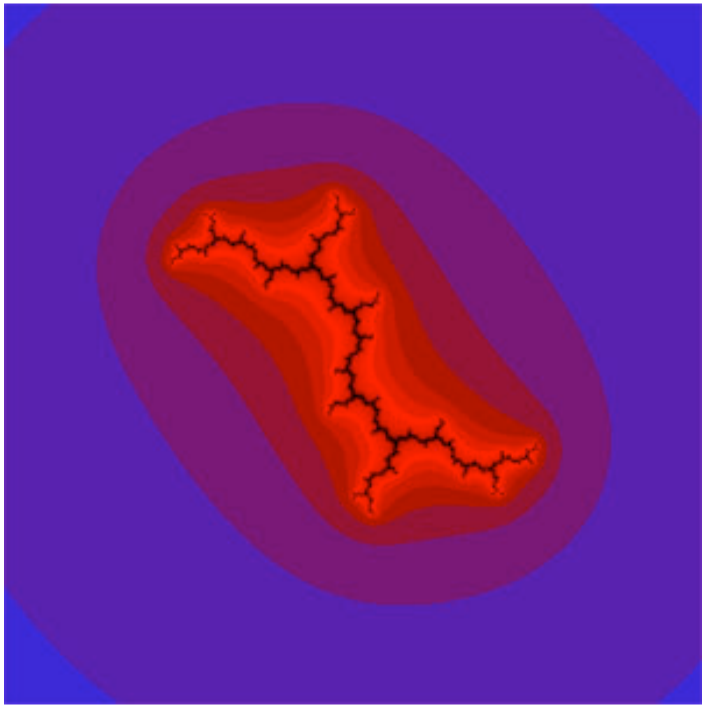
C = -0.12256+0.74486*i*.

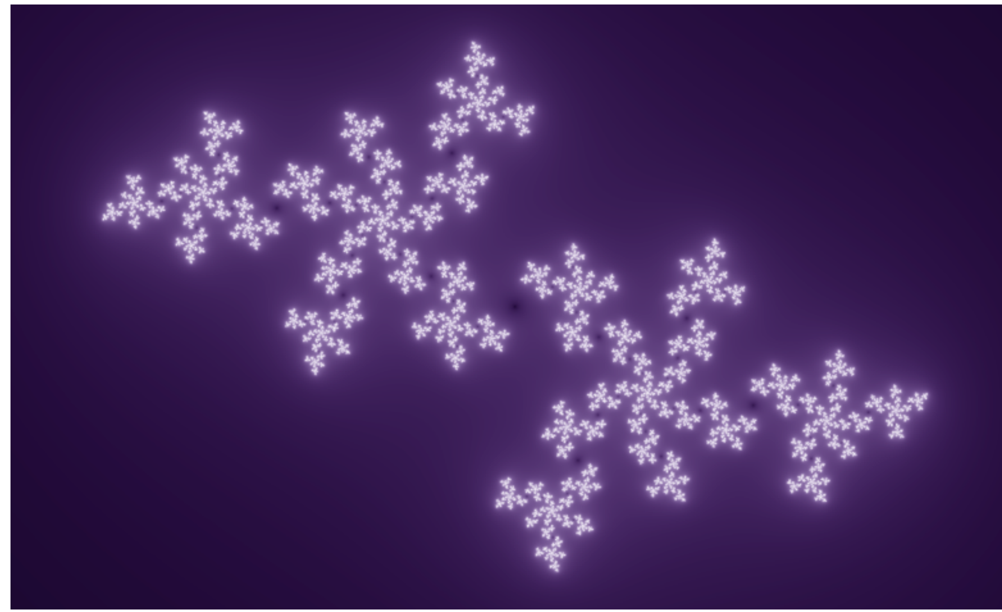
Dragon c=0.360284+0.100376i



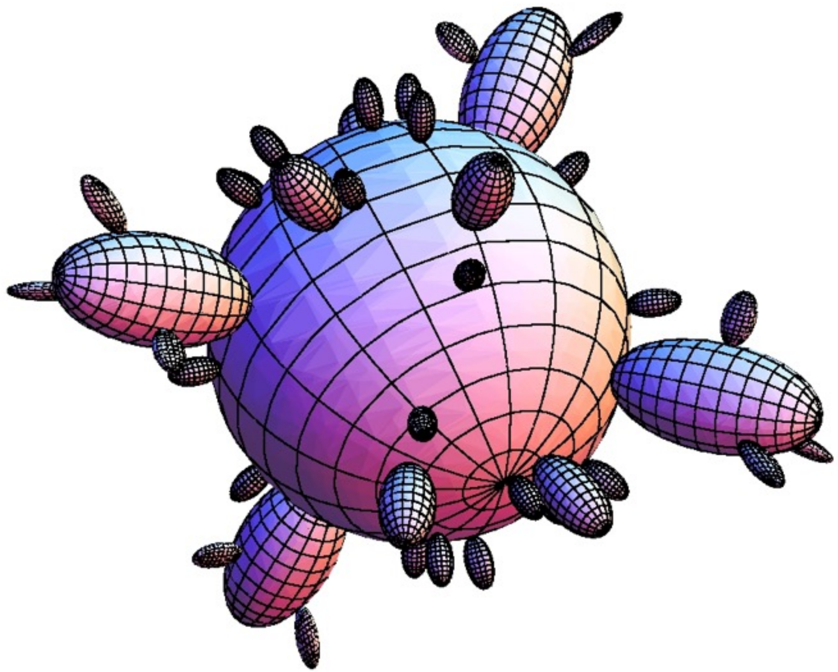


Dendrite and Cantor dust

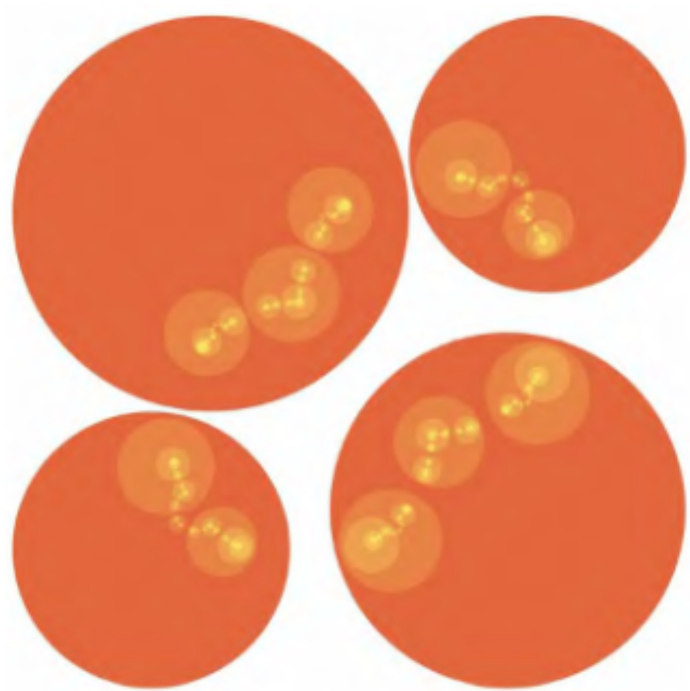




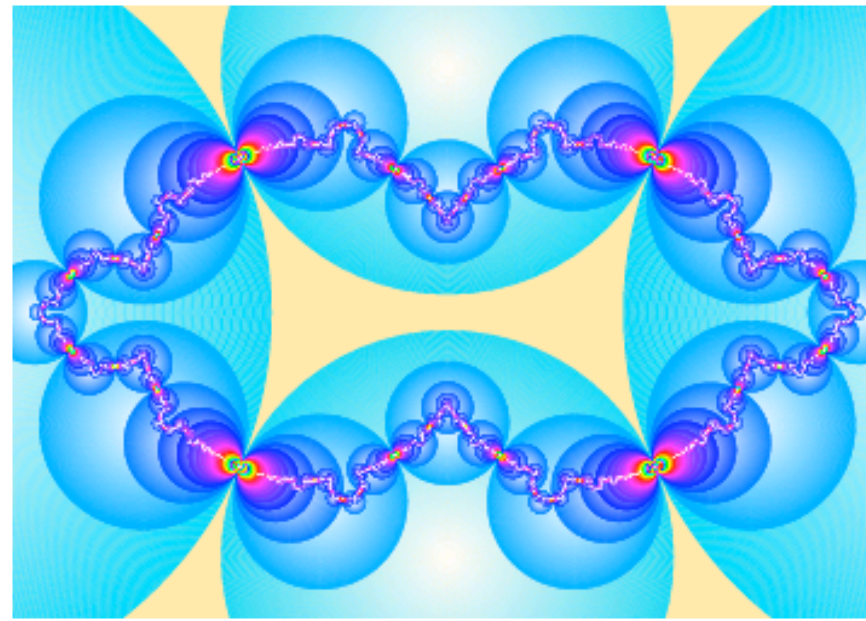
Higher dimensional analogues of complex polynomials (joint work in progress with G. Martin)



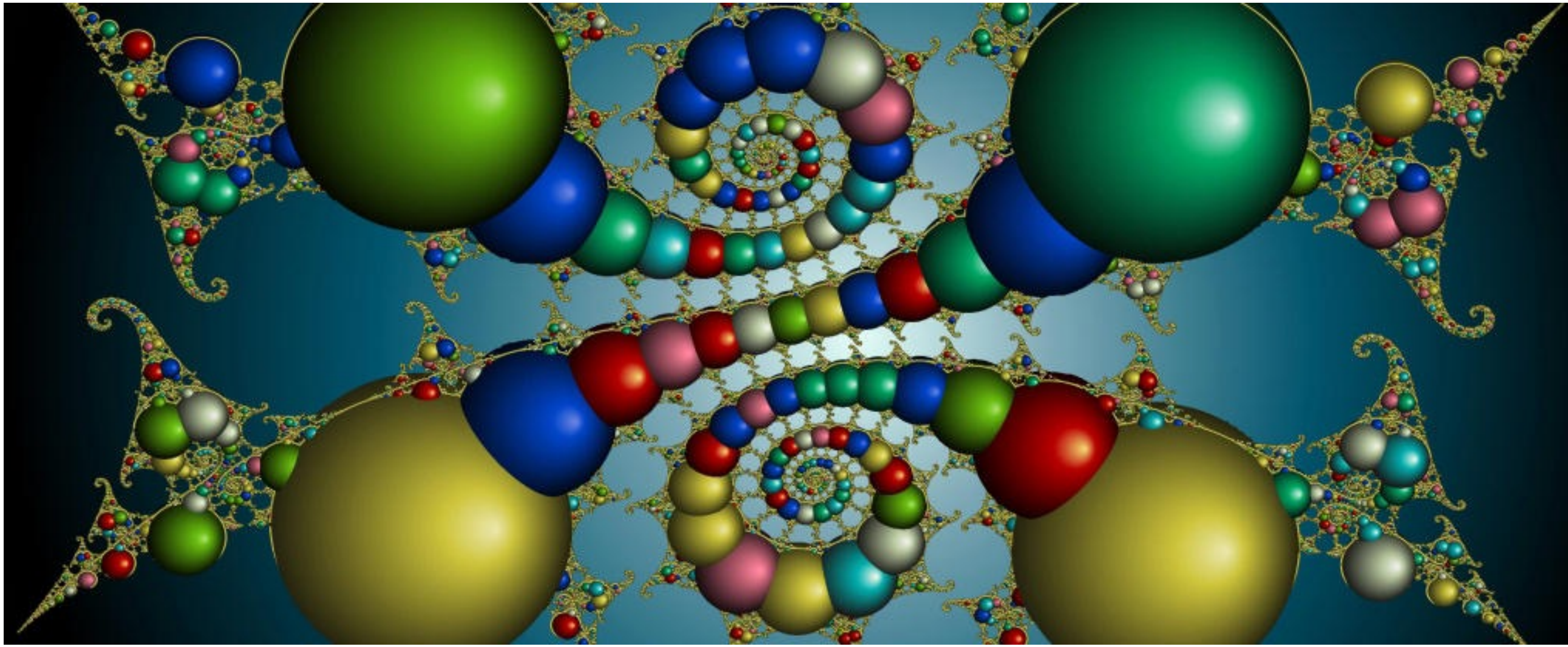
Kleinian groups



Ex: pairing of circles under Möbius transformations

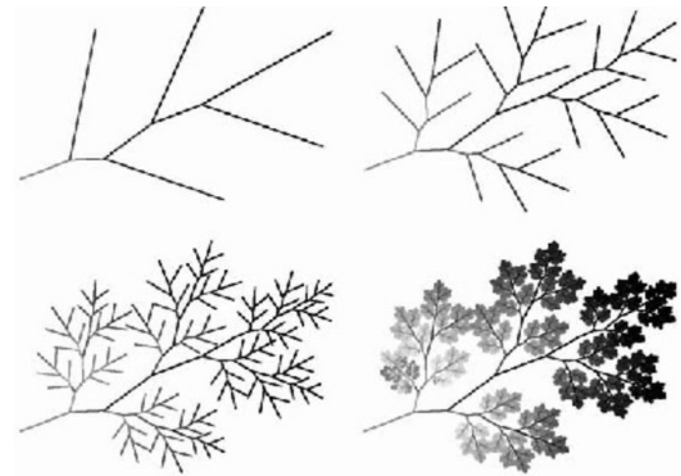


An artistic interpretation by Jos Leys

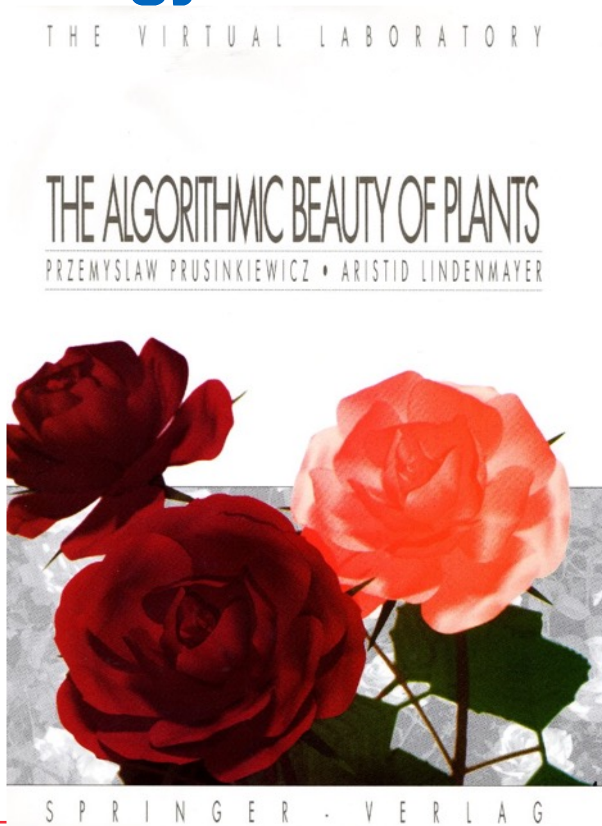


Fractals in approximating natural forms

Change from mechanical/geometrical to organic by using mathematical algorithm



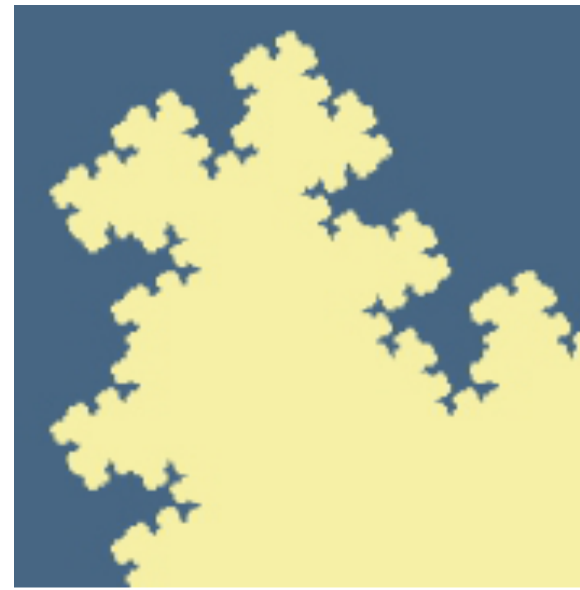
Aristid Lindenmayer 1925-1989 (L-systems) in plant biology



Artistic inventions of fractals a bit earlier and its reproduction by a process called Iterated Function System IFS.

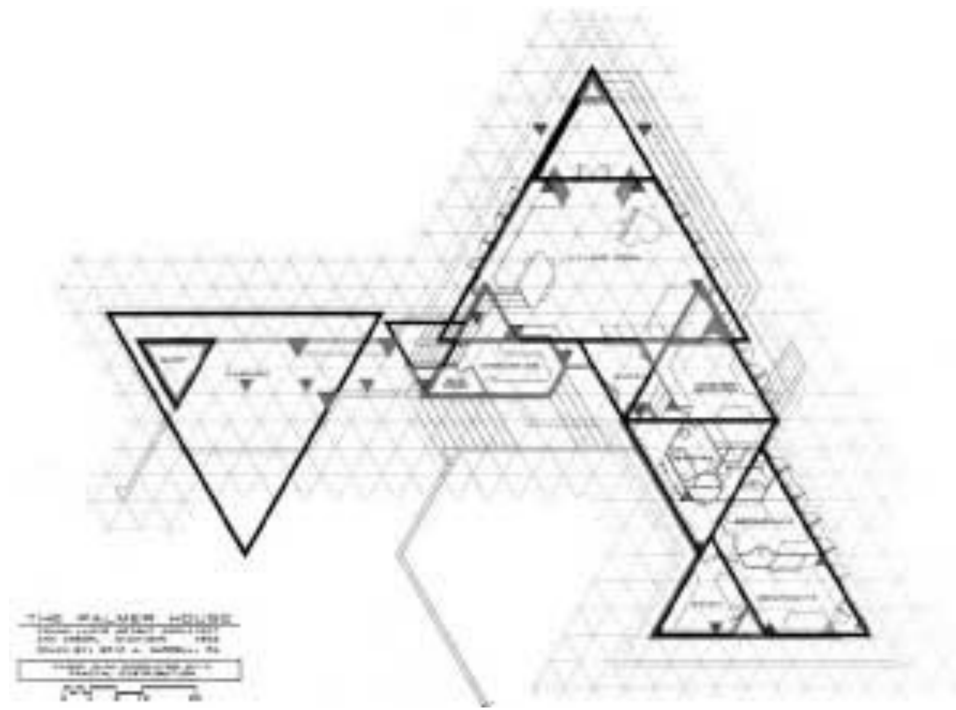
'Driving Rain' by Ando Hiroshige (1797-1858)

‘A Thousand Pictures of the Sea’ by Katsushika Hokusai (1817-1859) and IFS again

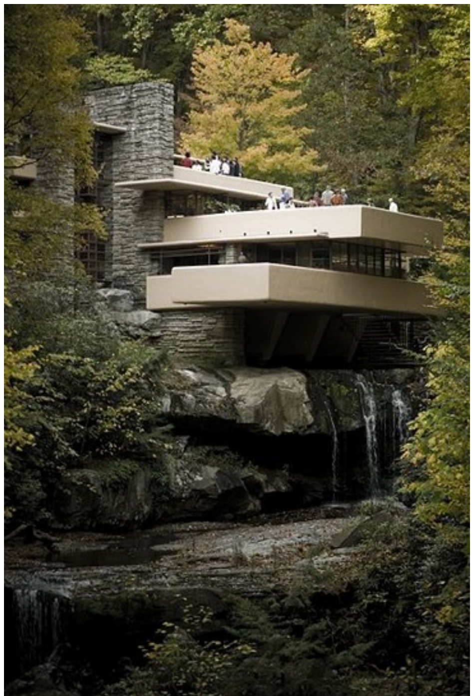


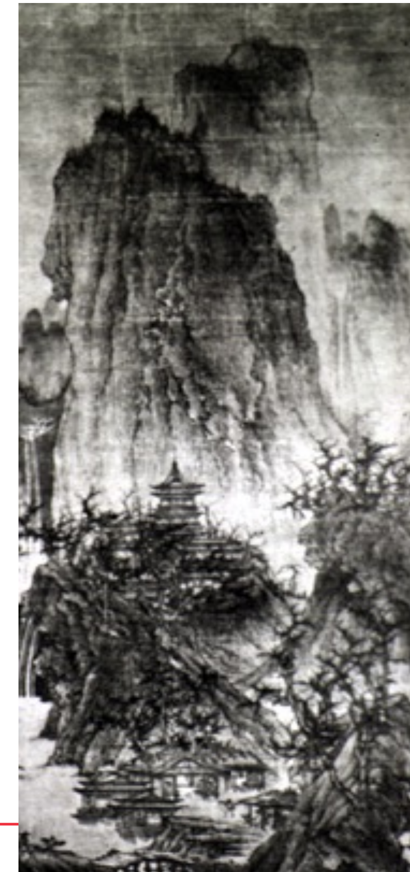
Frank Lloyd Wright (1867-1959)

Palmer house in Michigan (1950-51)



