Quantum Mechanics 2020

PHYS - C0252

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26.10.2020 Lecture I

ILOs

- Identify how course is technically implemented
- Identify Hilbert space and subspace of physical states
- Operate to state vectors by linear operators

Hilbert space

- Complete inner product space

1. $\mathcal{H} = \{|\psi\rangle\}$ is a vector space over a scalar field $\mathbb{C}$

2. There is a scalar product defined on $\mathcal{H}$ s.t. $\forall |\phi\rangle, |\psi\rangle \in \mathcal{H}$ we have

$$\langle\phi|\psi\rangle \equiv (|\phi\rangle, |\psi\rangle) \in \mathbb{C} \quad \text{s.t.}$$

(a) $\langle\psi|\phi\rangle = (\langle\phi|\psi\rangle)^* = \langle\phi|\psi\rangle^*$

(b) $(\psi|\, a\phi_1 + b\phi_2\rangle \equiv \langle\psi|\,(a|\phi_1\rangle + b|\phi_2\rangle)$
$= a\langle\psi|\phi_1\rangle + b\langle\psi|\phi_2\rangle$

(c) $\langle\psi|\psi\rangle \geq 0$; $\langle\psi|\psi\rangle = 0 \iff |\psi\rangle = 0$

Physical states are those elements of $\mathcal{H}$ for which norm is unity, i.e., $\langle\psi|\psi\rangle = 1$

Math recap on vector space

A vector space V over a scalar F is a set where $+$ is defined

$u, v, w \in V$, $\quad +: V \times V \to V$
$a, b \in F$

1. $u + (v + w) = (u + v) + w$
2. $u + v = v + u$
3. $\exists\, 0 \in V$ s.t. $v + 0 = v$
4. $\forall\, v \in V\ \exists\, -v \in V$ s.t. $v + (-v) = 0$
5. $a(bv) = (ab)v$
6. $1v = v$, when $1 \in F$
7. $a(u + v) = au + av$
8. $(a+b)v = av + bv$

$\Rightarrow$ we can define a norm $\|\psi\| \equiv \sqrt{\langle\psi|\psi\rangle} \geq 0$
$= \||\psi\rangle\|$

Ex. • $|\langle\psi|\phi\rangle| \leq \|\psi\|\,\|\phi\|$ Cauchy-Schwarz ineq

Ex. • $\||\psi\rangle + |\phi\rangle\| \leq \|\psi\| + \|\phi\|$ triangle inequality

3. All Cauchy sequences converge into $\mathcal{H}$.
That is if $\exists \{|\psi_i\rangle\}$ s.t. $\||\psi_n\rangle - |\psi_m\rangle\| \to 0$ for $n, m \to \infty$ then $\exists\, |\Psi\rangle \in \mathcal{H}$ s.t. $|\psi_m\rangle \to |\Psi\rangle$
for $m \to \infty$

not going to ask about this in the exam

Bra and ket vectors

- We define the ket vectors as elements of $\mathcal{H}$, i.e., $\mathcal{H} = \{|\psi\rangle\}$

- We define a bra vector $\langle\phi|$ through the inner product s.t.

  $\langle\phi|\psi\rangle = (|\phi\rangle, |\psi\rangle) \in \mathbb{C}$

  - observation: we can identify each bra vector $\langle\phi|$ with a linear and bounded functional $f_\phi \in \mathbb{C}$

    $\Rightarrow$ Riesz representation theorem

    ← not going to ask about this in exam

    tells that for each f_ϕ $\exists\, |\phi\rangle \in \mathcal{H}$

  $\Rightarrow$ There is a one-to-one correspondence between $\mathcal{H} \in \{|\psi\rangle\}$ and $\mathcal{H}^* \in \{\langle\phi|\}$

Math

- A functional f acting on space A is a mapping $f: A \rightarrow \mathbb{C}$

Linear operators

- $\hat{A}$ is a linear oper. on $\mathcal{H}$

  $\Leftrightarrow$ $\hat{A} : \mathcal{H} \to \mathcal{H}$ s.t. $\forall\, |\psi\rangle, |\phi\rangle \in \mathcal{H}$, $a, b \in \mathbb{C}$

  $$\hat{A}\left(a|\psi\rangle + b|\phi\rangle\right) = a\hat{A}|\psi\rangle + b\hat{A}|\phi\rangle$$

