Matrix Computations
MS-A0001
Hakula/Mirka
Problem Sheet 1, 2020

P

NOTE[1]

The due date is published on the course pages. Homework can be submitted only digitally. Instructions on labeling the "papers" can be found on the course pages.

1 Introductory Problems

INTRO 1 (a) Show that the two equations

$$\begin{cases} x = 3 + 2\tau \\ y = -1 - 3\tau \end{cases} \quad \text{and} \quad \begin{cases} x = -1 - 6\tau \\ y = 5 + 9\tau \end{cases}$$

represent the same line on the plane. (b) A plane is defined by two points $(0, 2, 1)$ and $(-3, 4, 1)$, and aligned with a vector $2\mathbf{i} - \mathbf{k}$. Find the parametric representation of the plane.

INTRO 2 Let

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -2 \\ 2 & 3 \end{pmatrix}, \quad b = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}.$$

Show that the linear system of equations $Ax = b$ has no solutions. Draw the lines in the coordinate system (plane $\mathbb{R}^2$) and discuss what would the best possible solution mean in this case.

INTRO 3 Solve the linear system of equations $Ax = b$ using Gaussian elimination, when

$$A = \begin{pmatrix} 1 & 2 & 2 \\ 1 & 4 & 6 \\ 3 & -2 & -1 \\ 2 & -5 & 3 \end{pmatrix}, \quad b = \begin{pmatrix} 2 \\ 0 \\ 5 \\ -4 \end{pmatrix}.$$

[1] Published on 2020-10-26 11:24:17+02:00.

INTRO 4 Compute the following matrix operations or indicate if they are not defined. Let

$$v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \end{pmatrix}, \quad u = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}.$$

(a) $A\,u$, (b) $B\,v$, (c) $u^T\,A$, (d) $v^T\,B$, (e) $u^T\,A\,u$, (f) $v^T\,B\,A$.

INTRO 5 Find all matrices B commuting with

$$A = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix},$$

that is, for which $AB = BA$.

INTRO 6 Let

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Compute the powers A^k, $k = 1, 2, 3, \dots, 2020$.

2 Homework Problems

EXERCISE 1 Three vectors are given: $\begin{pmatrix} 0 & 2 & -4 & 8 \end{pmatrix}^{\mathrm{T}}$, $\begin{pmatrix} 6 & 12 & 3 & 3 \end{pmatrix}^{\mathrm{T}}$, $\begin{pmatrix} 2 & 5 & -1 & 5 \end{pmatrix}^{\mathrm{T}}$. Is the vector $\begin{pmatrix} -2 & 0 & -9 & 15 \end{pmatrix}^{\mathrm{T}}$ a linear combination of the previous three? If so, is the combination unique?

EXERCISE 2 Three vectors are given: $a_1 = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}^{\mathrm{T}}$, $a_2 = \begin{pmatrix} 1 & 1 & 0 \end{pmatrix}^{\mathrm{T}}$, $a_3 = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}^{\mathrm{T}}$. Show that the vector $b = \begin{pmatrix} -2 & 0 & -9 \end{pmatrix}^{\mathrm{T}}$ a linear combination of the previous three. If $a_2 = \begin{pmatrix} \alpha & \beta & 0 \end{pmatrix}^{\mathrm{T}}$, is it possible to find the values for $\alpha, \beta \in \mathbb{R}$ so that b cannot be a linear combination of a_1, a_2, a_3?

EXERCISE 3 Using Gaussian elimination, solve the linear systems of equations, or show that the solutions do not exist.

$$\text{a)} \quad \begin{cases} x + y - z = 9 \\ 8y + 6z = -6 \\ -2x + 4y - 6z = 40 \end{cases}, \qquad \text{b)} \quad \begin{cases} 4y + 3z = 8 \\ 2x - z = 2 \\ 3x + 2y = 5 \end{cases}.$$

EXERCISE 4 Let

a) $A = \begin{pmatrix} 1 & 2 \\ -2 & 3 \\ 1 & -1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & -2 & -4 \\ -1 & 4 & 9 \end{pmatrix}, \qquad b = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix},$

b) $A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{pmatrix}, \qquad C = \frac{1}{10}\begin{pmatrix} 10 & 5 & 0 & -5 \\ -3 & -1 & 1 & 3 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}.$

Find the product CA. Examine, if the linear system of equations $Ax = b$ can be solved by multiplying from the left by C. What is the solution vector x? Is it the solution of the original problem $Ax = b$?

EXERCISE 5 Find $(AB)^k$, $k \in \mathbb{N}$, when

$$A = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 2 \\ 2 & 3 & 4 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & -2 & -2 \\ 0 & 8 & 4 \\ 0 & -5 & -2 \end{pmatrix}.$$

EXERCISE 6 Let A and B be matrices of size 10×10, with elements $\alpha_{ij} = i + j$, $\beta_{ij} = i - j$. Express the elements γ_{ij} of the product matrix $C = AB$ as functions of the indeces i, j.