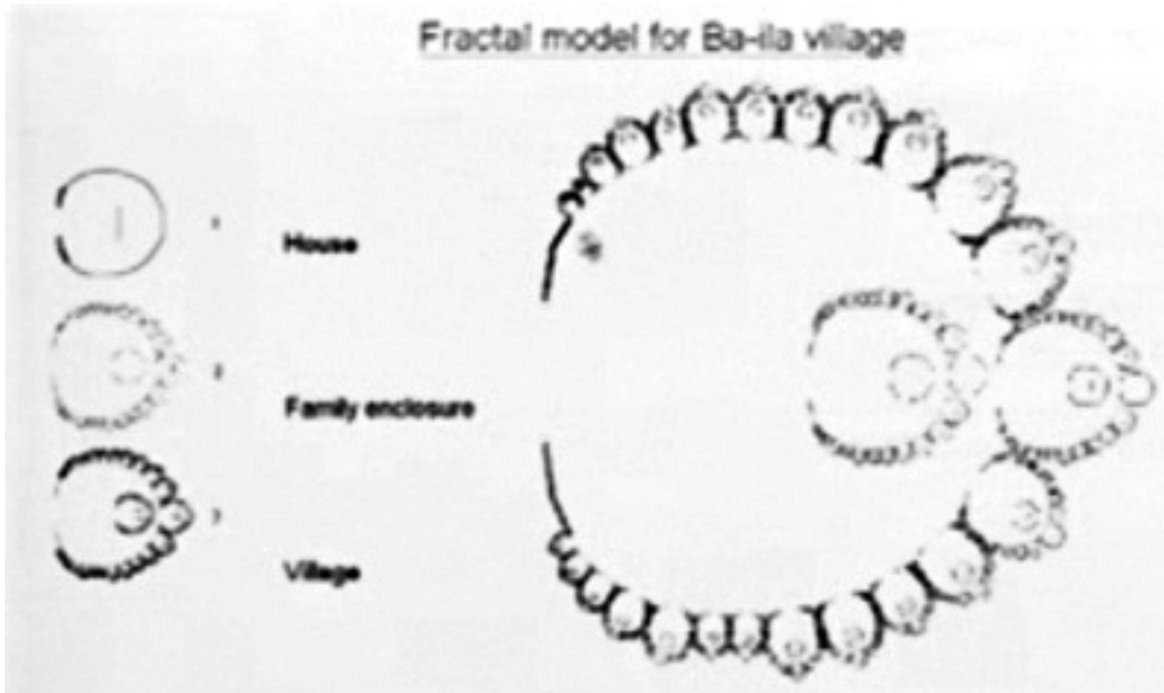
Fallingwater, Pennsylvania (1937) and Li Cheng (960-1127): Solitary Temple





African fractals: Ron Eglash

http//www.ted.com/talks/ron_eglash_on_african_fractals.html



Felix Christian Klein (1849-1925)

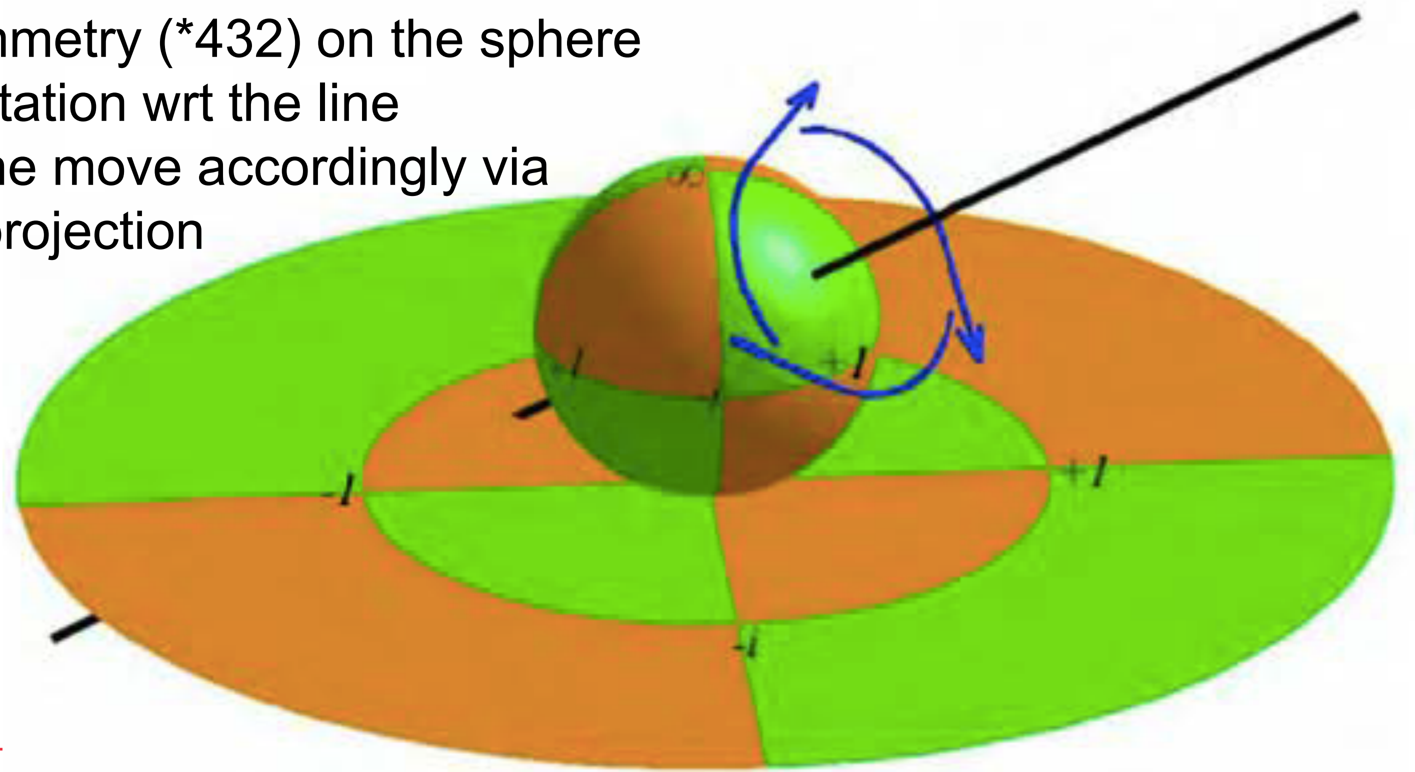
- **Erlangen program (1872):** Proposal for a radical extension of the view of symmetry
- Geometry = *study of the properties of a space which are invariant under a given group of transformations*
- **Ingredients** to study geometry:
  - Objects (triangles, circles, fractals,…)
  - *Movements* **(not just rigid motions of the Euclidean space)**

=> Much wider setups by using the ***group*** concept

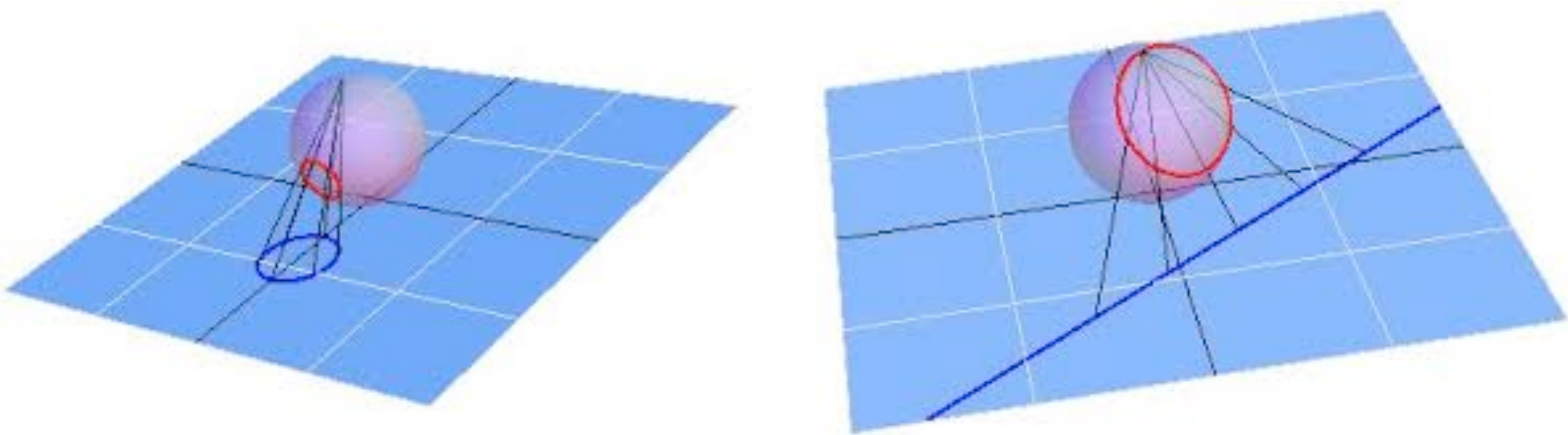
Groups, tiling, inversions and stereographic projection in one picture

- Octahedral symmetry (*432) on the sphere
- 120^{o} degree rotation wrt the line
- Tiles in the plane move accordingly via stereographic projection

=> Induces (new) symmetry for the plane



Circles and lines in the plane correspond circles on the sphere via stereographic projection



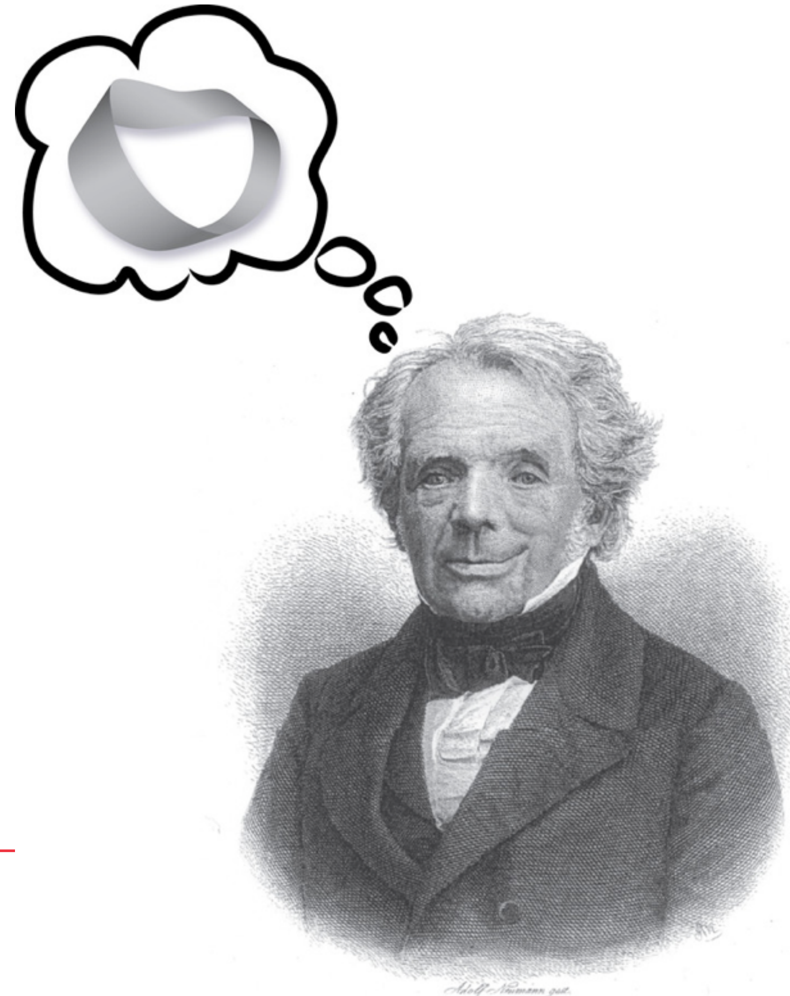
Symmetries generated by Möbius maps

All compositions of

- Translations
- Rotations
- Reflections wrt lines and circles
- Scalings

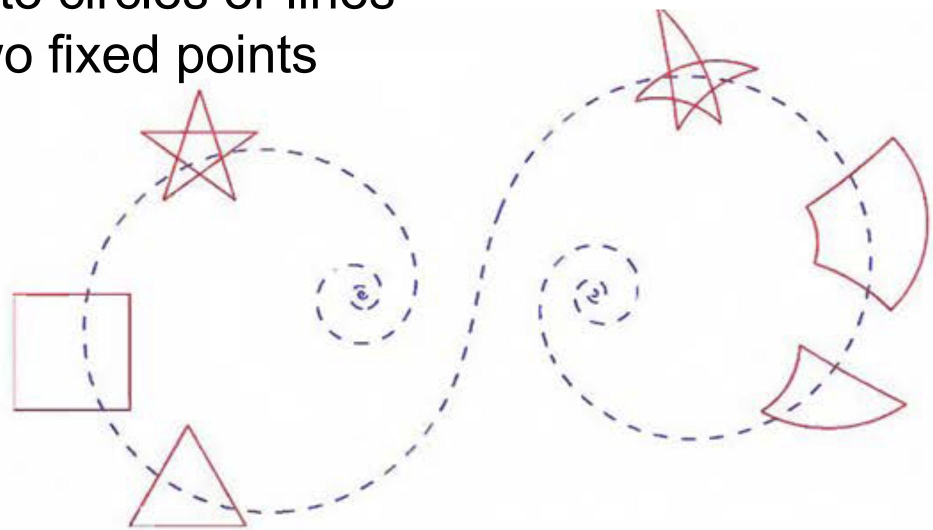
Note: These induce bijective maps (via stereographic projection) on the sphere !

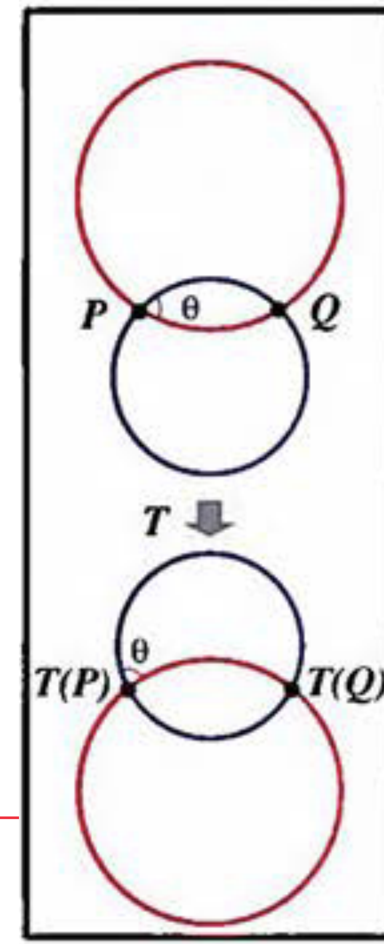
August Ferdinand Möbius (1790-1868)



Emphasis on geometrical and dynamical effects of Möbius maps

- Angle preserving maps (=conformal)
- Map circles to circles or lines
- Map lines to circles or lines
- At most two fixed points

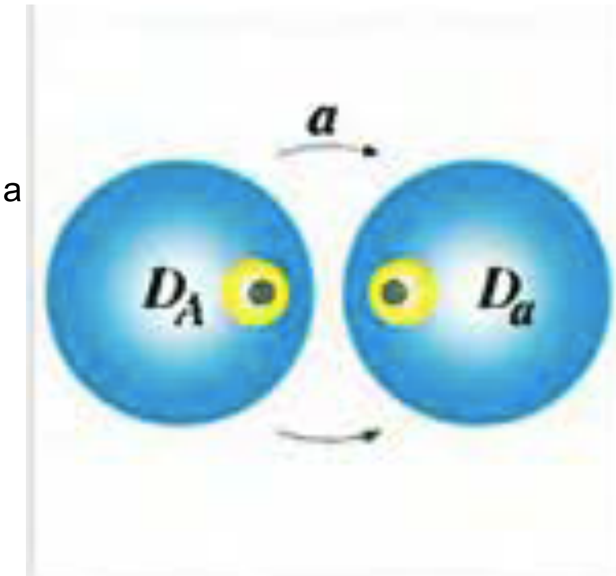




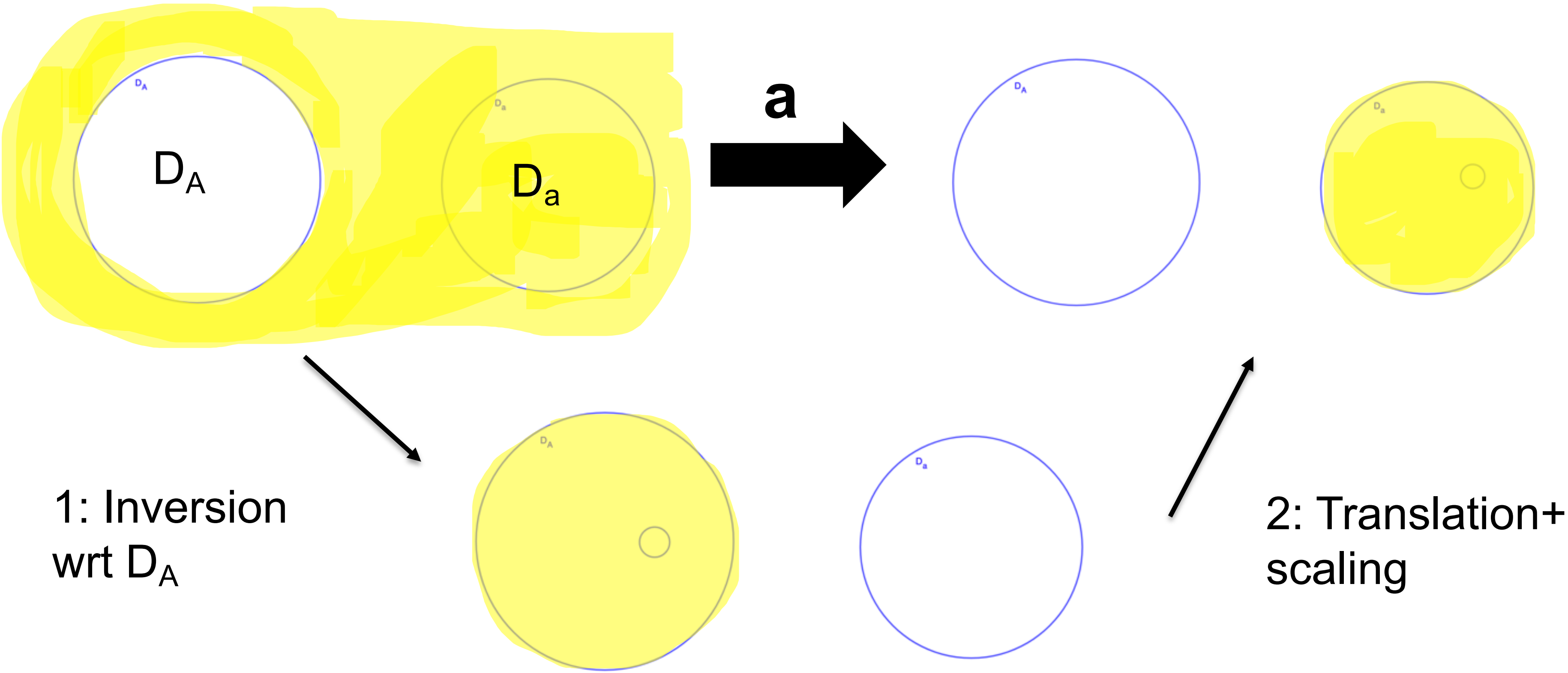
What patterns are simultaneously symmetrical under two Möbius maps ?