- We also define notation as

  $$\hat{A}(|\psi\rangle) \equiv \hat{A}|\psi\rangle \equiv |\hat{A}\psi\rangle$$

- It follows that $\forall\, \hat{A}, \hat{B} \in \mathcal{L}(\mathcal{H})$ we have

  space of linear operators on $\mathcal{H}$

  $$\hat{A}(\hat{B}|\psi\rangle) = (\hat{A}\hat{B})|\psi\rangle$$

Eigenvalues and eigenstates

- Eigen state $|\psi_\lambda\rangle \in \mathcal{H}$ related to the eigenvalue $\lambda \in \mathbb{C}$ of operator $\hat{A} \in \mathcal{L}(\mathcal{H})$ fulfills $\hat{A}|\psi_\lambda\rangle = \lambda|\psi_\lambda\rangle$

28.10.2020 Lecture II

ILOs

1. Use bases to represent operators
2. Identify the minimal mathematical structure to describe a physical system quantum mechanically
3. Differentiate between a measurement outcome and its expectation value ← moved to lecture 3

Outer product

We define the outer product $|\psi\rangle\langle\phi| : \mathcal{H} \to \mathcal{H}$, where $|\psi\rangle, |\phi\rangle \in \mathcal{H}$

s.t. $\forall\, |\chi\rangle \in \mathcal{H}$ we have

$$(|\psi\rangle\langle\phi|)\,|\chi\rangle = |\psi\rangle\langle\phi|\chi\rangle = \langle\phi|\chi\rangle\,|\psi\rangle$$

Bases of $\mathcal{H}$

- a set $\{|\phi_i\rangle\}_{i=1}^{N} \subset \mathcal{H}$ is called linearly independent if $\sum_{i=1}^{N} c_i|\phi_i\rangle = \vec{0}$ where $c_i \in \mathbb{C}$, $N \in \mathbb{Z}_+$, then $c_i = 0\ \forall i$
- $\mathrm{Dim}\{\mathcal{H}\}$ is the largest N for which such a linearly independent set of vectors exists
- The set is called complete if $\forall\, |\psi\rangle \in \mathcal{H}$ $\exists\, c_k \in \mathbb{C}$ s.t. $|\psi\rangle = \sum_{k=1}^{N} c_k |\phi_k\rangle$

  (similarly for infinite-dimensional space)
- A complete set of linearly independent vectors $\{|\phi_k\rangle\}$ is a basis of $\mathcal{H}$

- for orthonormal basis, $\{|\phi_k\rangle\}$ we have

$$\langle\phi_l|\phi_m\rangle = \delta_{lm} = \begin{cases} 0, & l \neq m \\ 1, & l = m \end{cases}$$

- Observation for orthonormal basis $\{|\phi_k\rangle\}$:

take an arbitrary $|\psi\rangle \in \mathcal{H}$.

we have $|\psi\rangle = \sum_k c_k |\phi_k\rangle$

$$\Big\Downarrow \ \langle \phi_m |$$

$$\langle \phi_m | \psi \rangle = \sum_k c_k \langle \phi_m | \phi_k \rangle = c_m$$

Thus $|\psi\rangle = \sum_m \langle \phi_m | \psi \rangle |\phi_m\rangle$

$$= \sum_m |\phi_m\rangle\langle \phi_m | \psi \rangle$$

$$= \underbrace{\left(\sum_m |\phi_m\rangle\langle \phi_m | \right)}_{=:\ \hat{I}} |\psi\rangle$$

states v.s. vectors

- fix an orthonormal basis $\{|\phi_k\rangle\}$

and a $|\psi\rangle \in \mathcal{H}$

$$|\psi\rangle = \sum_k c_k |\phi_k\rangle$$

$$\hat{=} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \end{pmatrix}$$

↑ represented by sign

operators v.s. matrices

- let $\hat{A} \in \mathcal{L}(\mathcal{H})$ and $\{|\phi_m\rangle\}$ an orthonormal basis

$$\hat{A} = \hat{I}\hat{A}\hat{I} = \left(\sum_m |\phi_m\rangle\langle\phi_m|\right)\hat{A}\left(\sum_k |\phi_k\rangle\langle\phi_k|\right)$$

$$= \sum_{m,k} |\phi_m\rangle \underbrace{\langle\phi_m|\hat{A}|\phi_k\rangle}_{A_{mk} \in \mathbb{C}} \langle\phi_k|$$

$$= \sum_{m,k} A_{mk} |\phi_m\rangle\langle\phi_k|$$

$$\hat{=} \begin{pmatrix} A_{11} & A_{22} & \cdots \\ A_{21} & A_{22} & \cdots \\ A_{31} & \ddots & \\ \vdots & & \end{pmatrix}$$

