ELEC-E8116 Model-based control systems
Intermediate exam 1. 22. 10. 2020 / Solutions

- *Write the name of the course, your name and student number to each answer sheet.*
- *There are three (3) problems and each one must be answered.*
- *Read the instructions in a separate file (Instructions), which is available in the Exam Assignment and which has also been published in advance.*
- *In problem 0 sign with your name (typesetting is enough if you use computer document) in which you assure that you follow the exam regulations.*

Each problem 1-3 gives the maximum of 5 points.

0. Write your signature.

1. Do the following

a. Let G be the (multivariable) process model and F_y the feedback controller. Prove that it generally holds

$$(I+GF_y)^{-1}GF_y = GF_y(I+GF_y)^{-1} \qquad \text{(2 p)}$$

You may use some known mathematical formulas or results here.

b. Consider a SISO system and let $L = GF_y$. Draw a hypothetical example of a frequency response plot (Nyquist diagram) of L on the complex plane. Then show the points on the complex plane where i. the absolute value of the sensitivity function has the value $1/\sqrt{2}$, ii. the absolute value of the complementary sensitivity function has the value $1/\sqrt{2}$. Identify and explain three definitions of *bandwidth* by using the schema you have drawn.

Hint: Write $L(j\omega) = x(\omega) + jy(\omega)$ and calculate. (3 p)

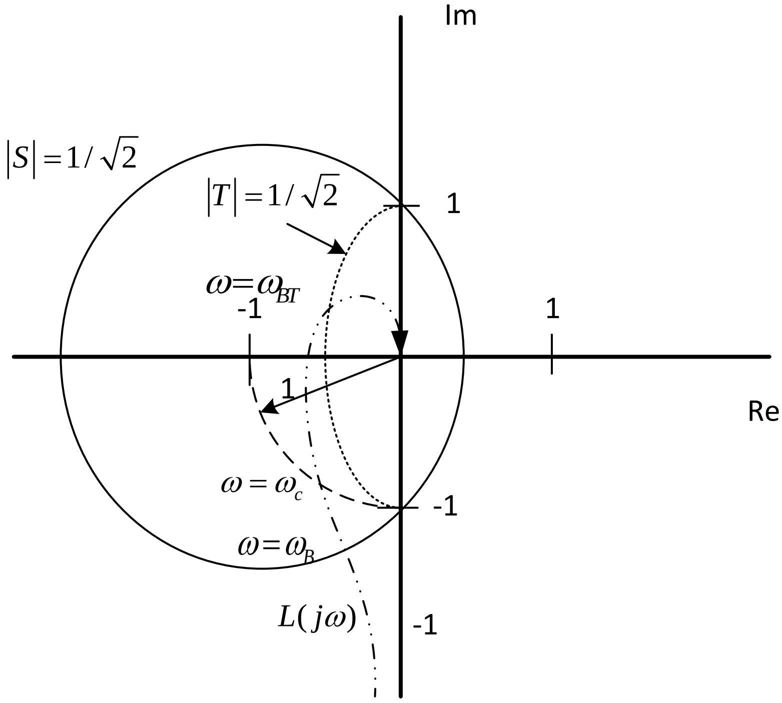
Solution: a. Use the push-through rule $(I+AB)^{-1}A = A(I+BA)^{-1}$ 2 times

$(I+GF_y)^{-1}GF_y = G(I+F_yG)^{-1}F_y = GF_y(I+GF_y)^{-1}$**.** Even cleverer is to choose $A = GF_y,\ B = I$ which gives the result directly by the push-through rule.

Solution: b.

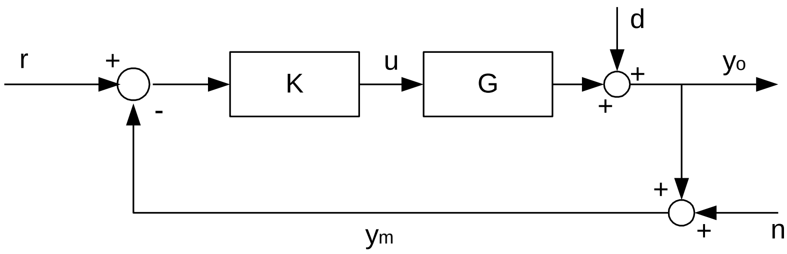
$$S(j\omega)=\frac{1}{1+L(j\omega)}=\frac{1}{1+x+jy}$$
$$\Rightarrow |S|^2=\frac{1}{(1+x)^2+y^2}=(\frac{1}{\sqrt{2}})^2=\frac{1}{2}\Rightarrow (1+x)^2+y^2=2$$
$$T(j\omega)=\frac{L(j\omega)}{1+L(j\omega)}=\frac{x+jy}{1+x+jy}$$
$$\Rightarrow |T|^2=\left|\frac{L(j\omega)}{1+L(j\omega)}\right|^2=\frac{|L|^2}{|1+L|^2}=\frac{x^2+y^2}{(1+x)^2+y^2}=(\frac{1}{\sqrt{2}})^2=\frac{1}{2}$$
$$\Rightarrow \ldots \Rightarrow (x-1)^2+y^2=2$$

The magnitudes of both sensitivity functions are circles on the complex plane. The center points are (-1,0) and (1,0), respectively. Both have the radius $\sqrt{2}\approx 1.4$.