Step 1: Circle pairing

- Map *outside* of D_A onto the *inside* of D_a
- Induces an attracting fixed point inside D_a and a repelling fixed point inside D_A
- Iterating a shrinks D_a to smaller disks containing the attracting fixed point
- Denote $A=a^{-1}$
- Disk D_x will then contain attracting fixed point of x

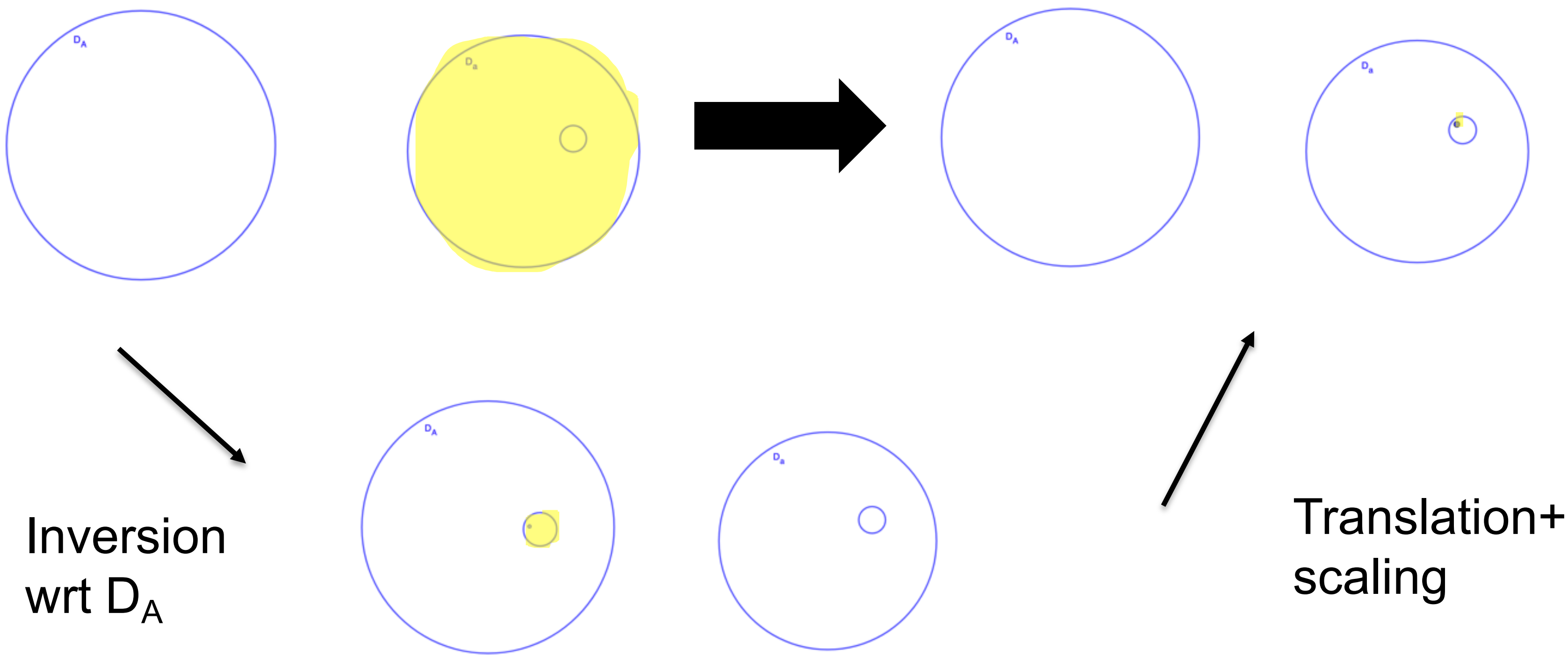


a: Map outside D_A to inside D_a



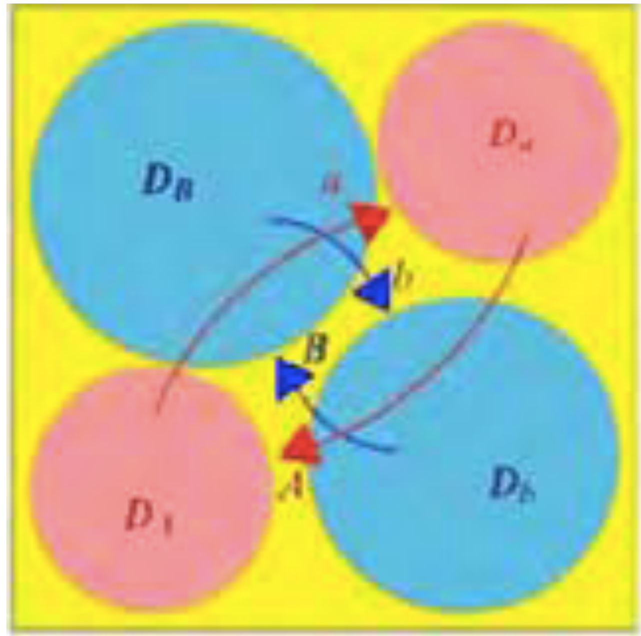
$A=a^{-1}$ maps outside D_a inside D_A

Repeat



Step 2: Two pairs of disk pairs

a: Map *outside* of D_A onto the *inside* of D_a
b: Map *outside* of D_B onto the *inside* of D_b

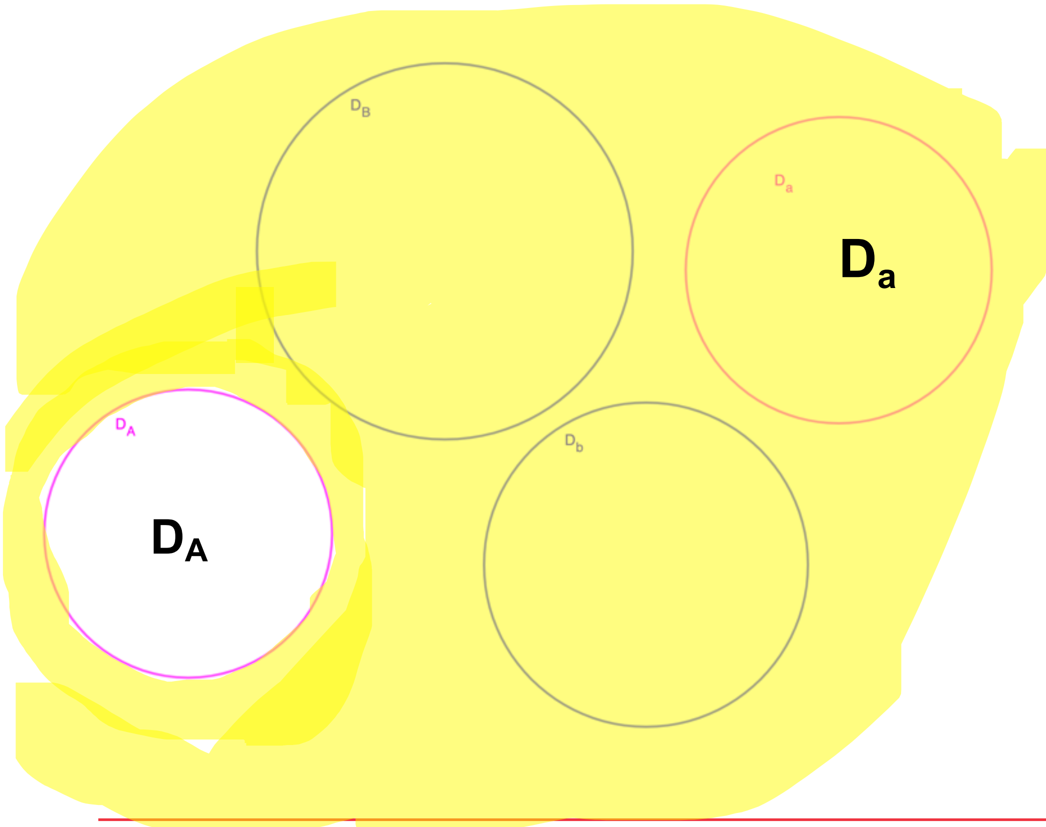


⇒ **Schottky group of two generators:**

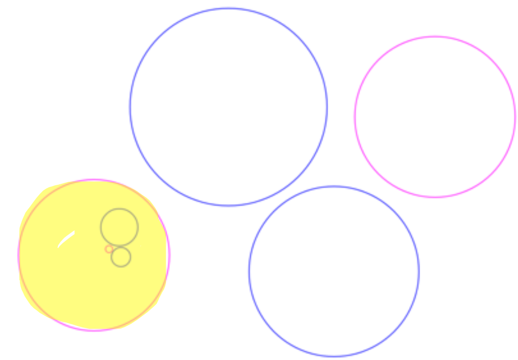
- All possible compositions of a and b and their inverses $A=a^{-1}$ and $B=b^{-1}$

Friedrich Schottky (1851-1935)

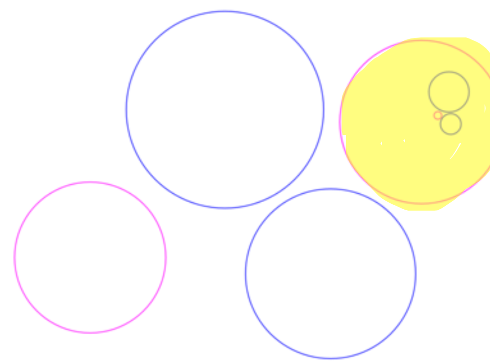
a: Map outside D_A to inside D_a



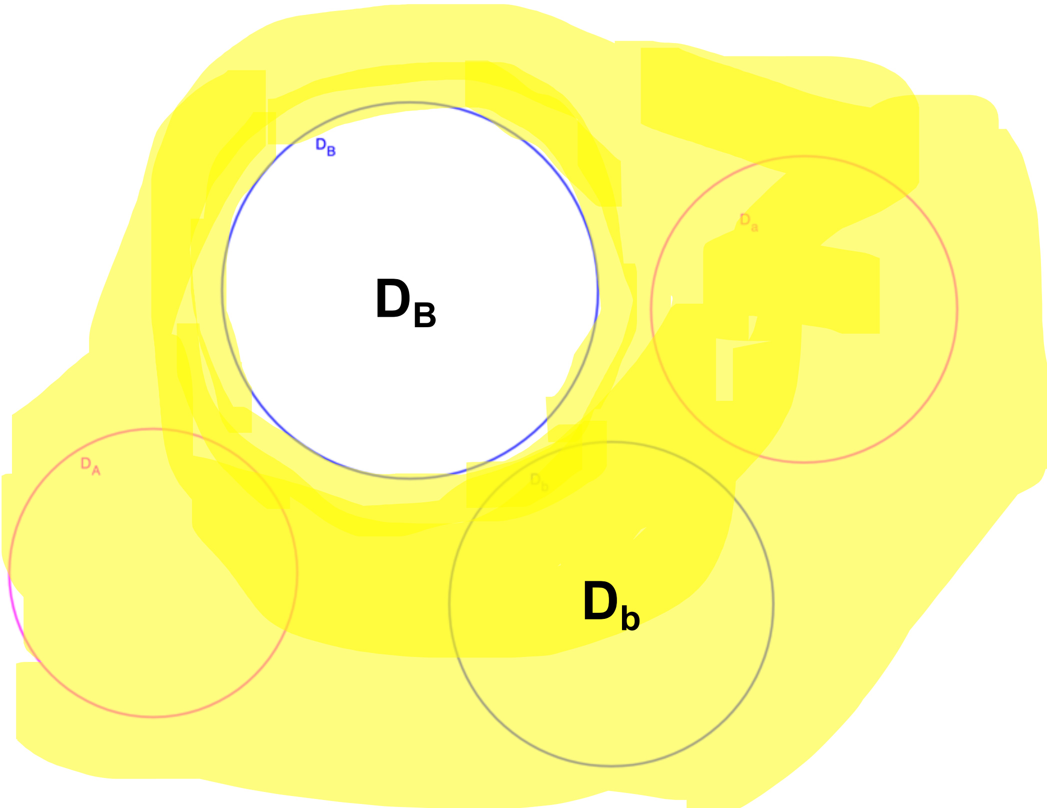
Step1: Inversion wrt D_A



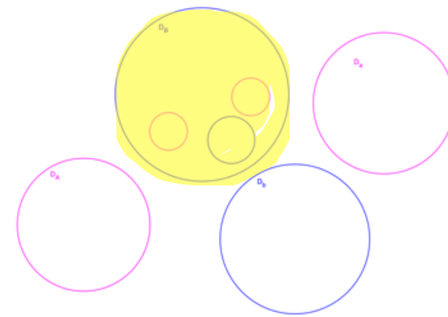
Step2: Move&scale D_A to D_a



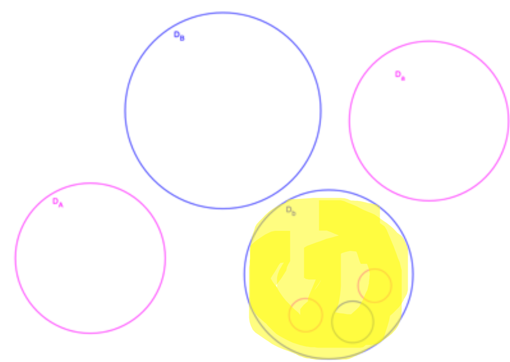
b: Map outside D_B to inside D_b



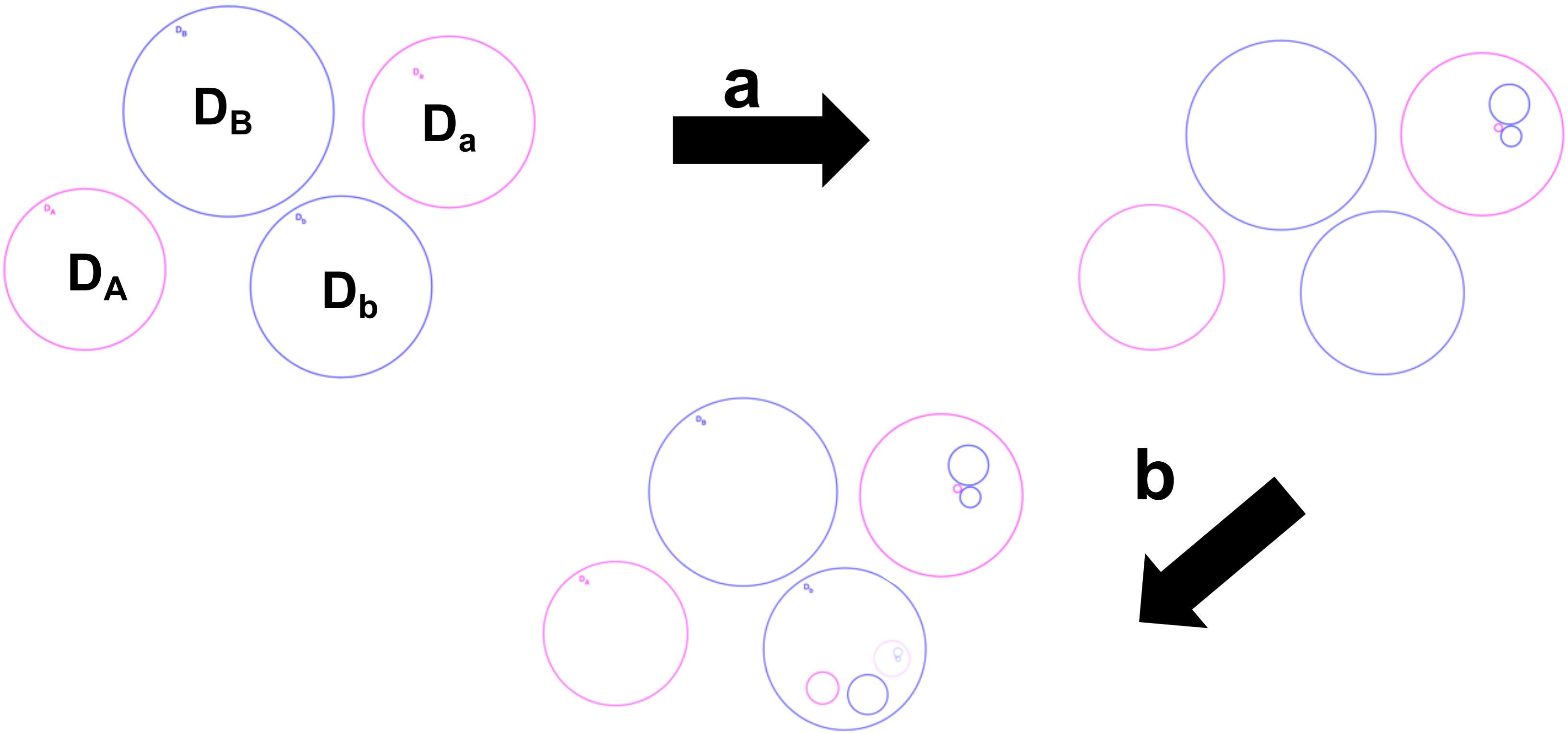
Step1: Inversion wrt D_B

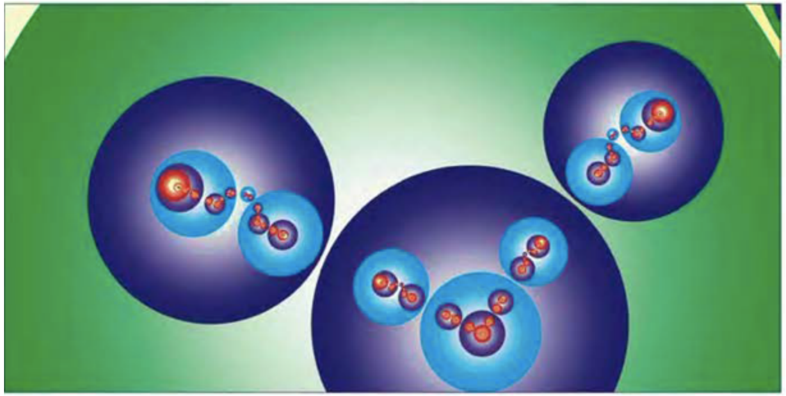


Step2: Move&scale D_B to D_b

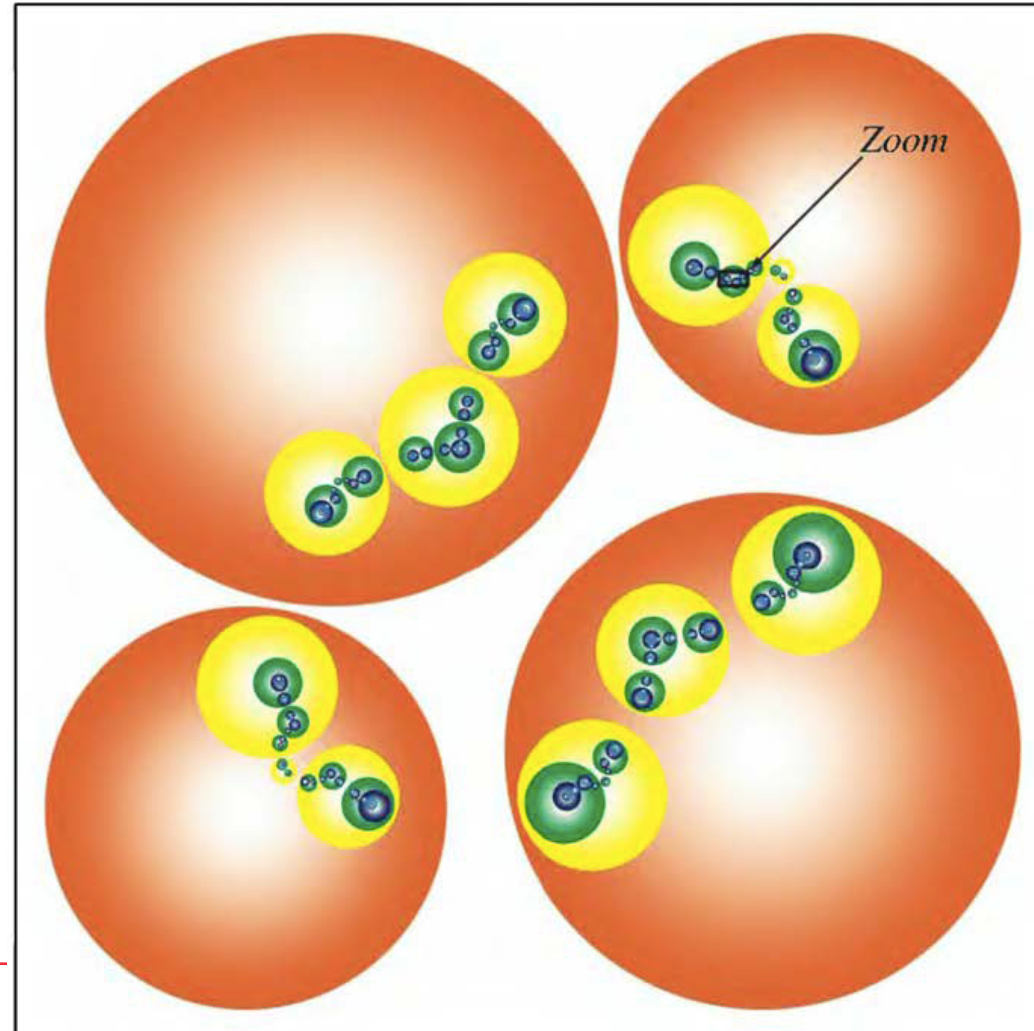


First a and then b





- Limit set = the collection of those points that belong to disks at every level
- What is the *symmetrical tiling* of the group ?
- What is the *orbifold of this tiling?*