- operation

$$\hat{A}|\psi\rangle = \sum_{m,k,l} A_{mk}|\phi_m\rangle\langle\phi_k| c_l|\phi_l\rangle$$

$$= \sum_{m,k} A_{mk} c_k |\phi_m\rangle$$

$$\langle\phi_m|\hat{A}|\psi\rangle = \sum_k A_{mk} c_k$$

$$= (\overbrace{0\ 0 \cdots}^{m-1 \text{ zeroes}} 1\ 0\ 0 \cdots) \begin{pmatrix} A_{11} & A_{12} & \cdots \\ A_{23} & \ddots & \\ \vdots & & \ddots \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \end{pmatrix}$$

Math on complex conjugation

$\forall z \in \mathbb{C}$ we have $x, y \in \mathbb{R}$

$z = x + iy$, i is imag. unit

$z^* = x - iy$

Adjungate

Fix $\hat{A} \in \mathcal{L}(\mathcal{H})$. We defi $\hat{A}: \mathcal{H}^* \to \mathcal{H}^*$
s.t. $\forall |\phi\rangle, |\psi\rangle \in \mathcal{H}$

$$(\langle\phi|\hat{A})|\psi\rangle = \langle\phi|(\hat{A}|\psi\rangle)$$

We observe that the operation

$\langle\phi|\hat{A}$ is a linear functional on $\mathcal{H}$ and is bounded since $\hat{A}$ is bounded.

Thus it follows from Riesz representation theorem that $\exists\, |\phi'\rangle \in \mathcal{H}$ s.t.

$$\langle\phi'| = \langle\phi|\hat{A}$$

This defines also a linear operator

$$\hat{A}^\dagger|\phi\rangle = |\phi'\rangle$$

↑
adjungate of $\hat{A}$

ex. $\langle\phi|c|\psi\rangle = (|\phi\rangle, c|\psi\rangle)$

$= c(|\phi\rangle, |\psi\rangle)$

$= (c^*|\phi\rangle, |\psi\rangle)$

$\Rightarrow c^\dagger = c^* \in \mathbb{C}$

Properties of adjugate

$(\hat{A}^\dagger)^\dagger = \hat{A}$ (not proved here)

$(a\hat{A})^\dagger = a^*\hat{A}^\dagger$

$(\hat{A}\hat{B})^\dagger = \hat{B}^\dagger\hat{A}^\dagger$

$(|\psi\rangle\langle\phi|)^\dagger = |\phi\rangle\langle\psi|$

About notation

$\langle\phi|\hat{A}|\psi\rangle = (|\phi\rangle, \hat{A}|\psi\rangle) = (\hat{A}^\dagger|\phi\rangle, |\psi\rangle)$

$= (\hat{A}^\dagger\phi|\psi\rangle$ ← sometimes denoted like this

Hermitian operators

- $\hat{A} \in \mathcal{L}(\mathcal{H})$ is defined Hermitian iff $\hat{A}^\dagger = \hat{A}$

- The ~~se~~ (generalized) spectral theorem that states that $\forall \hat{A}^\dagger = \hat{A} \in \mathcal{L}(\mathcal{H})$ $\exists \{|\psi_k\rangle\} \subset \mathcal{H}$ which is a complete $\underbrace{\text{basis}}^{\text{(orthonormal)}}$ of $\mathcal{H}$ and $\{\lambda_k\} \subset \mathbb{R}$ s.t.

$$\hat{A}|\psi_k\rangle = \lambda_k|\psi_k\rangle$$

- Thus we can express $\hat{A} = \sum_k \lambda_k |\psi_k\rangle\langle\psi_k|$

Postulates of quantum Mechanics

we talk about this level

I) For each physical system there exists a corresponding (Rigged) Hilbert space

II) Each physical state of this system can be represented by a quantum state $|\psi\rangle \in \mathcal{H}$, where $\langle\psi|\psi\rangle = 1$

(in case of partial description of the full system, or subsystem may be described by a so-called density operator)