The above figure shows the required plots. It also demonstrates the fact that under a mild condition (phase margin less than 90 degrees) it holds $\omega_B<\omega_c<\omega_{BT}$, where ω_B denotes the bandwidth, where L crosses $|S|=1/\sqrt{2}\approx -3$ dB from below, ω_c is the gain crossover frequency $|L|=1$ and ω_{BT} denotes the bandwidth, where L crosses $|T|=1/\sqrt{2}\approx -3$ dB from above.

2. Consider a multivariable control configuration.



Write the equations describing the system and identify

a. the closed loop transfer function (1 p)
b. the sensitivity function (1 p)
c. the complementary sensitivity function. Show that *S* + *T* = *I* and explain the result. (1 p)
d. Derive the equation for the error $e = r - y_0$. What are the requirements for the transfer functions in a-c in order that the system would perform well? (2 p)

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Solution: By direct calculation from the figure

$$
\begin{aligned}
& y_0 = d + GK\left[r - (n + y_0)\right] = d + GKr - GKn - GKy_0 \\
& \Rightarrow (I + GK)y_0 = d + GKr - GKn \\
& \Rightarrow y_0 = (I + GK)^{-1}d + (I + GK)^{-1}GKr - (I + GK)^{-1}GKn \\
& \qquad = Sd + G_c r - G_c n = Sd + G_c r - Tn
\end{aligned}
$$

where *S* is the sensitivity function, G_c is the closed loop transfer function and *T* is the complementary sensitivity function. Note that in a one-degree-of-freedom system $G_c = T$. Further $S + T = (I + GK)^{-1} + (I + GK)^{-1}GK = (I + GK)^{-1}\left[I + GK\right] = I$, which states that at each frequency the sum of the sensitivity and complementary sensitivity functions is always identity (matrix). This completes parts a,b, and c in the problem.

For d:
$$
\begin{aligned}
& e = r - y_0 = r - Sd - G_c r + Tn = (I - G_c)r - Sd + Tn \\
& \quad = (I - T)r - Sd + Tn = Sr - Sd + Tn
\end{aligned}
$$

For the error *e* to be small in spite of disturbances in *r*, *d* and *n* the sensitivity functions should be small. At any frequency this cannot be.

3. Find the poles, zeros and a minimal realization to the system

$$
G(s) = \begin{bmatrix} \dfrac{2}{s+1} & \dfrac{2s-3}{(s+1)(s+2)} \\ \dfrac{s-2}{s+1} & \dfrac{s}{s+2} \end{bmatrix}
$$

Note: Solution obtained by computer is not accepted. Computer can be used in verification, but that is only for yourself to check the results.

Solution: The minors are

$$\frac{2}{s+1}, \frac{2s-3}{(s+1)(s+2)}, \frac{s-2}{s+1}, \frac{s}{s+2},$$

$$\det\,(G) = \frac{2s}{(s+1)(s+2)} - \frac{2s-3}{(s+1)(s+2)} \cdot \frac{s-2}{s+1} = \ldots = \frac{3(3s-2)}{(s+1)^2(s+2)}$$

The smallest polynomial that contains all denominator polynomials of the minors is the pole polynomial. It is

$$p(s) = (s+1)^2(s+2)$$

The system has three poles, -1, -1 and -2 and so the minimal realization has three states. To determine the zero polynomial only the maximal minors count. Here only one maximal minor exists, and it has already the pole polynomial as its denominator. Then the zero polynomial is

$$z(s) = 3(3s-2)$$

and the system has one zero, 2/3 (RHP zero).

To determine a minimal realization consider

$$G(s) = \begin{bmatrix} \frac{2}{s+1} & \frac{2s-3}{(s+1)(s+2)} \\ \frac{s-2}{s+1} & \frac{s}{s+2} \end{bmatrix} = \begin{bmatrix} \frac{2}{s+1} & \frac{-5}{s+1} + \frac{7}{s+2} \\ 1 - \frac{3}{s+1} & 1 - \frac{2}{s+2} \end{bmatrix}$$

$$Y_1(s) = \frac{2}{s+1}U_1(s) - \frac{5}{s+1}U_2(s) + \frac{7}{s+2}U_2(s)$$
$$Y_2(s) = U_1(s) - \frac{3}{s+1}U_1(s) + U_2(s) - \frac{2}{s+2}U_2(s)$$

and choose for example

$$X_1(s) = \frac{U_1(s)}{s+1} \Rightarrow \dot{x}_1 + x_1 = u_1$$
$$X_2(s) = \frac{U_2(s)}{s+1} \Rightarrow \dot{x}_2 + x_2 = u_2$$
$$X_3(s) = \frac{U_2(s)}{s+2} \Rightarrow \dot{x}_3 + 2x_3 = u_2$$

The state-space representation follows immediately

$$\dot{x} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 2 & -5 & 7 \\ -3 & 0 & -2 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} u$$

Note that this is not necessarily the representation Matlab would give. The system has an infinite number of realizations, even minimal realizations.