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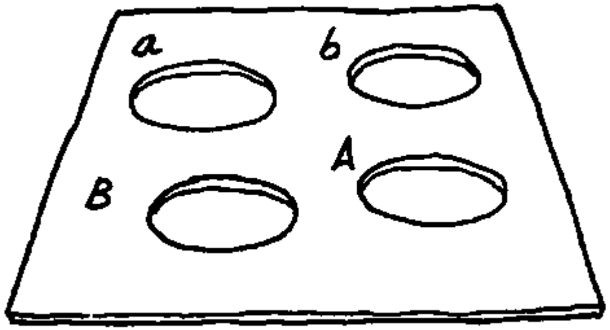
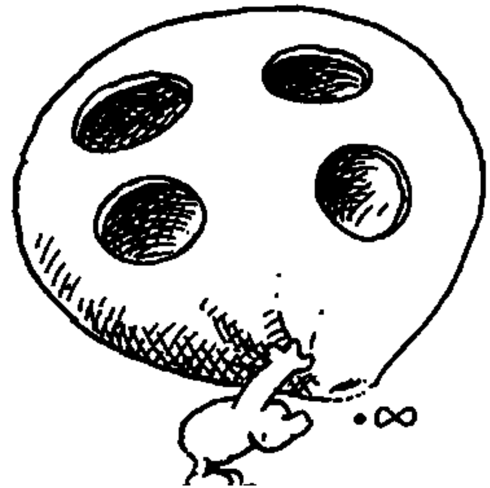

a
b
A
B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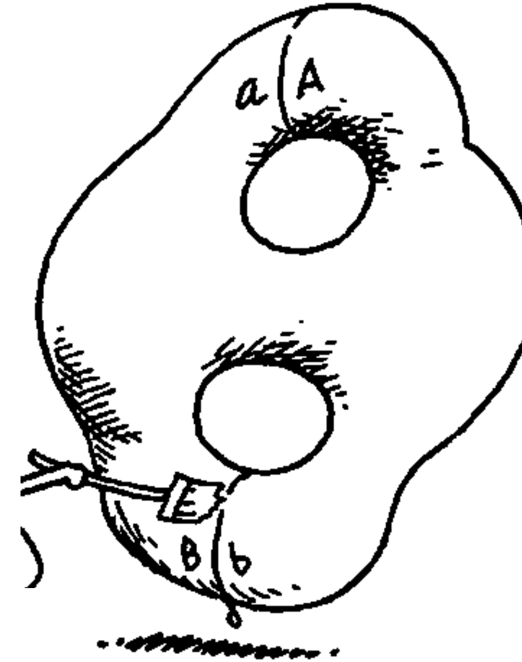
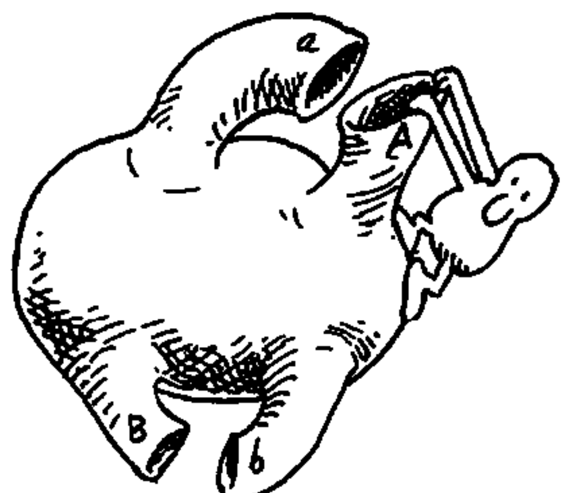
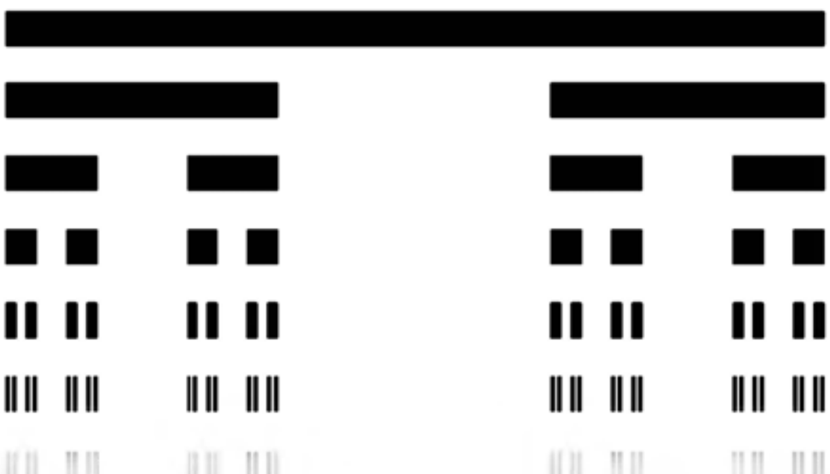

a
A
B
b

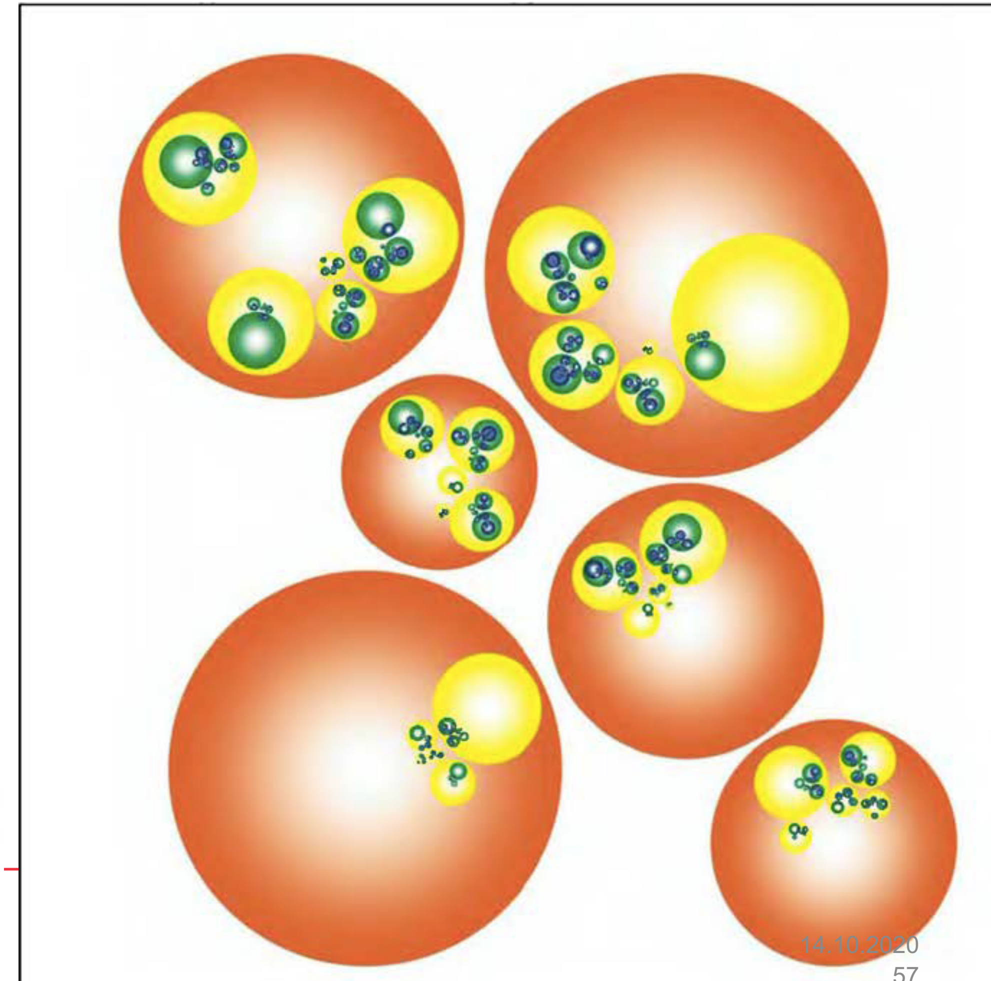

a
A
B
b


a
A
B
b

A Schottky group with 3 generators

- Orbifold is a genus 3 surface
- Limit set (=chaotic set) is 'fractal dust'
- Compare to Cantor middle third

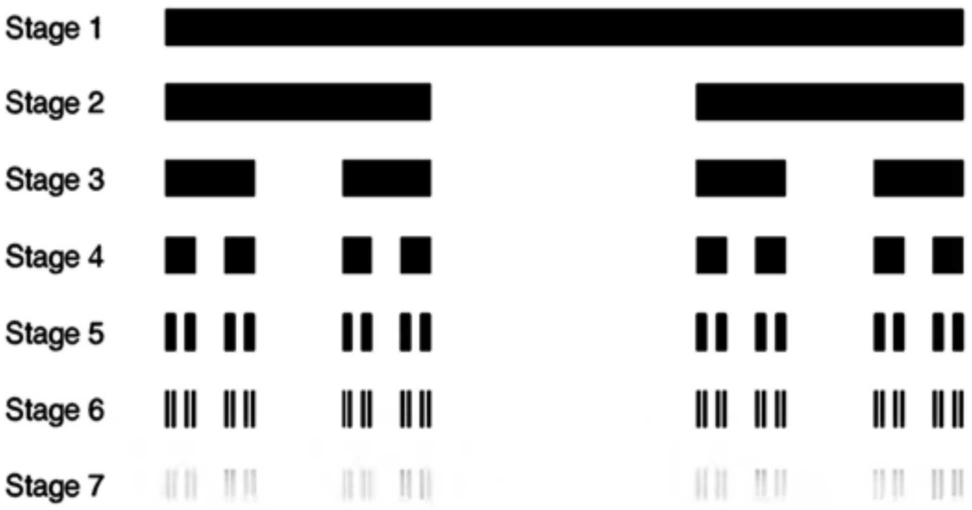




Felix Hausdorff (1868-1942) and his dimension (1919)

- **Idea of a d-dimensional measurement where d is not an integer**
- **Basic properties**
  - Breaking an object apart into pieces keeps the measure of the whole equal to the sum of the parts
  - Expanding/contracting object by factor k changes the size by factor k^d
  - d=1 length, d=2 area, d=3 volume

Hausdorff dimension of the Cantor middle third set



- x_N = amount of **dust** in each level N segment => $\mathbf{x_N=2x_{N+1}}$
- Expanding a level N+1 subsegment by a factor 3 expands the dust in level N+1 to dust in level N (self similarity) => $\mathbf{x_N=3^d x_{N+1}}$

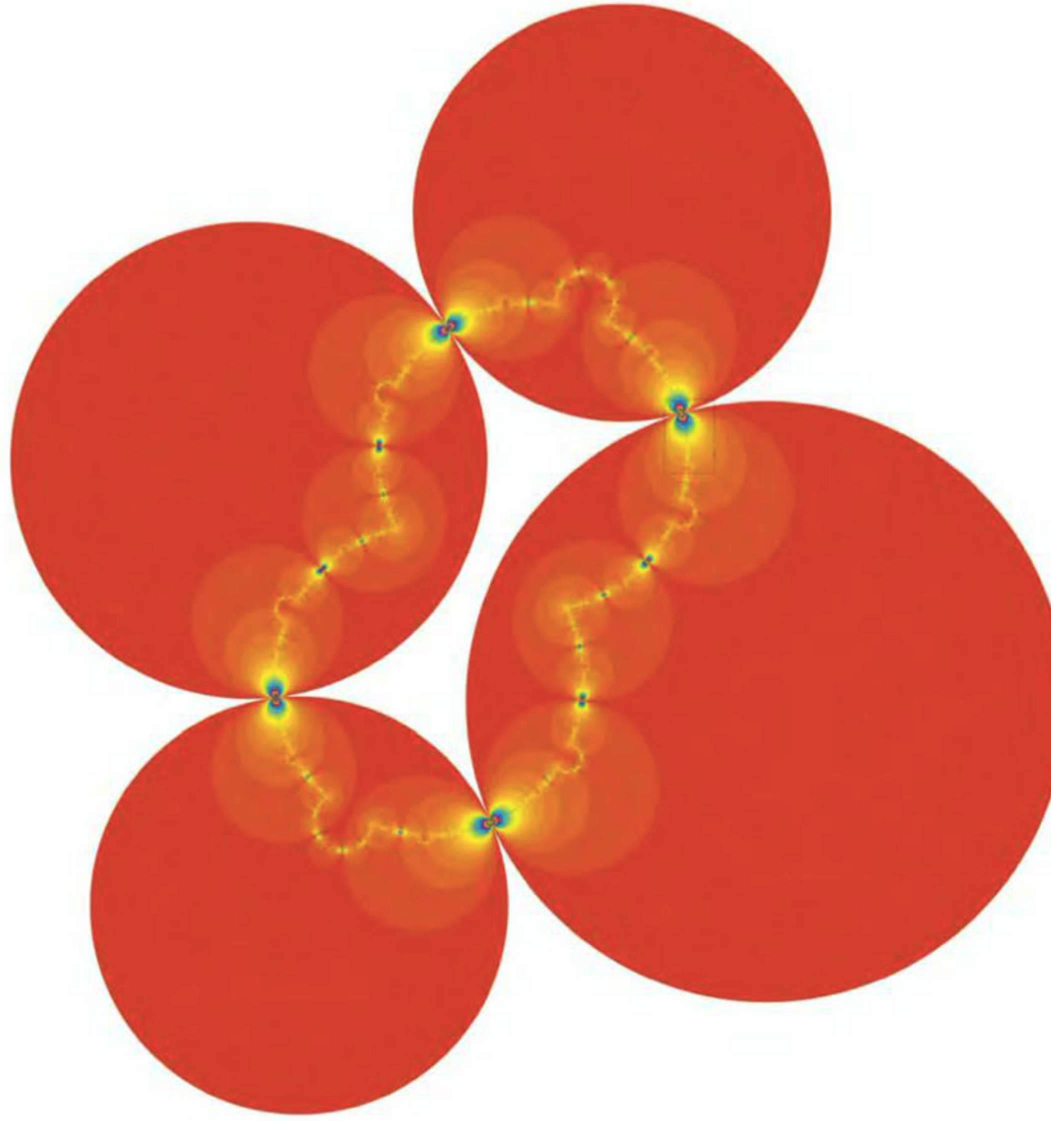
=> $3^d=2$

=> $d= \log(2)/\log(3) \approx 0.63$

Indra's necklace

From dust to necklace

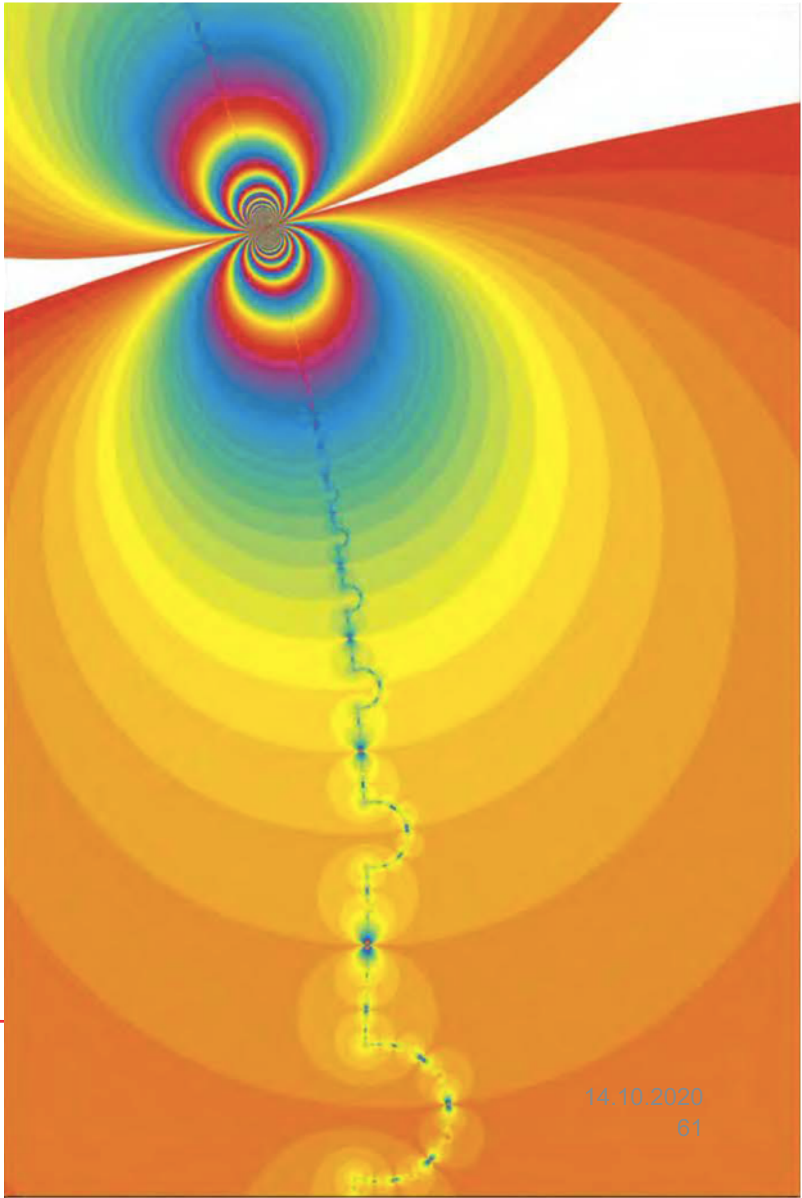
- Symmetry generated by two maps as before
- Choose *tangential* circles



Limit circle

- **Continuous loop**
- **Quasicircle**
- **Quasifuchsian group**

(Lazarus Fuchs, 1833 – 1902)



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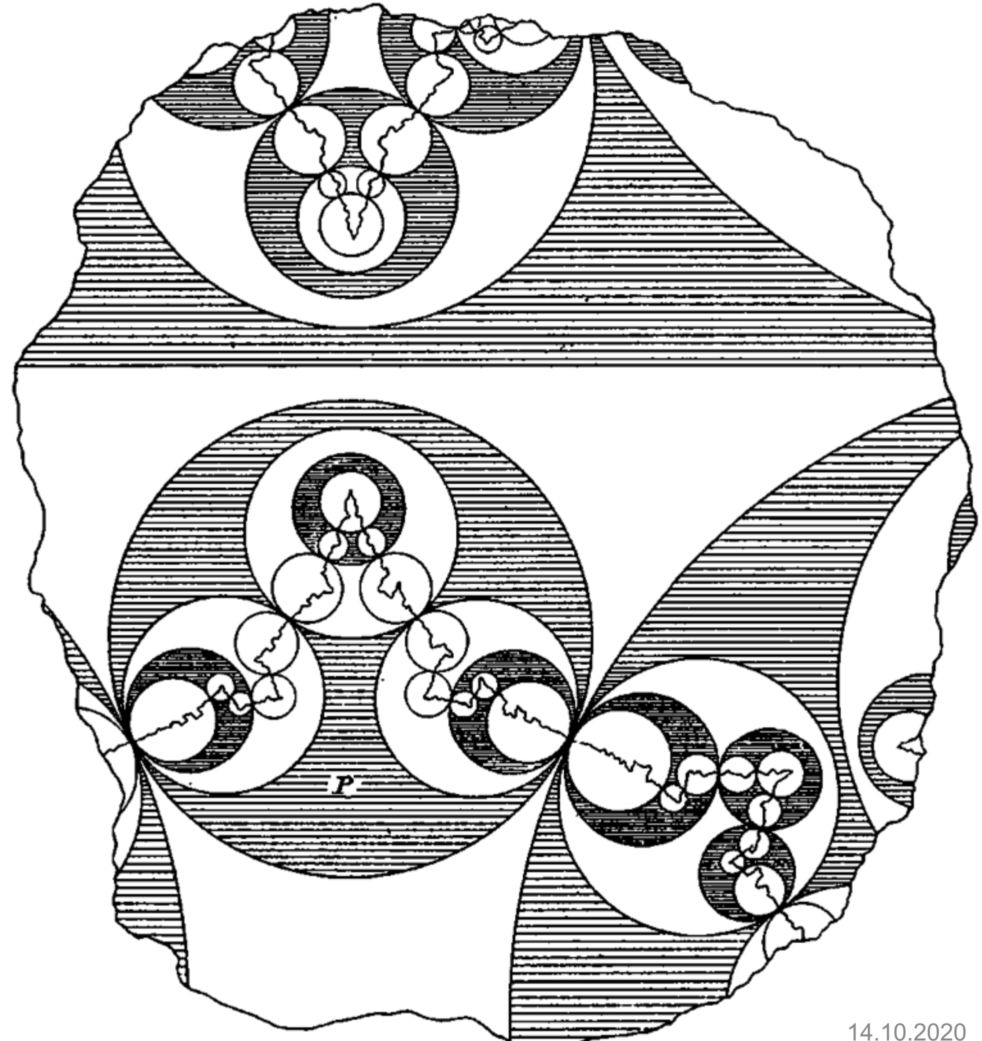
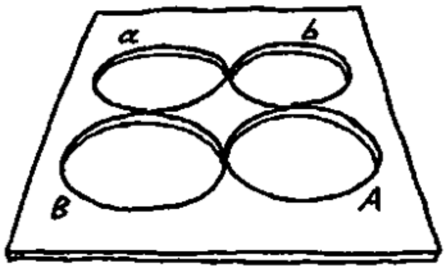
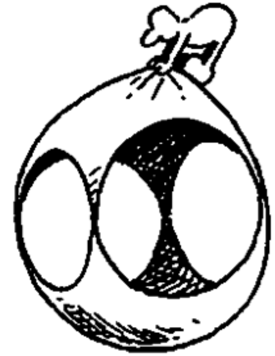
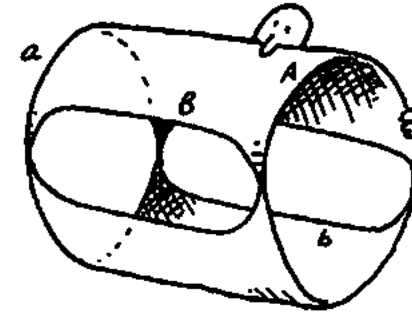
Fricke and Klein, 1897