III) For each measurable quantity of the system A we have $\hat{A} \in \mathcal{L}(\mathcal{H})$ s.t. $\hat{A}^{\dagger} = \hat{A}$

- In an ideal measurement any measurement outcome equals be an eigenvalue of $\hat{A}$

Measurement

- Let $|\psi\rangle \in \mathcal{H}$
- Let $\hat{A}^{\dagger} = \hat{A} \in \mathcal{L}(\mathcal{H})$ with a discrete spectrum $\{a_n\}_n$ and the corresponding eigenstates $\{|\phi_{n,i}\rangle\}_{n,i=1}^{g_n}$, where g_n equals the amount of degeneracy.

  Thus $\hat{A}|\phi_{n,i}\rangle = a_n|\phi_{n,i}\rangle$.
- Let $\{|\phi_{n,i}\}$ orthonormal complete basis of $\mathcal{H}$

- The probability of obtaining a measurement result a_n is given by

$$P(a_n) = \sum_{i=1}^{g_n} |\langle\phi_{n,i}|\psi\rangle|^2$$

After measurement

- If we measure a_n which is an eigenvalue of $\hat{A}$ for $|\psi\rangle \in \mathcal{H}$ (all definitions as above), the state of the system collapses into

$$|\psi'\rangle = \frac{\hat{P}_n|\psi\rangle}{\|\hat{P}_n|\psi\rangle\|}, \quad \hat{P}_n = \sum_{i=1}^{g_n} |\phi_{n,i}\rangle\langle\phi_{n,i}|$$

this is a projector into the subspace corresponding to the, subspace spanned by the eigenstates $\{|\phi_{n,i}\rangle\}_{i=1}^{g_n}$

Def

$\hat{P} \in \mathcal{L}(\mathcal{H})$ is a projector iff $\hat{P}^2 = \hat{P}$

Temporal evolution

Let $|\psi\rangle \in \mathcal{H}$ and $\hat{H} \in \mathcal{L}(\mathcal{H})$ s.t. H is classical Hamiltonian which is typically equal to the total energy of the system in question. Thus we have

$$i\hbar\,\partial_t|\psi(t)\rangle = \hat{H}|\psi(t)\rangle$$

← Schrödinger equation

- Note that $\hat{H}$ may depend on time through temporally dependent parameters $\{\alpha_i(t)\}$. That is $\hat{H} = \hat{H}[\alpha_1(t), \dots]$

2.11.2020 Lecture III

ILOs

1. Differentiate between measurement outcome and its expectation value
2. Identify continuous bases for Hilbert spaces
3. Apply Lagrangian formalism to quantize physical systems

Expectation values

- Let $\hat{A} \in \mathcal{L}(\mathcal{H})$ and $|\psi\rangle \in \mathcal{H}$
- We define the expectation value of $\hat{A}$ in the state $|\psi\rangle$ is defined by

$$\langle A \rangle \equiv \langle \psi | \hat{A} | \psi \rangle, \quad \langle \hat{A} \rangle \equiv \langle \psi | \hat{A} | \psi \rangle$$

- Let A be an observable $\Rightarrow \hat{A} = \hat{A}^{\dagger}$

$\Rightarrow \exists \{|\phi_k\rangle\}$ s.t. $\langle \phi_k | \phi_m \rangle = \delta_{km}$

and $\hat{A}|\phi_k\rangle = a_k|\phi_k\rangle, \quad a_k \in \mathbb{R}$

$\Rightarrow |\psi\rangle = \sum_k c_k |\phi_k\rangle$

$$\langle \psi | \hat{A} | \psi \rangle = \Big(\sum_k c_k |\phi_k\rangle\Big)^{\dagger} \hat{A} \sum_k c_k |\phi_k\rangle$$

$$= \Big(\sum_m c_k |\phi_k\rangle\Big)^{\dagger} \sum_k c_k a_k |\phi_k\rangle$$

$$= \sum_{m,k} c_m^{*} a_k c_k \langle \phi_m | \phi_k \rangle$$

$$= \sum_k a_k \underbrace{|c_k|^2}_{P(a_k)} = \sum_k a_k P(a_k)$$

Variance

- We define the variance of $\hat{A}$ in $|\psi\rangle$

$$\Delta A^2 = \langle \psi | (\hat{A} - \langle \psi | \hat{A} | \psi \rangle)^2 | \psi \rangle$$

$$= \langle \psi | \hat{A}^2 | \psi \rangle - (\langle \psi | \hat{A} | \psi \rangle)^2$$

$$= \sum_k a_k^2 P_k - \Big(\sum_k a_k P_k\Big)^2$$

Continuous bases

Math $f \in C^\infty$

$$\int dx \, \{ \delta(x) f(x) \} = f(0)$$

- Continuous bases are sometimes required, for example if continuous variables are used.
- let $\{|\psi_\alpha\rangle\}$ where $\alpha \in \mathbb{R}$ $\alpha' \in \mathbb{R}$ s.t.

$$\langle \psi_\alpha | \psi_{\alpha'} \rangle = \delta(\alpha - \alpha')$$

  - Note that $\langle \psi_\alpha | \psi_\alpha \rangle = \delta(0) = \infty$ thus $|\psi_\alpha\rangle$ is normalizable.

    ⇒ we work with Rigged Hilbert space that allows these ← Maybe more on this later

  - In any case, we can use these as

$$|\psi\rangle = \int c_\alpha |\psi_\alpha\rangle \, d\alpha = \int \langle \psi_\alpha | \psi \rangle |\psi_\alpha\rangle \, d\alpha$$

$$\overset{\uparrow}{c_\alpha} = \langle \psi_\alpha | \psi \rangle \equiv \psi(\alpha) \leftarrow \text{wave function}$$

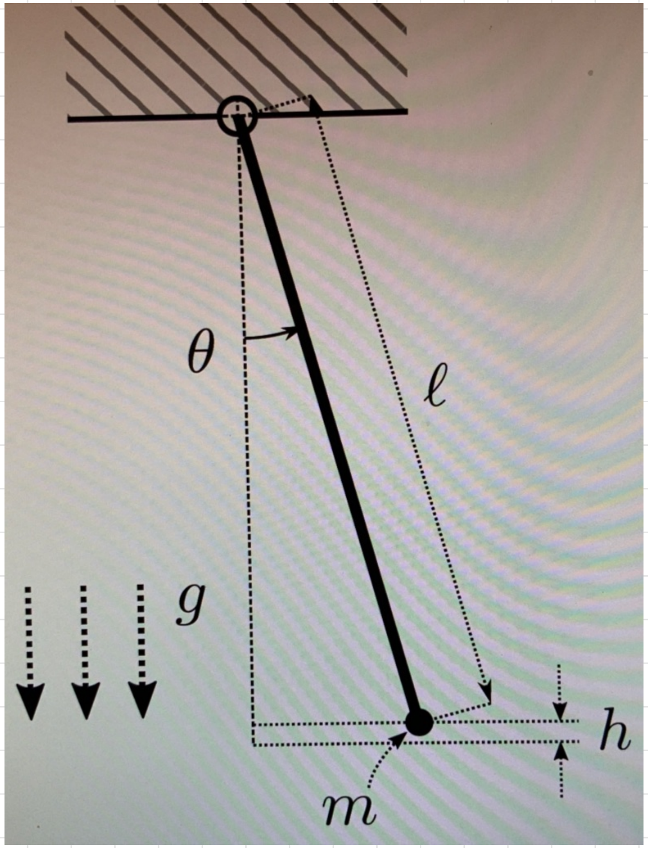
$$= \int |\psi_\alpha\rangle\langle\psi_\alpha | \psi \rangle \, d\alpha = \underbrace{\left(\int |\psi_\alpha\rangle\langle\psi_\alpha| \, d\alpha \right)}_{= \hat{1}} |\psi\rangle \, , \quad \forall |\psi\rangle \in \mathcal{H}$$

- Measurement probability of the measurement outcome $[\alpha, \alpha + d\alpha]$ is defined by

$$dP(\alpha) = |\langle \psi_\alpha | \psi \rangle|^2 \, d\alpha$$

Quantization of quantum systems:

classical pendelum



- Pick our generalized position to be

$$q = l\theta$$

- The potential energy is described by q (not $\dot q$). thus it assumes the form

$$V = gmh = mgl(1-\cos\theta) \approx \frac{1}{2} mgl\theta^2 = \frac{mg}{2l} q^2$$

- The kinetic energy is described by $\dot q$. Thus we have

$$T = \frac{1}{2} m v^2 = \frac{1}{2} m l^2 \dot\theta^2 = \frac{1}{2} m \dot q^2$$