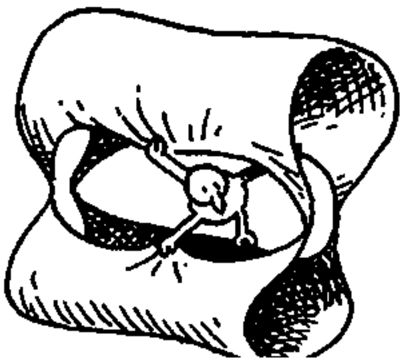


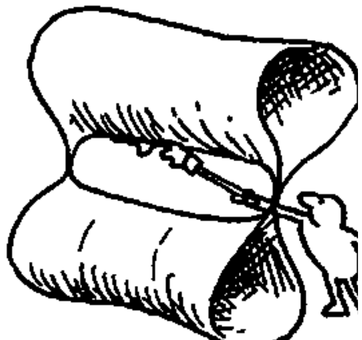
Building the orbifold with a cusp



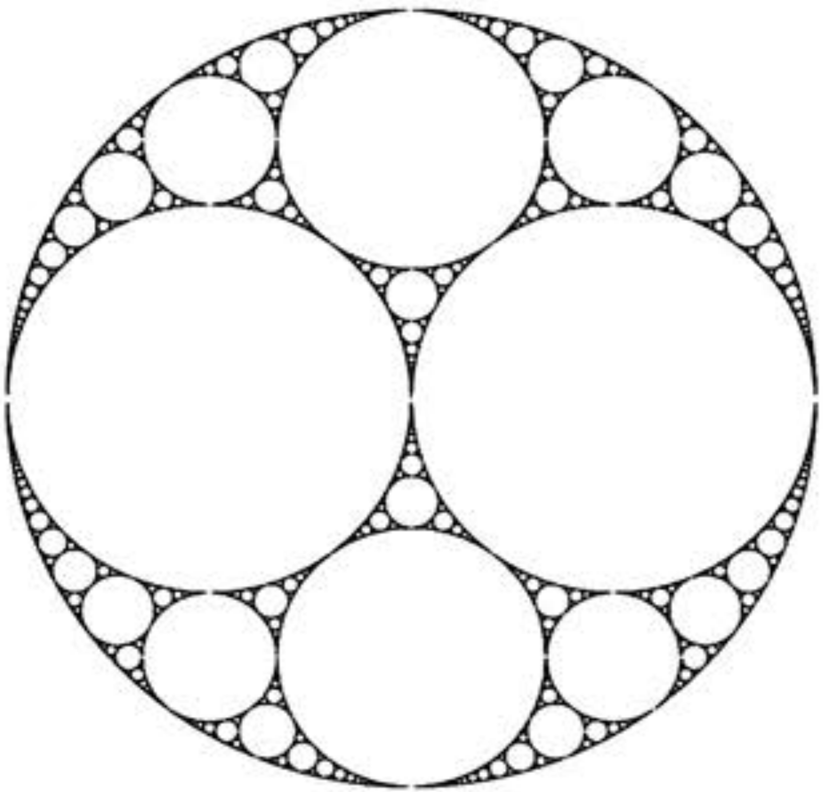


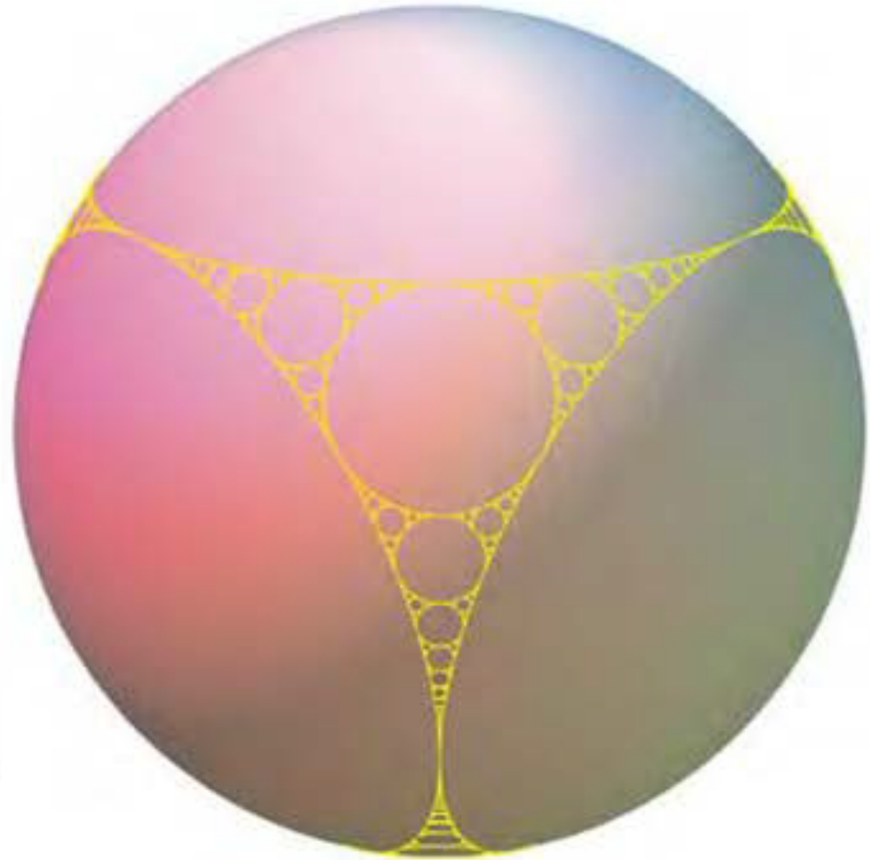






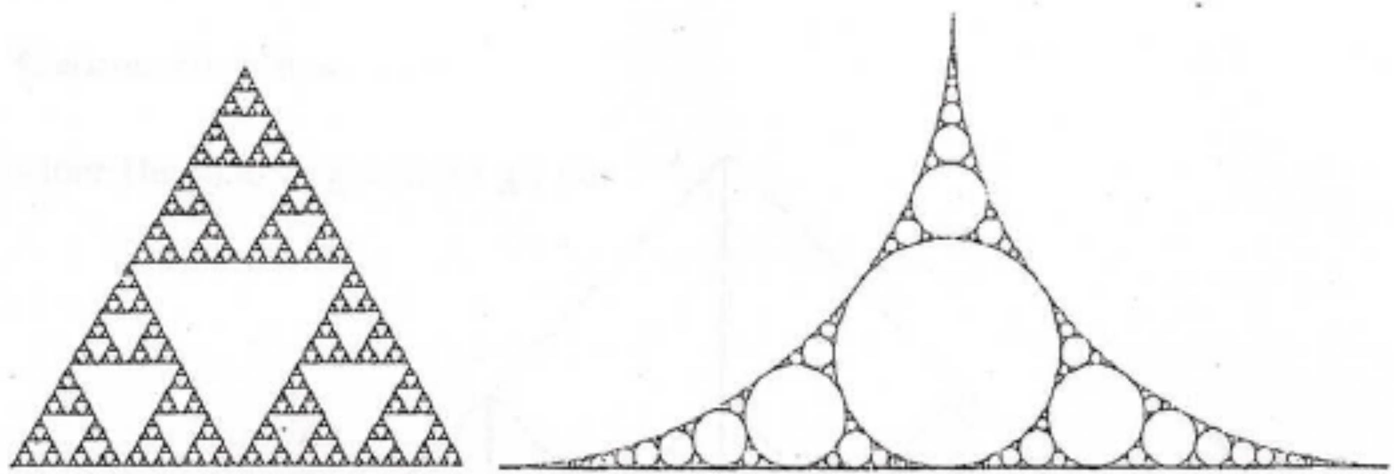
Apollonian gasket





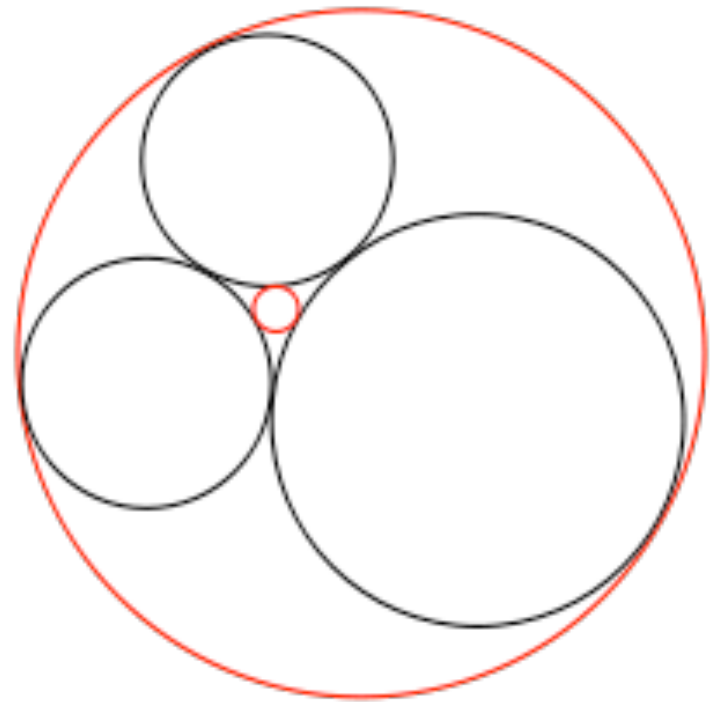
Plane vs Sphere Curt McMullen $d \approx 1{,}306688$

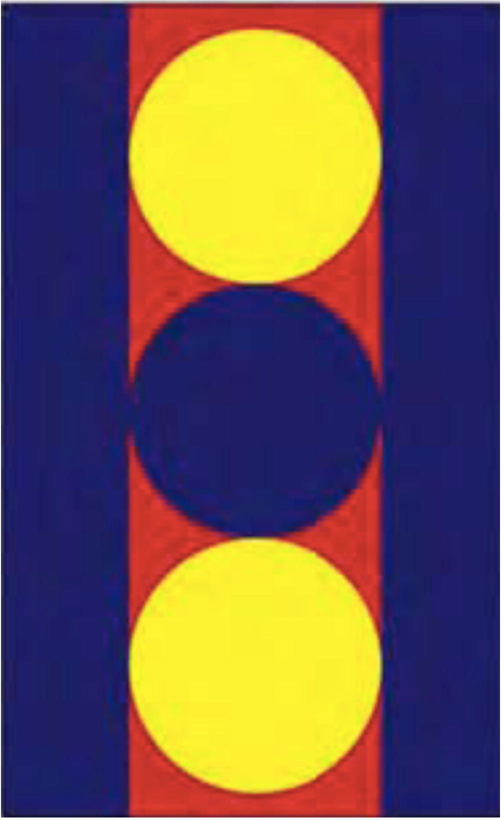
Apollonius meet Sierpinski

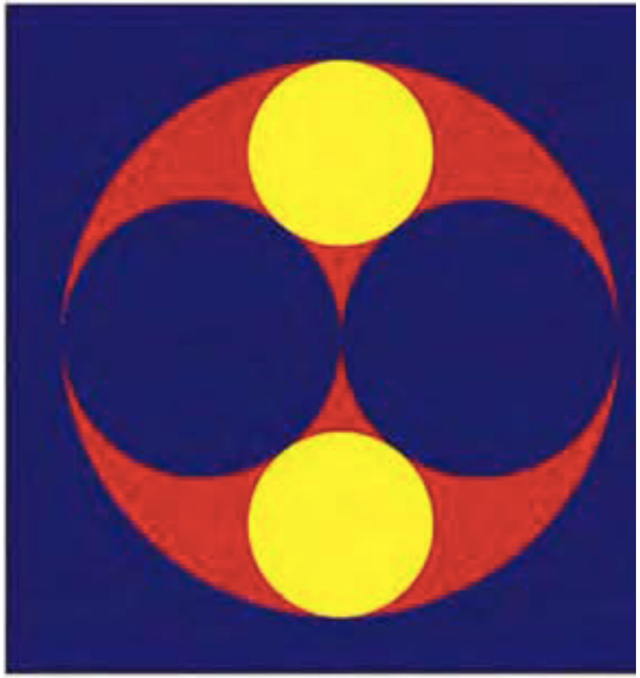


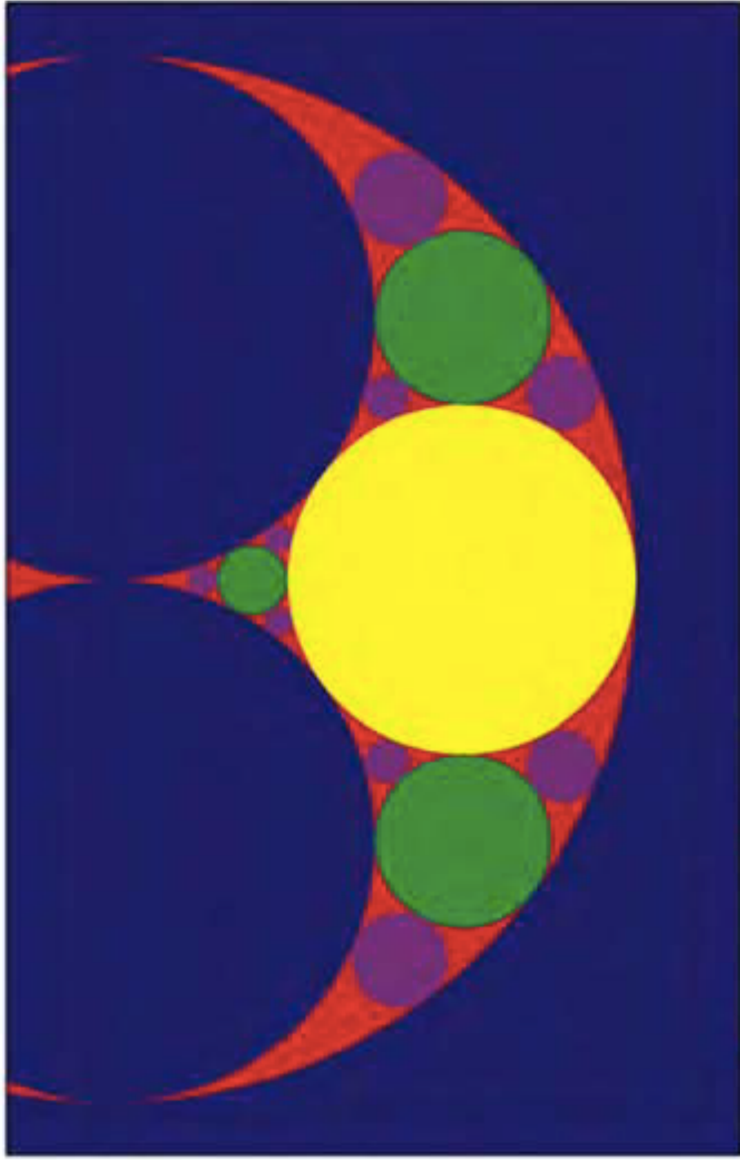
Classical construction of Apollonian circles

- 3 tangent circles bound an ***ideal triangle*** (zero angles)
- unique ***incircle*** tangent to the given 3 outer circles
- **Note:** construction on the sphere does not distinguish inside from outside





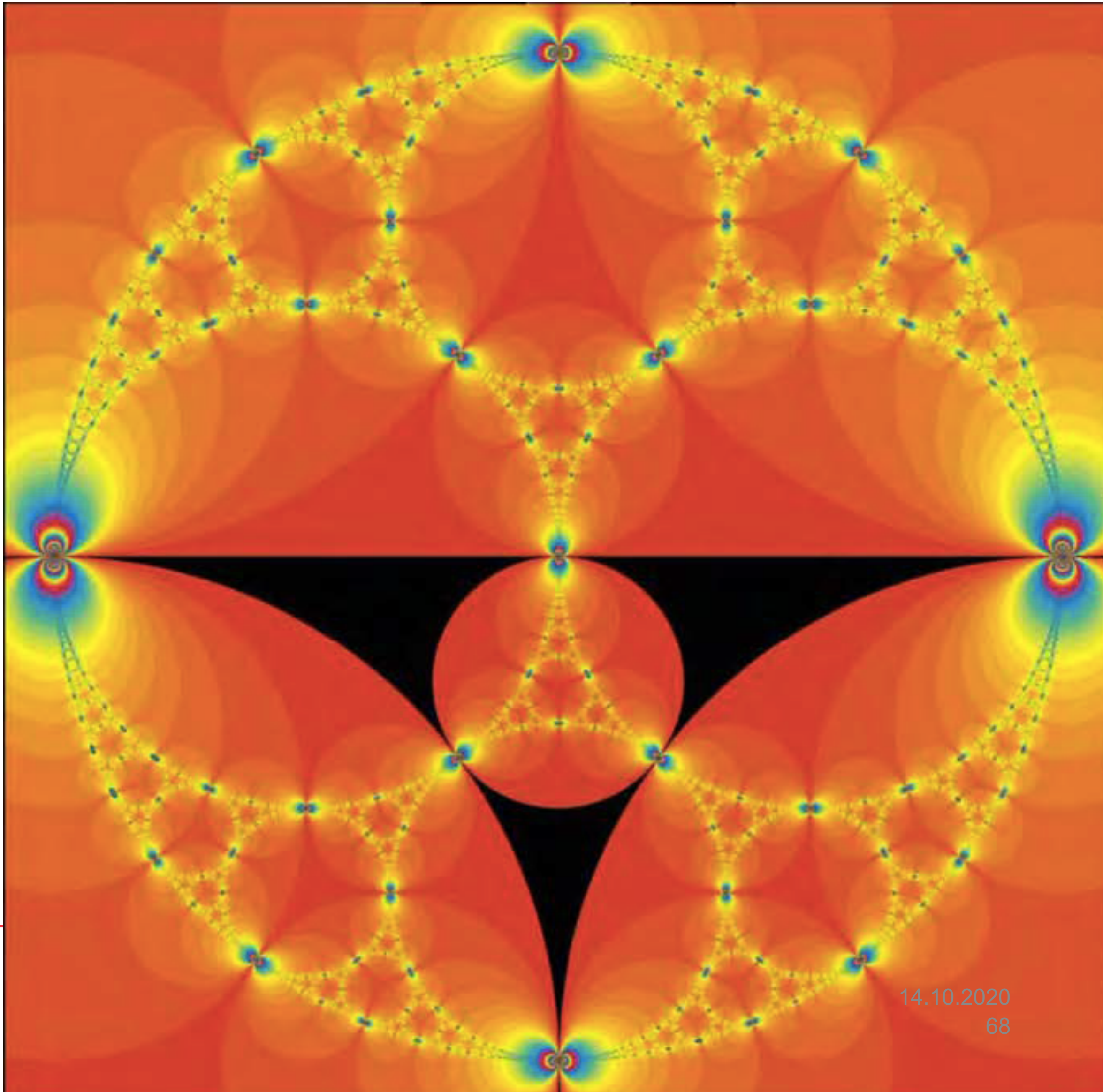




Lines interpreted as circles going through infinity

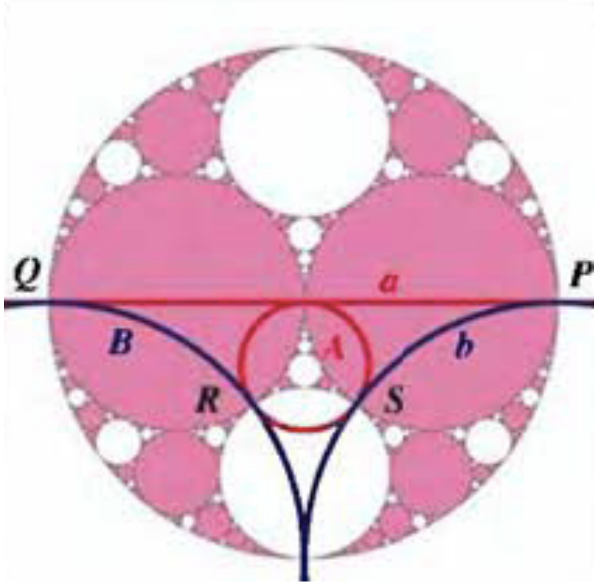
The glowing gasket

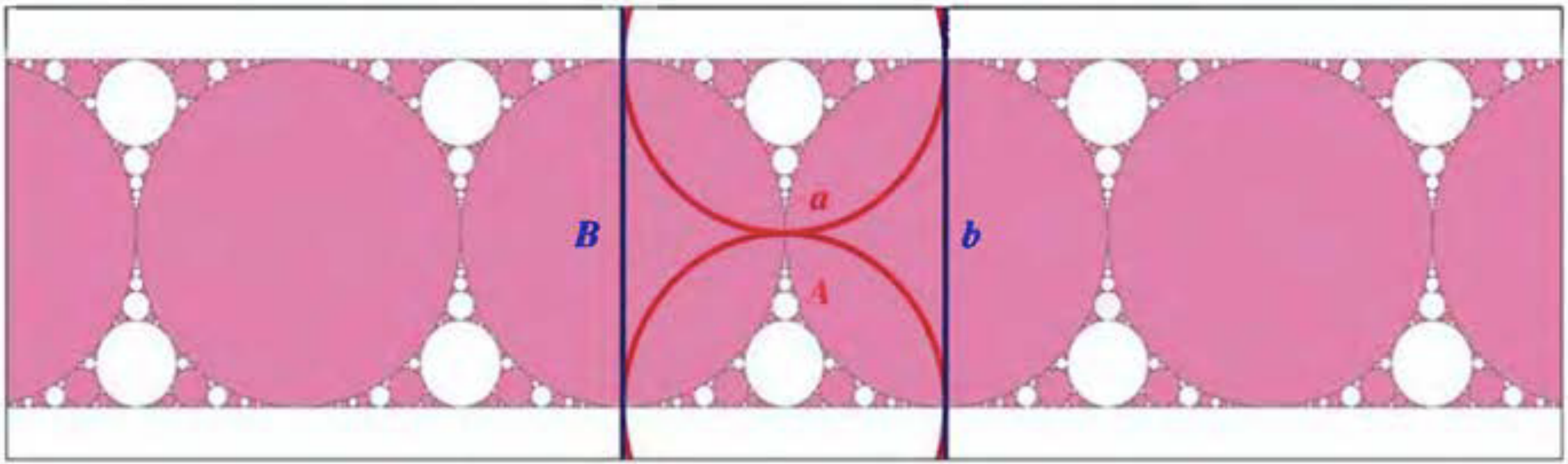
- Generating 4 solid red disks as Schottky pairs
- Traditional construction activated by the dual (yellow) circles