- Lagrangian is defined as

$$L = T - V = \frac{1}{2} m l^2 \dot\theta^2 - \frac{1}{2} mgl\theta^2 = \frac{1}{2} m\dot q^2 - \frac{mg}{2l} q^2$$

- Generalized momentum is defined as

$$p = \frac{\partial L}{\partial \dot q} = \frac{\partial}{\partial \dot q}\left(\frac{1}{2} m\dot q^2 - \frac{mg}{2l} q^2\right) = m\dot q$$

← consider q and $\dot q$ as independent variables here

Math Taylor series

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots$$

- The Hamiltonian is defined by

$$H = \dot q p - L = m\dot q^2 - \left(\frac{1}{2} m\dot q^2 - \frac{mg}{2l} q^2\right) = \underbrace{\frac{p^2}{2m} + \frac{mg}{2l} q^2}_{\text{this is the Hamiltonian of 1D harmonic oscillator}} = T + V$$

- Quantization proceeds as follows:

1. operator substitution
   $q \to \hat q : \mathcal{H} \to \mathcal{H}$, $\hat q = \hat q^\dagger$
   $p \to \hat p : \mathcal{H} \to \mathcal{H}$, $\hat p = \hat p^\dagger$
2. $[\hat p, \hat q] \equiv \hat p \hat q - \hat q \hat p = -i\hbar$
3. $H \to \hat H$ with the help of 1.
   [for above H we have $\hat H = \frac{\hat p^2}{2m} + \frac{mg}{2l} \hat q^2$]
4. $i\hbar \partial_t |\psi\rangle = \hat H |\psi\rangle$

Lecture 4

ILOs

1. Apply operator exponential to symbolically solve the Schrödinger eq.
2. Differentiate between a qubit and a general quantum system
3. Represent qubit state on the Bloch sphere

Unitary temporal evolution

- Let $|\psi(0)\rangle \in \mathcal{H}$ and $\hat{H} \in \mathcal{L}(\mathcal{H})$ be the Hamiltonian. Schrödinger equation yields the temporal evolution as $\exists\, |\psi(t)\rangle \in \mathcal{H}$ s.t. $|\psi(t=0)\rangle = |\psi(0)\rangle$

and $i\hbar\,\partial_t|\psi(t)\rangle = \hat{H}|\psi(t)\rangle \iff \partial_t|\psi(t)\rangle = -\frac{i\hat{H}}{\hbar}|\psi(t)\rangle$

- Note that we assumed that $\hat{H}$ is independent of time

- Let us define $e^{\hat{A}} = \sum_{n=0}^{\infty} \frac{\hat{A}^n}{n!}$, for $\forall \hat{A} \in \mathcal{L}(\mathcal{H})$

Thus

$$\partial_t e^{\hat{A}t} = \partial_t\left(\sum_{n=0}^{\infty}\frac{\hat{A}^n t^n}{n!}\right) = \sum_{n=1}^{\infty}\frac{\hat{A}^n\, n t^{n-1}}{n!}$$

$$= \hat{A}\sum_{n=1}^{\infty}\frac{(\hat{A}t)^{n-1}}{(n-1)!} = \hat{A}\sum_{n=0}^{\infty}\frac{(\hat{A}t)^n}{n!}$$

$$= \hat{A}\, e^{\hat{A}t}$$

- Thus

$$|\psi(t)\rangle = \underbrace{e^{-i\hat{H}t/\hbar}}_{\equiv\, \hat{U}(t)}|\psi(0)\rangle$$

Math

$$\partial_x f(x) = \lambda f(x) \Rightarrow f(x) = A e^{\lambda x} \quad \Big| \quad e^{-x}e^{x} = 1$$

- We observe that $\hat{U}(t)^{\dagger}\hat{U}(t)$

$$= e^{i\hat{H}t/\hbar}e^{-i\hat{H}t/\hbar} = \hat{U}^{-1}\hat{U} = \hat{I}$$

↑ ex.

which means by definition the $\hat{U}$ is unitary

- Let $\{|\psi_m\rangle\} \subset \mathcal{H}$ be eigenbasis of $\hat{H}$, i.e., $\hat{H}|\psi_n\rangle = a_m|\psi\rangle_m$ where $\{a_m\}_{m=0}^{\infty} \in \mathbb{R}$

Thus we can expand

$$|\psi(0)\rangle = \sum_{m=0}^{\infty} c_m|\psi_m\rangle, \quad \begin{array}{c}\langle\psi_m|\psi\rangle\\ \shortparallel\\ c_m\end{array} \in \mathbb{C}$$

$$\Rightarrow |\psi(t)\rangle = e^{-i\hat{H}t/\hbar}|\psi(0)\rangle$$

$$= \left(\sum_{n=0}^{\infty}\frac{(-i\hat{H}t)^n}{n!}\right)\left(\sum_{m=0}^{\infty}c_m|\psi_m\rangle\right)$$

$$= \sum_{m=0}^{\infty}\left(\sum_{n=0}^{\infty}\frac{c_m(-i\hat{H}t)^n}{n!}|\psi_m\rangle\right) = \sum_{m=0}^{\infty} e^{-ia_m t/\hbar}c_m|\psi_m\rangle$$

$$= \sum_{m=0}^{\infty} c_m e^{-iat/\hbar}|\psi_m\rangle$$

Case of temporally dependent Hamiltonian

- Still the Schrödinger equation is obeyed

$$i\partial_t|\psi(t)\rangle = \hat{H}(t)|\psi(t)\rangle$$

and the evolution is unitary

Thus $\exists \{\hat{U}(t)\} \in \mathcal{L}(\mathcal{H})$ s.t.

$$\hat{U}(t)|\psi(0)\rangle = |\psi(t)\rangle \;, \quad \forall |\psi(0)\rangle \in \mathcal{H}$$

$$\Rightarrow i\partial_t\left(\hat{U}(t)|\psi(0)\rangle\right) = \hat{H}(t)\left(\hat{U}(t)|\psi(0)\rangle\right)$$

$$\Rightarrow \quad i\partial_t \hat{U}(t) = \hat{H}(t)\,\hat{U}(t)$$

is equivalent to Schrödinger eq.

ex.

Build $U(t)$ for $H(t)$

Properties of unitary operators

- $(\hat{U}_1\hat{U}_2)^\dagger = \hat{U}_2^\dagger \hat{U}_1^\dagger = \hat{U}_2^{-1}\hat{U}_1^{-1} = (U_1U_2)^{-1}$

$\Rightarrow \hat{U}_1\hat{U}_2$ is also unitary

- Let $|\psi\rangle, |\phi\rangle \in \mathcal{H}$ and $\hat{U}^\dagger = \hat{U}^{-1} \in \mathcal{L}($

we define $|\psi'\rangle = \hat{U}|\psi\rangle$

$|\phi'\rangle = \hat{U}|\phi\rangle$

$$\langle\psi|\phi\rangle = \langle\psi|\hat{I}|\phi\rangle = \langle\psi|\hat{U}^{-1}\hat{U}|\phi\rangle$$

$$= \langle\psi|\hat{U}^\dagger\hat{U}|\phi\rangle = (\hat{U}|\psi\rangle, \hat{U}|\phi\rangle)$$

$$= \langle\psi'|\phi'\rangle$$

- Unitary operators can be considered as rotations (sometimes reflections as well)

Qubit

- A qubit can refer either to a physical system or to a mathematical construction. In either case it modeled by a two-level quantum system as follows;

- Let $\mathcal{H}_2 = \text{span}\{|\tilde{0}\rangle, |\tilde{1}\rangle\}$, where $\langle\tilde{0}|\tilde{0}\rangle = 1 = \langle\tilde{1}|\tilde{1}\rangle$ and $\langle\tilde{0}|\tilde{1}\rangle = 0$

- $\mathcal{H}_2$ fully describes all possible states of the qubit where

$$|\psi\rangle \in \mathcal{H}_2 \quad \text{and} \quad \| |\psi\rangle \| = 1$$

- Thus the Hamiltonian $\hat{H}$ has just two eigenvalues $\varepsilon_1 \leq \varepsilon_2 \in \mathbb{R}$ and the corresponding eigenvectors are $|g\rangle$ and $|e\rangle$, respectively.