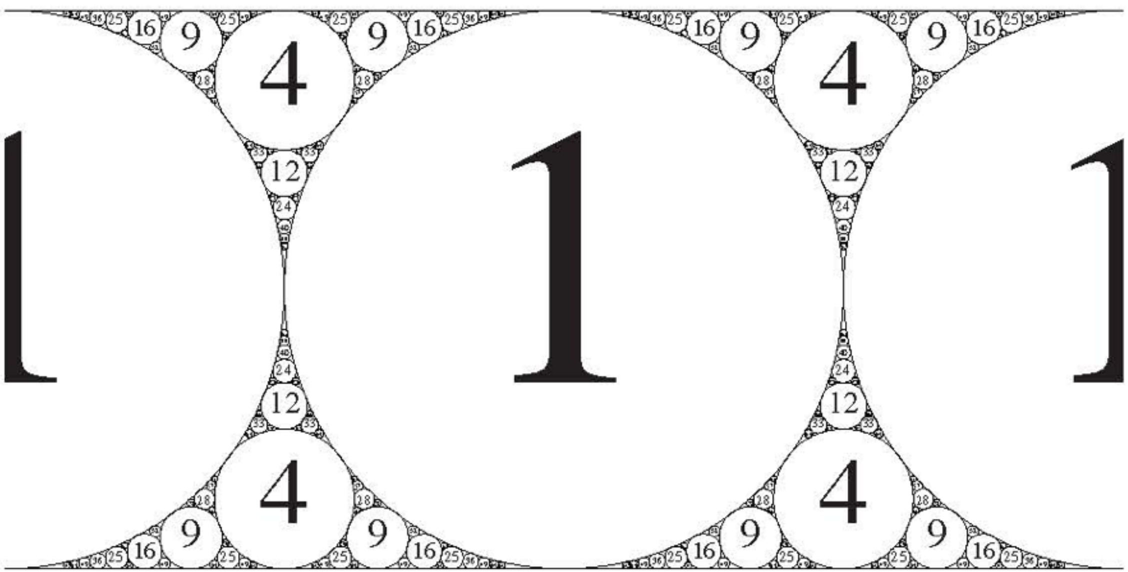
Strip gasket and other variants

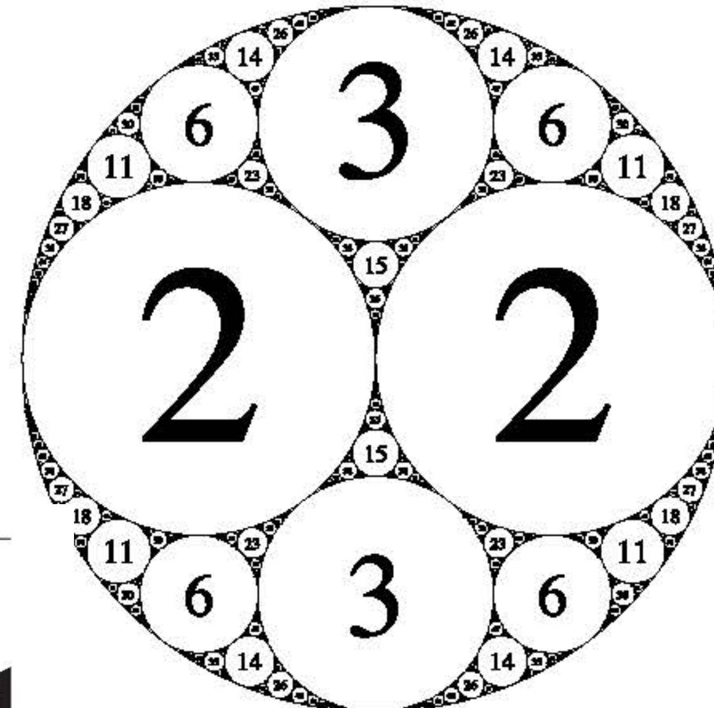
In fact they are all the same up to conjugation by a Möbius transformation….





Numbers in the gasket ?





Soddy circles *(Nature 1936)*

Four circles to the kissing come.
The smaller are the benter.
The bend is just the inverse of
The distance from the center.
Though their intrigue left Euclid dumb
There's now no need for rule of thumb.
Since zero bend's a dead straight line
And concave bends have minus sign,
The sum of the squares of all four bends
Is half the square of their sum.

$$\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)^2 = 2\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{d^2}\right)$$

Frederick Soddy (1877-1956)

- **Chemistry Nobel 1921**
- **Packing spherical atoms**

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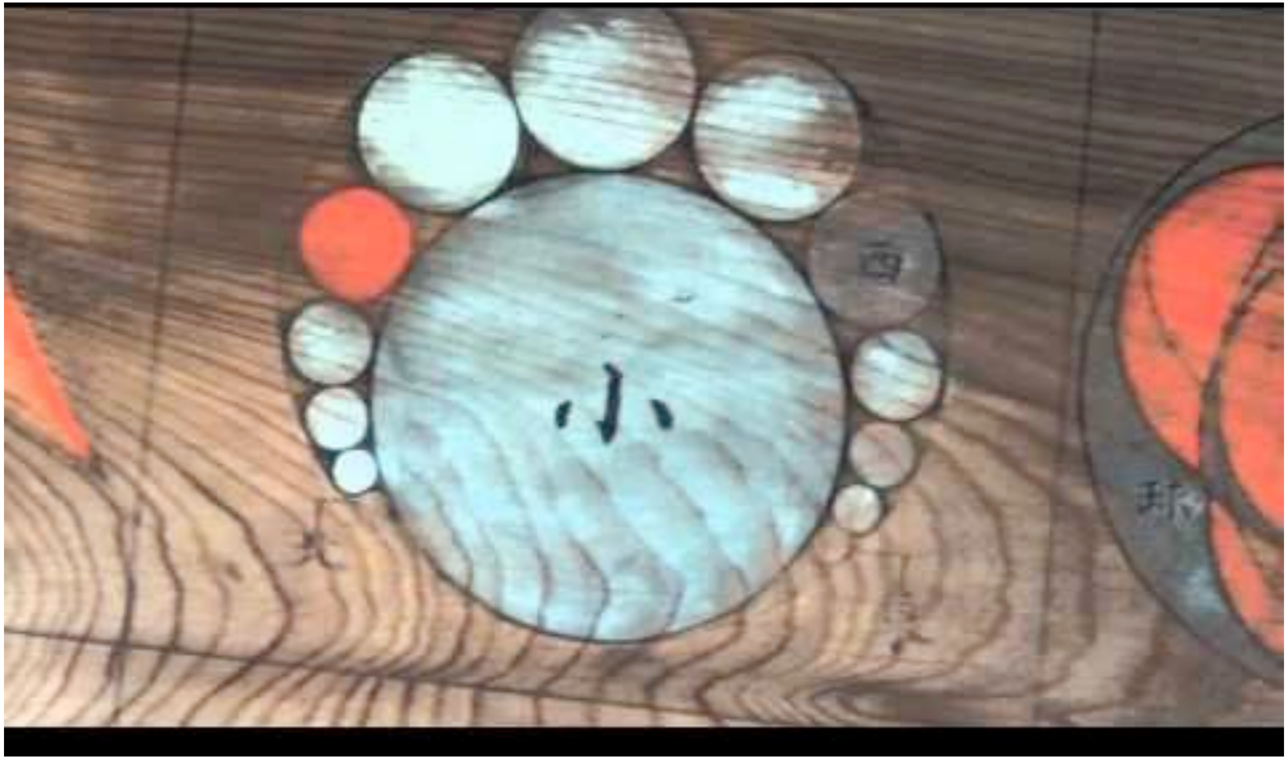
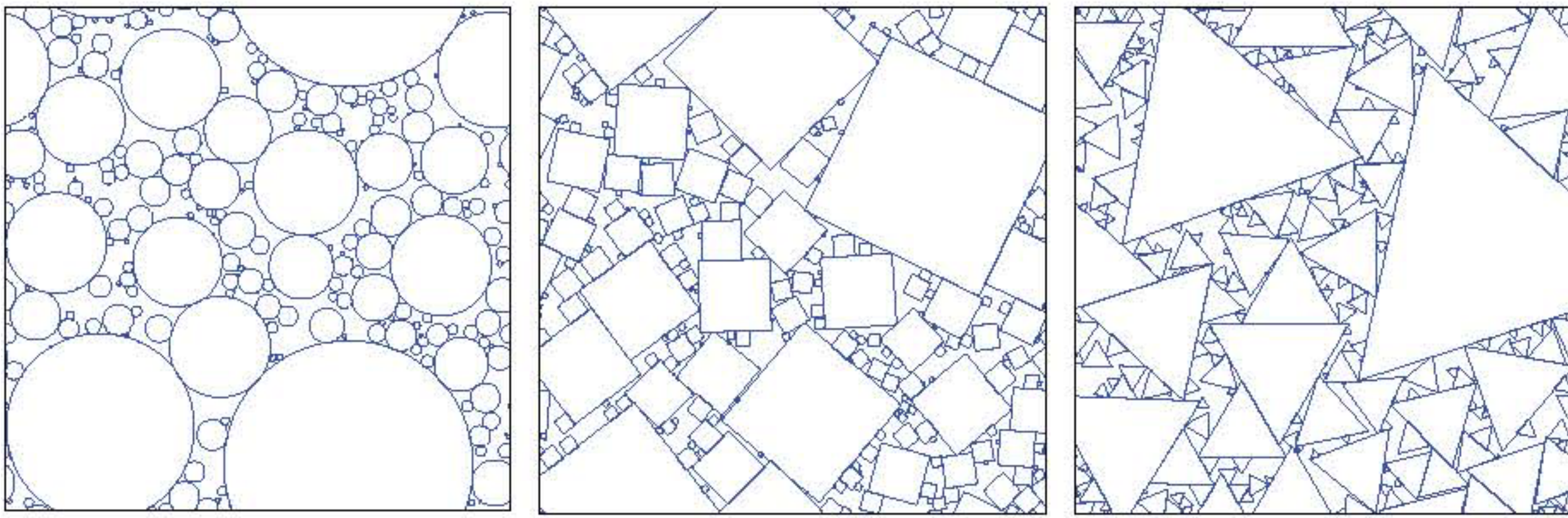
Note: Formula known also for Rene Descartes (1643) etc.

Apollonian circles in Japanese Sangaku tradition (Edo period 1603-1868)

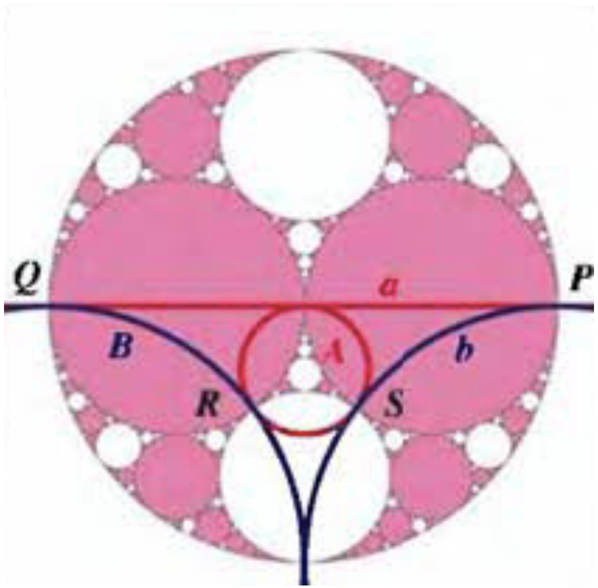


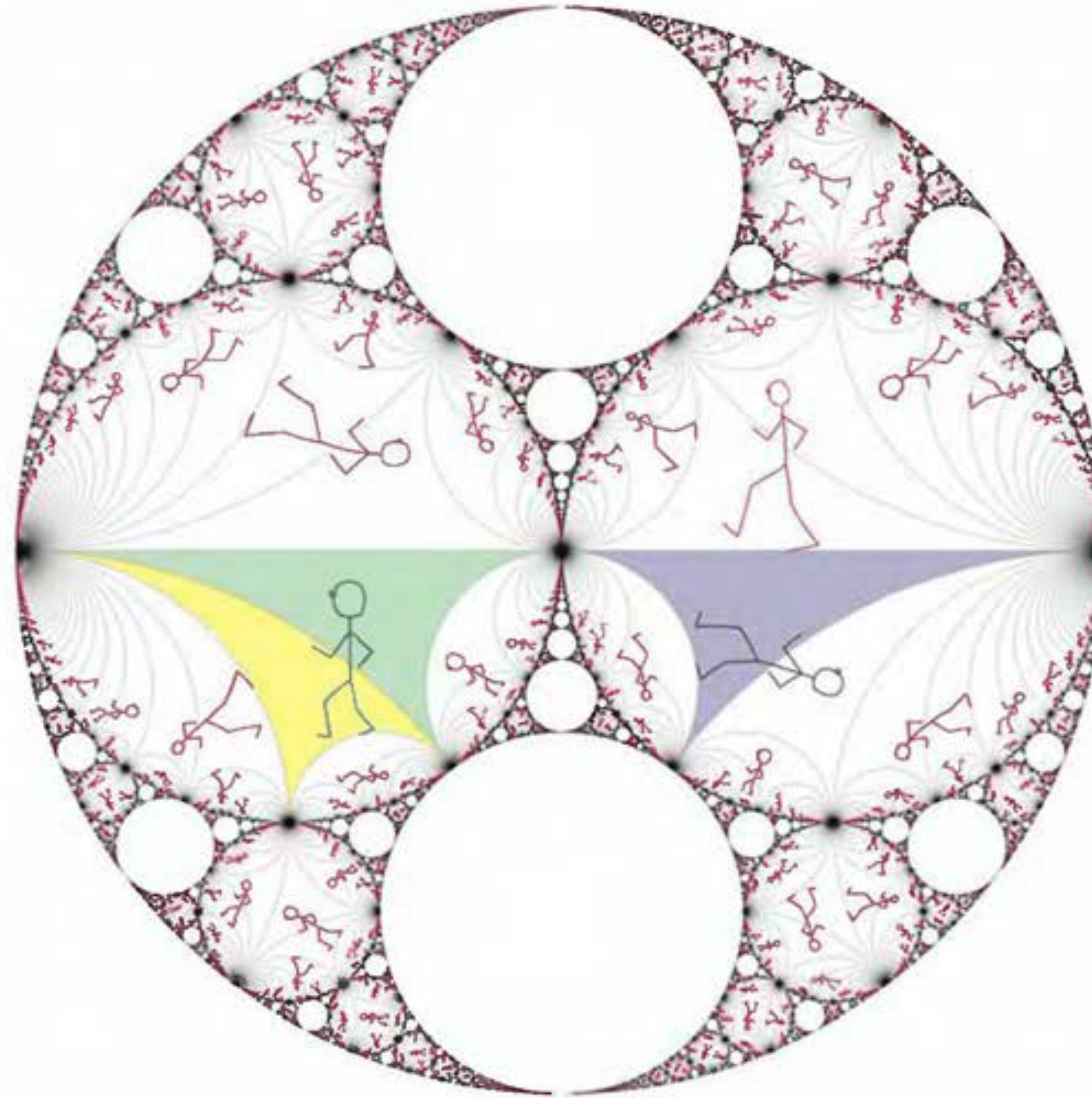
Random Apollonian packings in physics

- Models for foams and powders
- Ex: Number of bubbles of radius bigger than r in the foam $\sim \mathrm{r}^d$

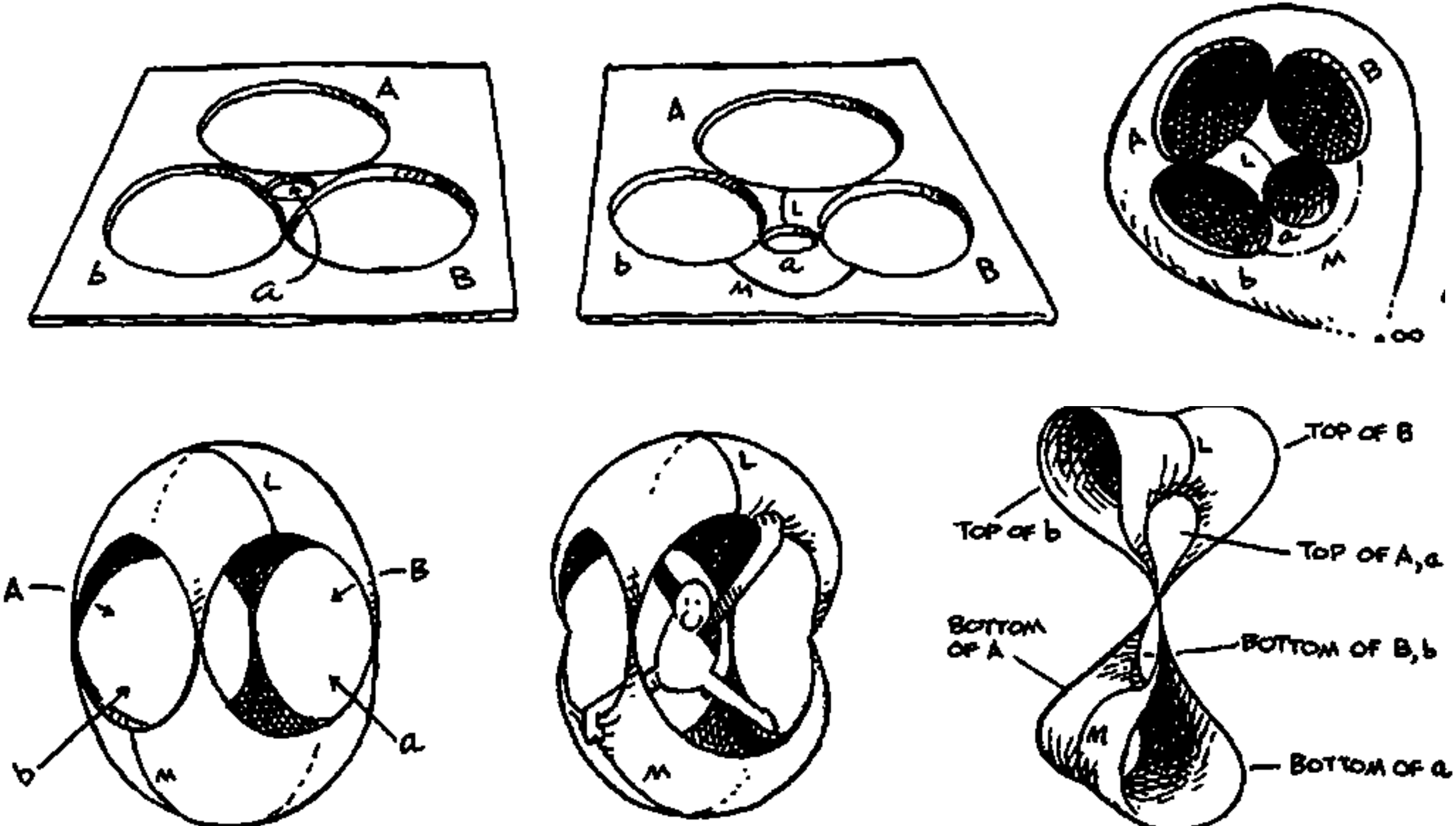


Tiling related to the glowing gasket

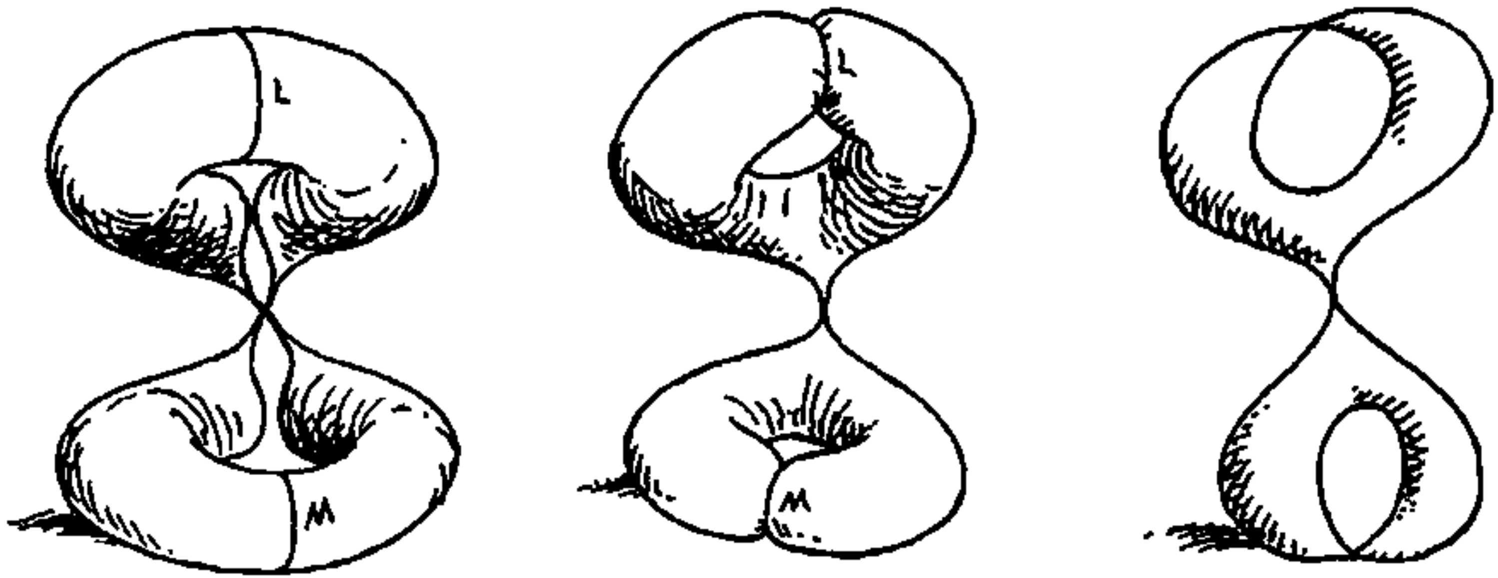




Building the corresponding orbifold….

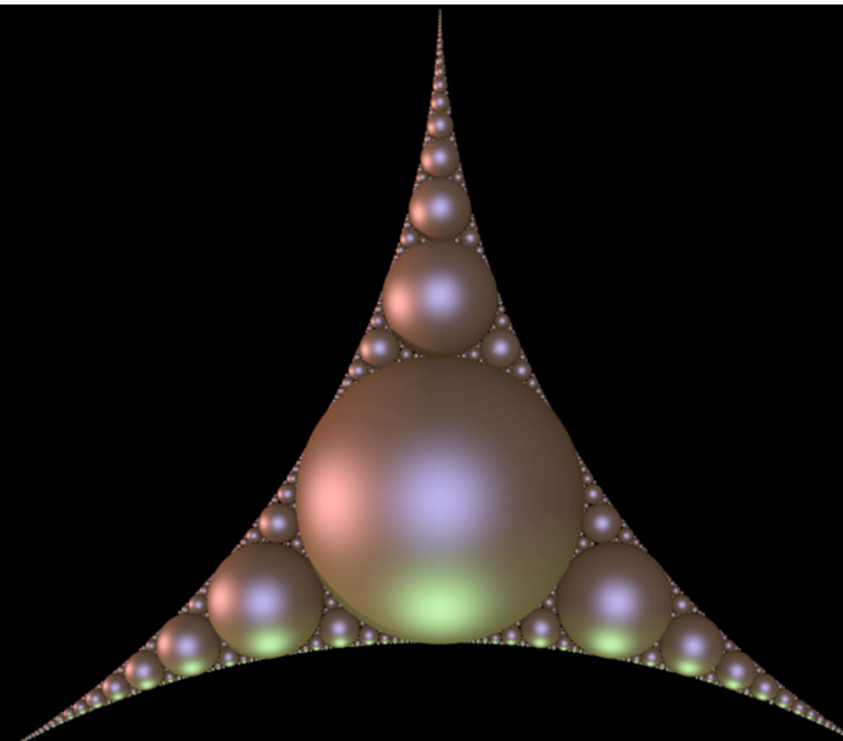


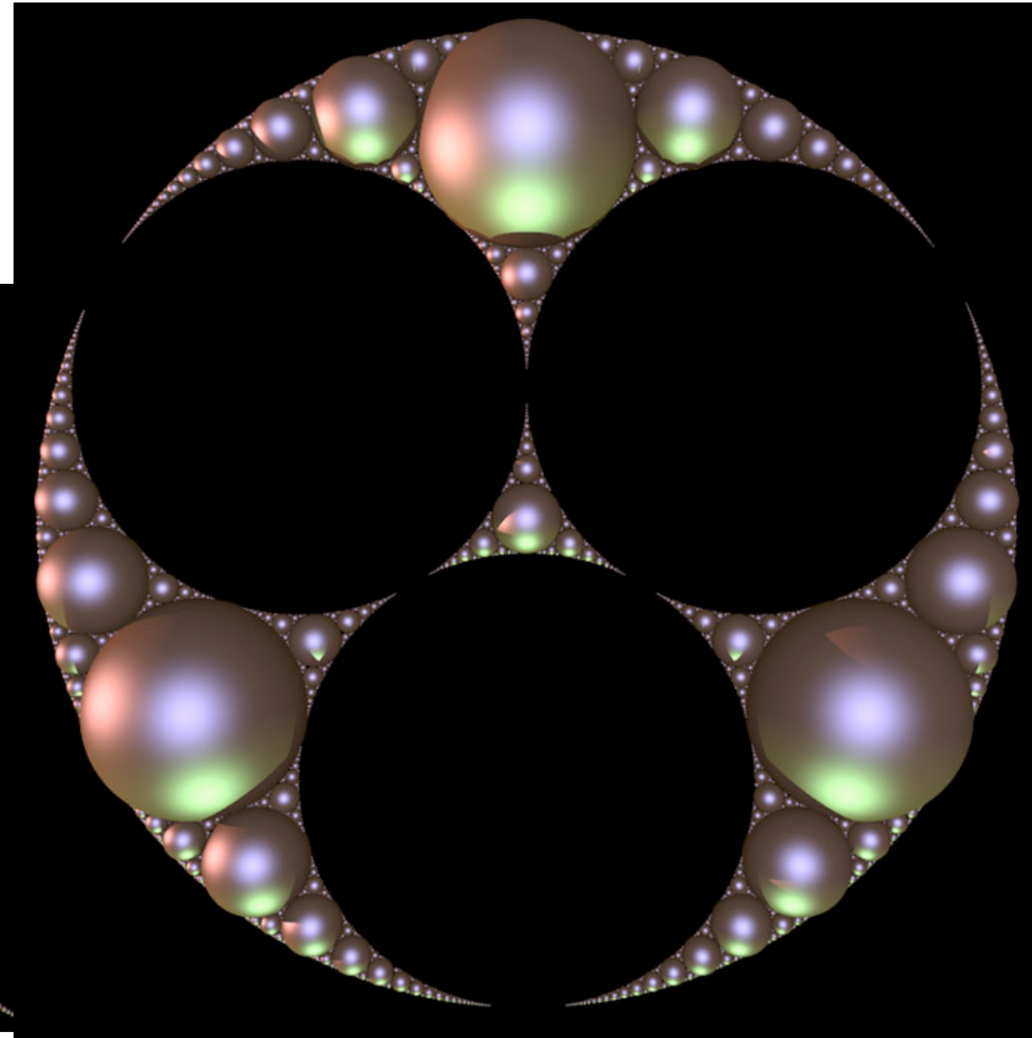
Two triply punctured spheres

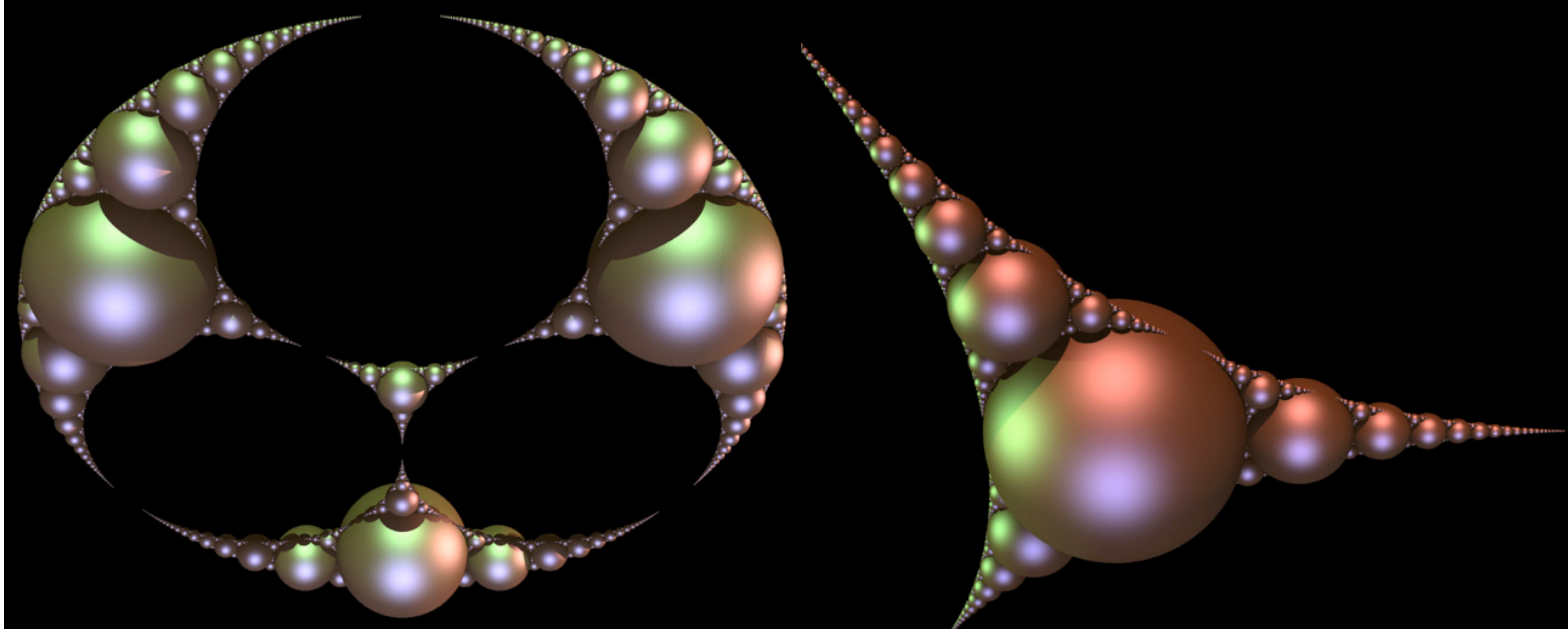


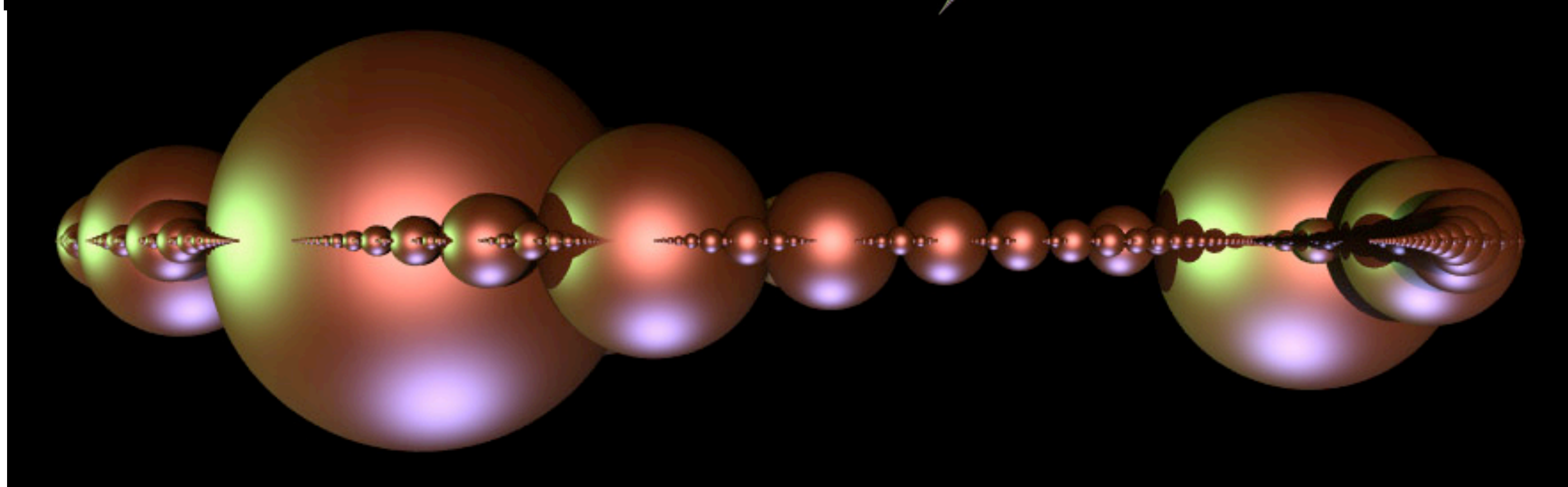
3D variants of Apollonian gasket

- By just turning circles to spheres

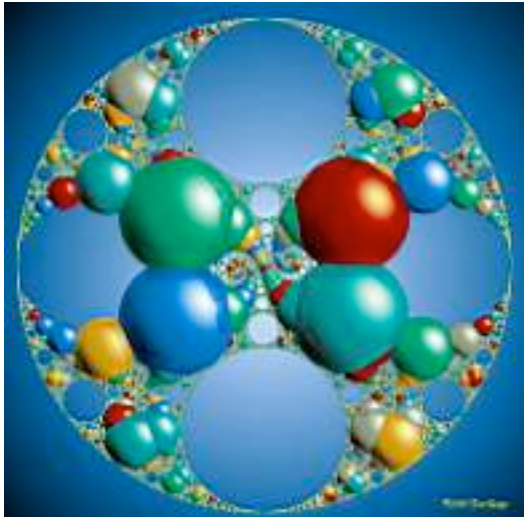


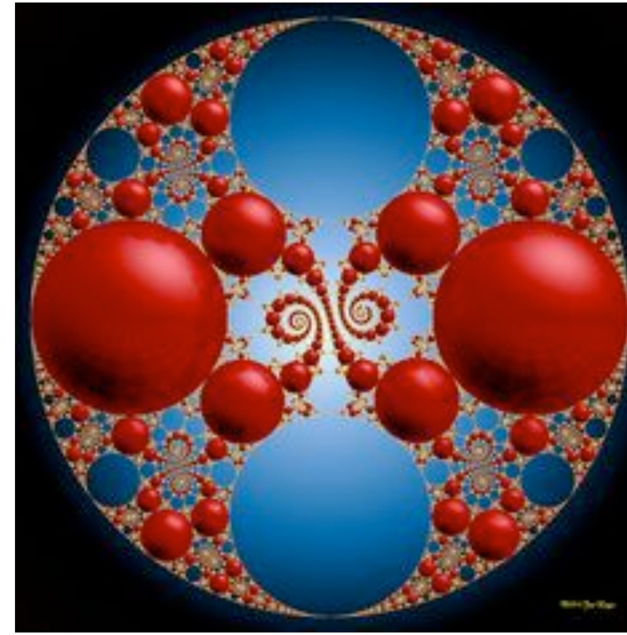


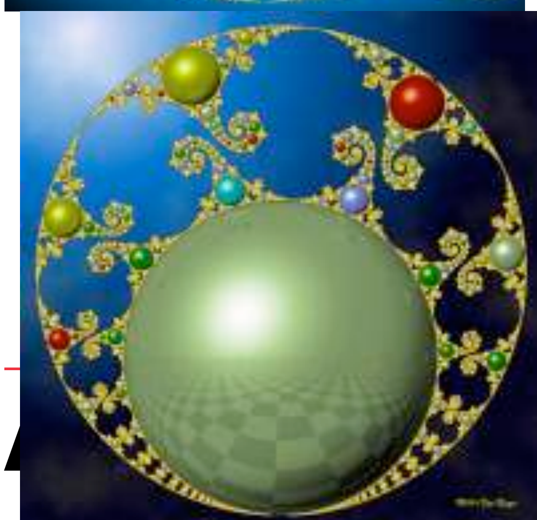


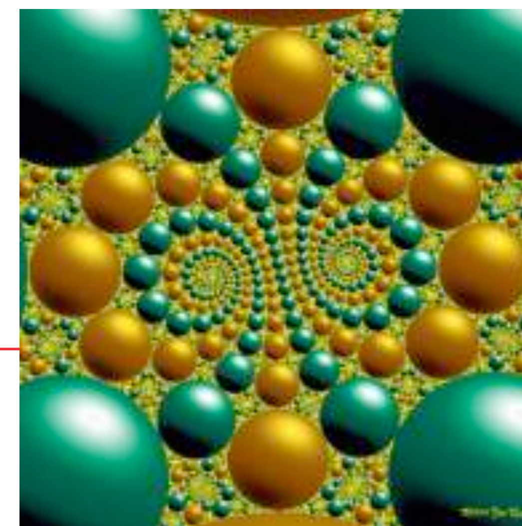


http://www.josleys.com/

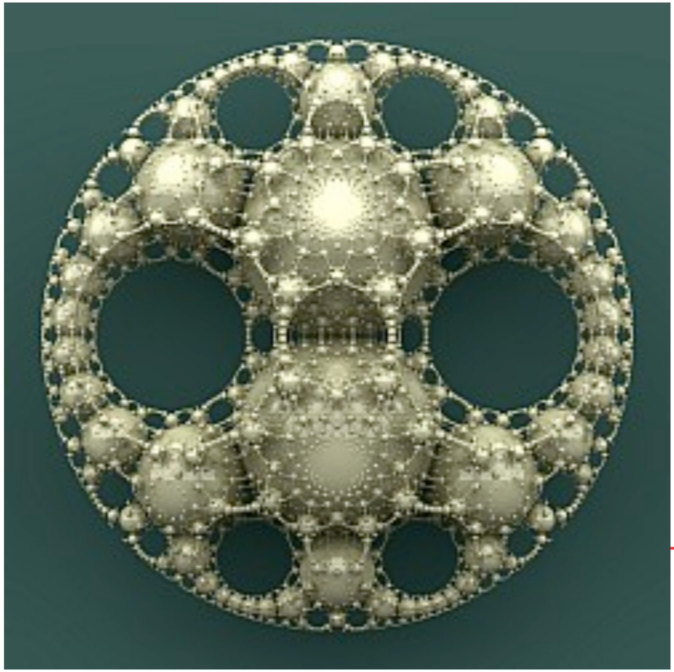


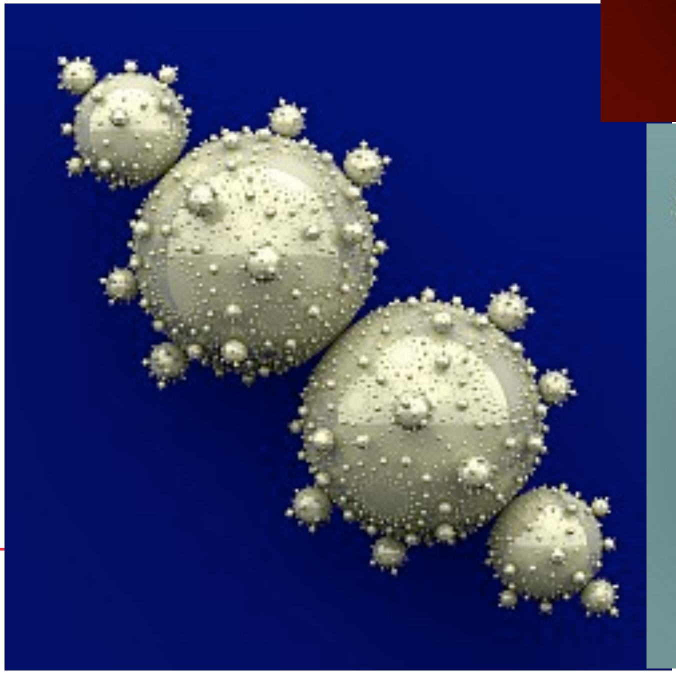


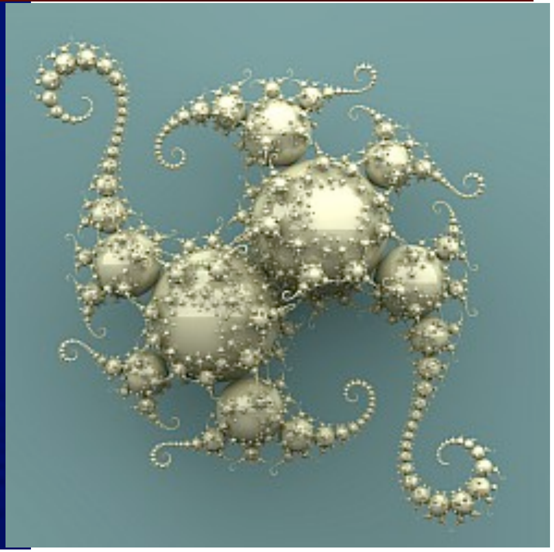




Genuinely 3D Kleinian group interpretations

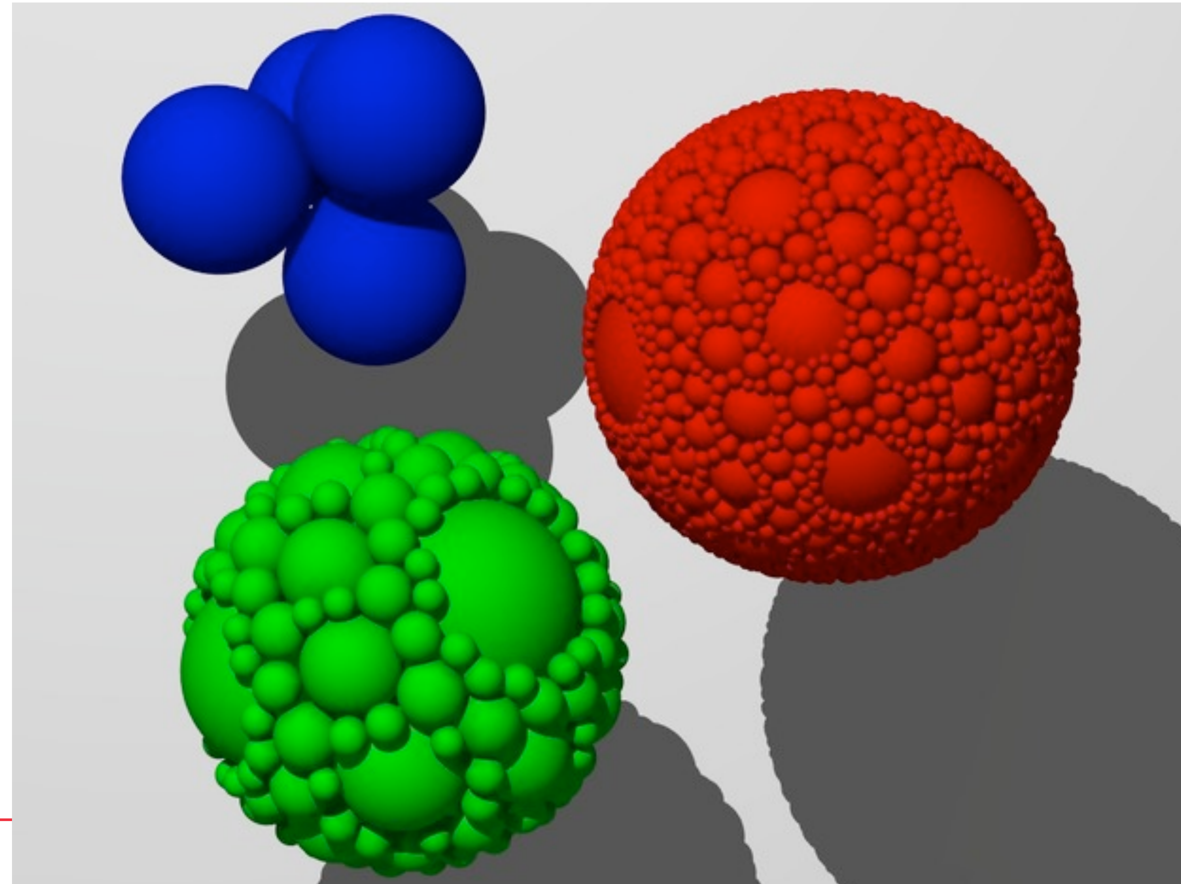




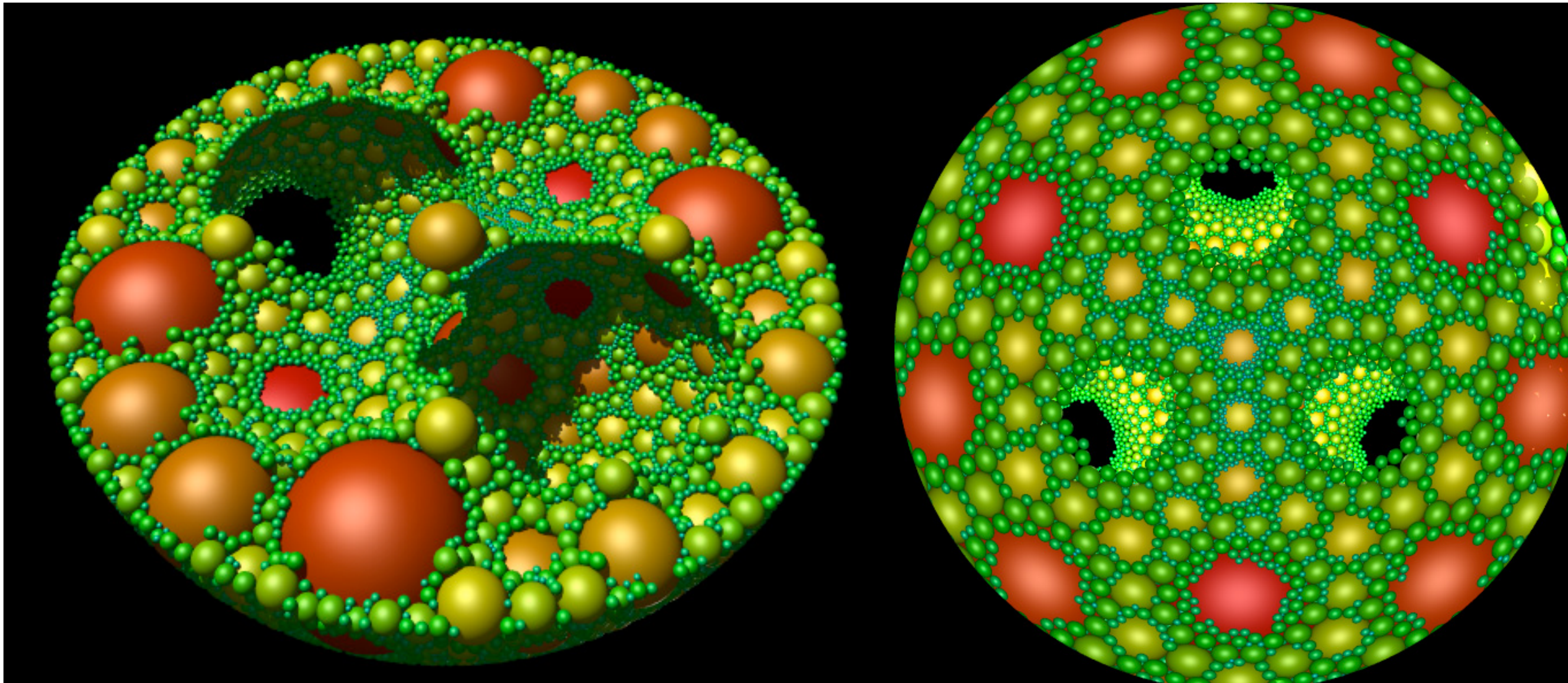


Apollonian sphere packing

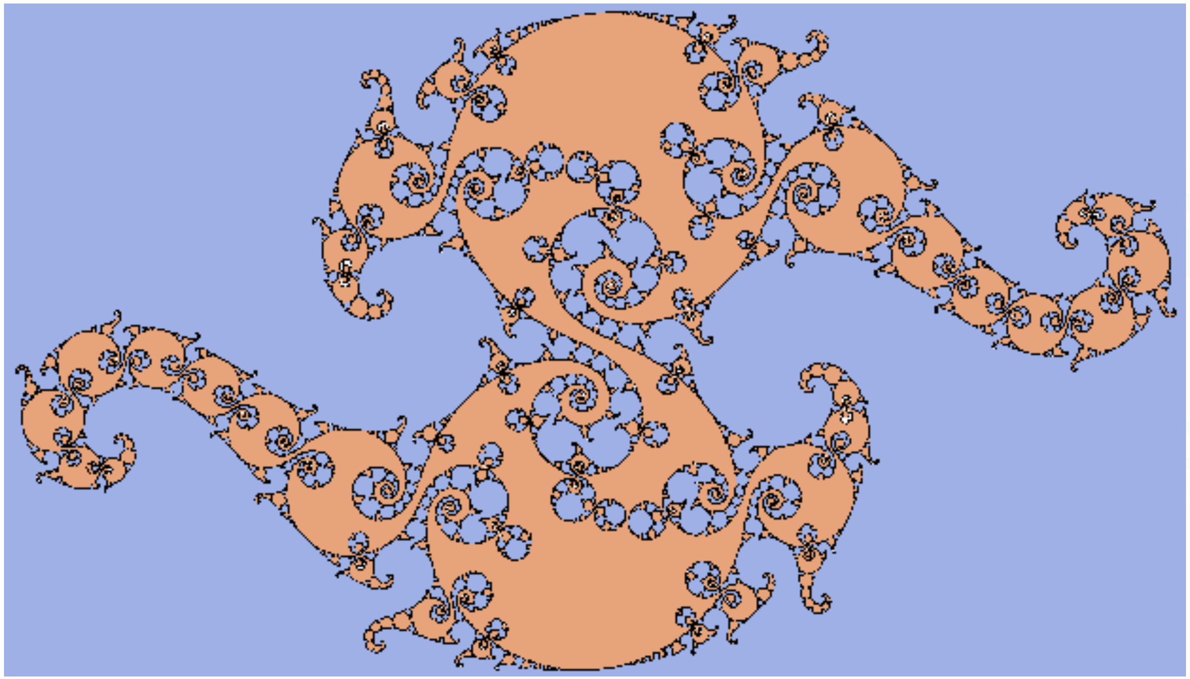
- Tetrahedral base
- d ≈ $2{,}473946$

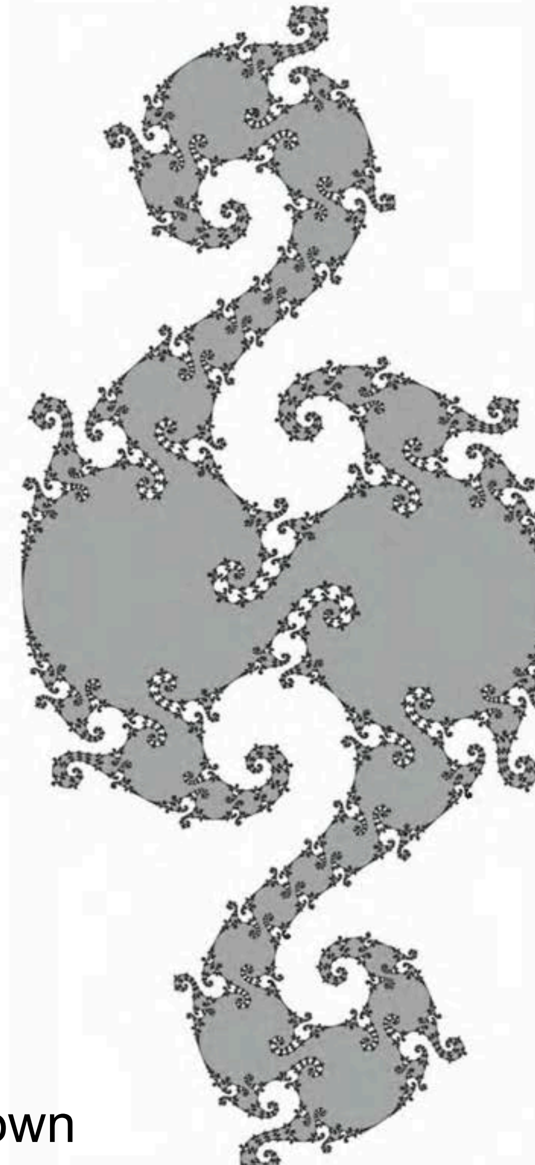


Approximation by 40 000 spheres



Dance of spirals and Jordan curve theorem





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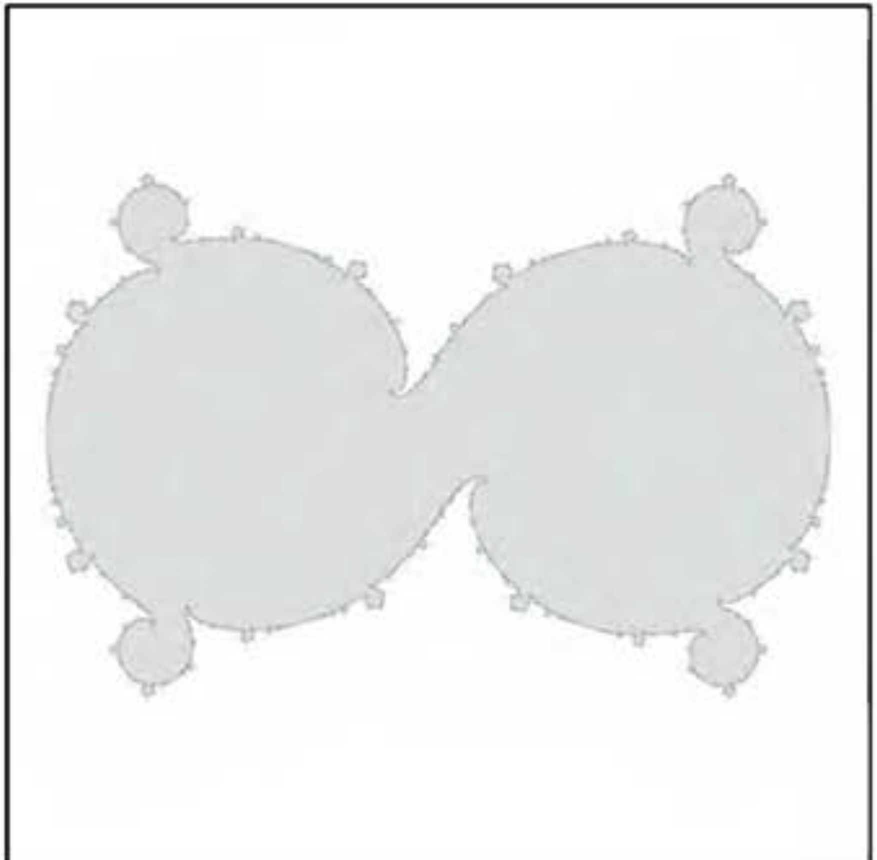
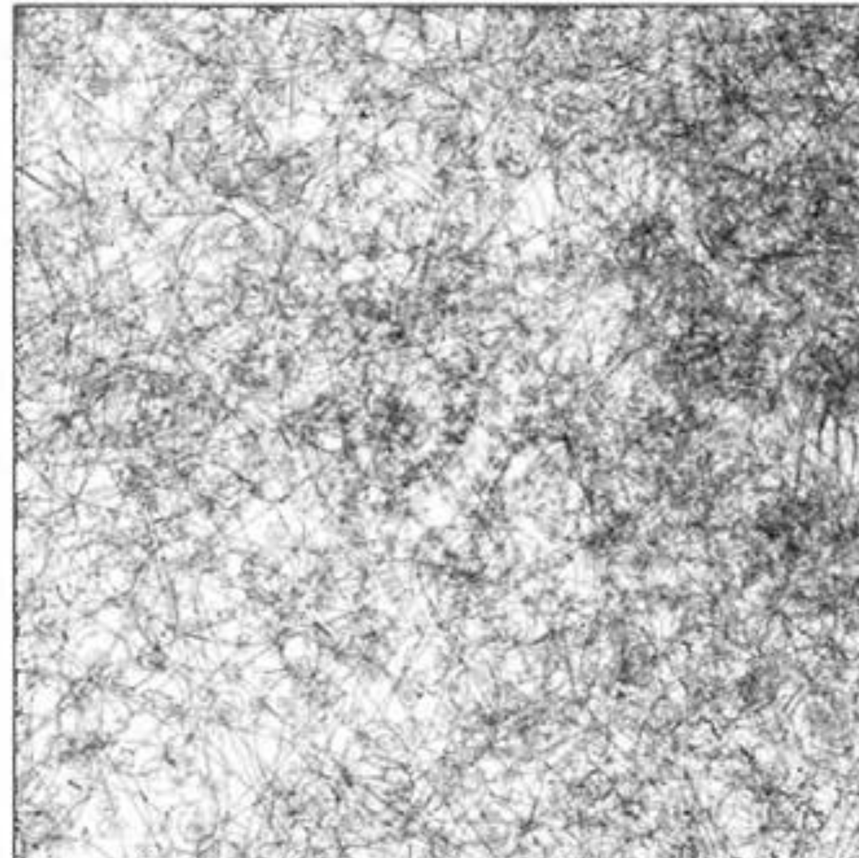
RHS generated by two maps, circle pairs not known

Playing with the parameters

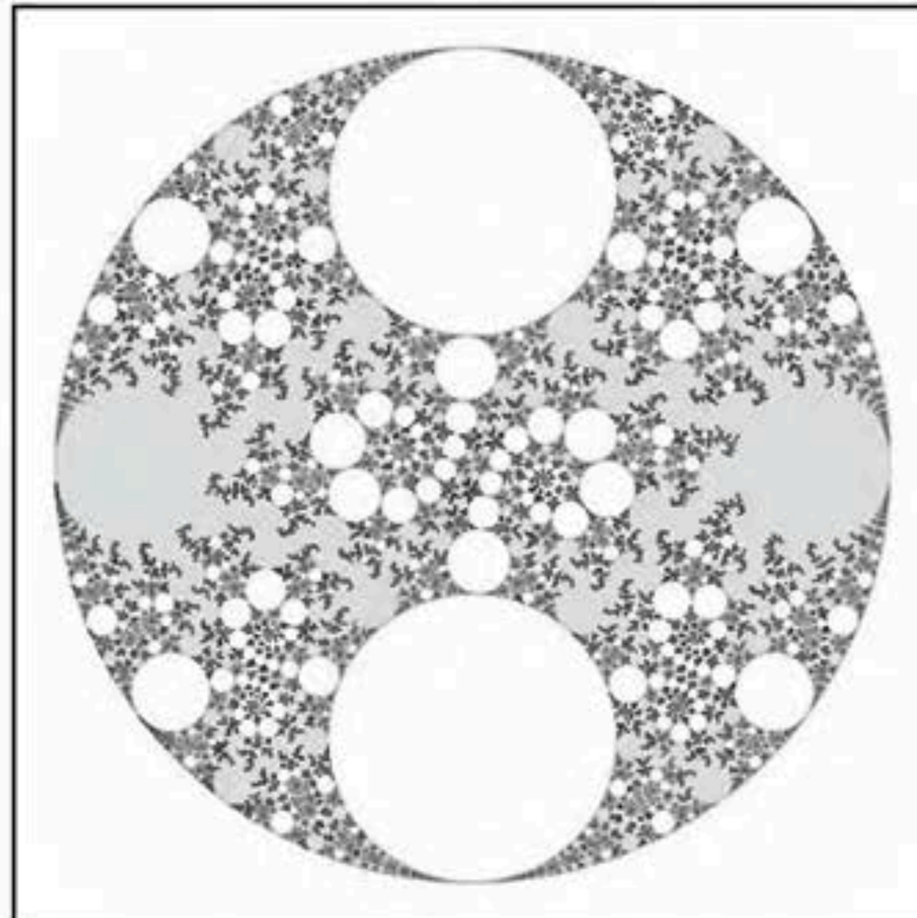


(iii) $t_a = 2.0 + 0.05i,\ t_b = 3$



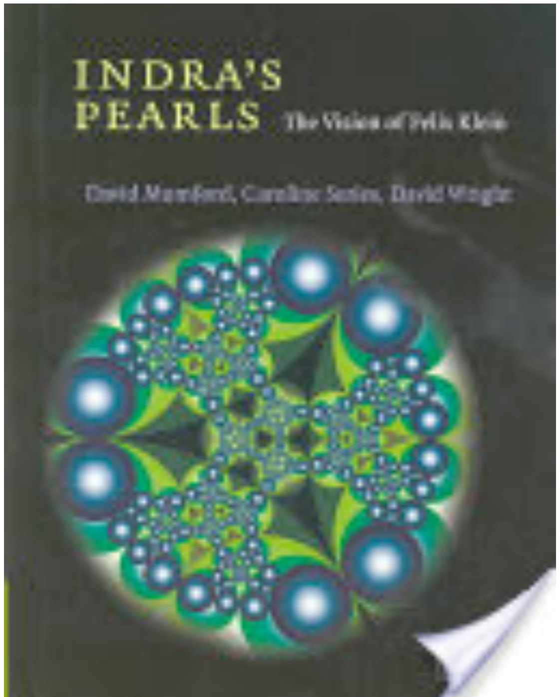
(iv) $t_a = 1.9 + 0.05i,\ t_b = 3$

(iii) $t_a = 1.888 + 0.05i$, $t_b = 2$



(iv) $t_a = 1.887 + 0.05i$, $t_b = 2$

Caroline Series



Major reference for stuff related to Kleinian groups!

Webropol feedback

Please remember to answer !

Extra 2 points for giving feedback ☺

Please also note the forthcoming Math & Arts Minor courses

https://into.aalto.fi/display/ensivuaineet2020/Aalto+Math+and+Arts

UWAS-C1400 Spatial structures (period II)

MS-E1000 Crystal Flowers in Halls of Mirrors: Mathematics meets Art and Architecture (periods III-V)