- Thus $\hat{H} = \varepsilon_1 |g\rangle\langle g| + \varepsilon_2 |e\rangle\langle e|$

$$= \frac{\varepsilon}{2}\left(-|g\rangle\langle g| + |e\rangle\langle e|\right)$$

$$+ \frac{(\varepsilon_1 - \varepsilon_2)}{2} |g\rangle\langle g|$$

$$+ \frac{(\varepsilon_1 - \varepsilon_2)}{2} |e\rangle\langle e|$$

$\varepsilon = \varepsilon_2 - \varepsilon_1$

Math Let $\{|\psi_m\rangle\} \subset \mathcal{H}$

$$\text{span}(\{|\psi_m\rangle\}) = \left\{ \sum_m c_m |\psi_m\rangle \,\middle|\, c_m \in \mathbb{C} \right\}$$

$$= \frac{\varepsilon}{2}\left(-|g\rangle\langle g| + |e\rangle\langle e|\right) + \underbrace{\frac{\varepsilon_1 - e_2}{2}\hat{I}}$$

- Thus $\hat{H}_q = -\frac{\varepsilon}{2}\left(|g\rangle\langle g| - |e\rangle\langle e|\right)$

We can drop this since it just changes the phase of all $|\psi\rangle \in \mathcal{H}$

- We can define the qubit states $|0\rangle \equiv |g\rangle$

$$|1\rangle \equiv |e\rangle$$

- Thus $\hat{H}_q = -\frac{\varepsilon}{2}\hat{\sigma}_z$, $\hat{\sigma}_z = |0\rangle\langle 0| - |1\rangle\langle 1|$

$$\hat{=} \begin{pmatrix} -\frac{\varepsilon}{2} & 0 \\ 0 & +\frac{\varepsilon}{2} \end{pmatrix}$$

- The temporal evolution is given by

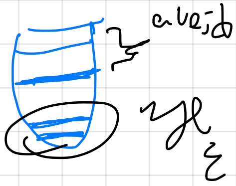
$$|\psi(t)\rangle = e^{-i\hat{H}t/\hbar}|\psi(0)\rangle, \quad |\psi(0)\rangle = c_0|0\rangle + c_1|1\rangle$$

$$= e^{+i\frac{\varepsilon}{2}\hat{\sigma}_z t/\hbar}|\psi(0)\rangle$$

$$= e^{i\frac{\varepsilon}{2}t/\hbar} c_0|0\rangle + e^{-i\frac{\varepsilon}{2}t/\hbar} c_1|1\rangle$$

How to get a qubit from a physical system

- Confine dynamics to the subspace of two states
- For example, a spin is a natural two-level system, but confined (for example in atoms)
- For example, non-linear system where $\varepsilon_0 < \varepsilon_1 < \varepsilon_3$ --- are eigenvalues of H s.t. $\varepsilon_1 - \varepsilon_0 \neq \varepsilon_2 - \varepsilon_1$.



Pauli operators

Def. $\hat{\sigma}_z = |0\rangle\langle 1| - |1\rangle\langle 1|$

$\hat{\sigma}_x = |0\rangle\langle 1| + |1\rangle\langle 0|$

$\hat{\sigma}_y = i|0\rangle\langle 1| - i|1\rangle\langle 0|$

Properties

$\hat{\sigma}_\alpha^2 = I, \quad \hat{\sigma}_\alpha^\dagger = \hat{\sigma}_\alpha$

$$[\hat{\sigma}_i, \hat{\sigma}_j] = \sum_{k\in\{x,y,z\}} 2i\hat{\sigma}_k \varepsilon_{ijk}, \quad i,j \in \{x,y,z\}$$

where $\varepsilon_{ijk} = \begin{cases} 1, & \varepsilon_{xyz} = \varepsilon_{zxy} = \varepsilon_{yzx} \\ -1, & \varepsilon_{yxz} = \varepsilon_{zyx} = \varepsilon_{xzy} \\ 0, & \text{otherwise} \end{cases}$

- Def. $\hat{\sigma}^- = |0\rangle\langle 1|$

$\hat{\sigma}^+ = (\hat{\sigma}^-)^\dagger = |1\rangle\langle 0|$

ex. show $e^{i\varphi \vec{a}\cdot\vec{\sigma}} = I\cos\varphi + \vec{a}\cdot\vec{\sigma}\sin\varphi$

where $\|\vec{a}\| = 1$

$\vec{a}\cdot\vec{\sigma} = a_x\hat{\sigma}_x + a_y\hat{\sigma}_y + a_z\hat{\sigma}_z$

Bloch Sphere

Qubit state can always be expressed as

$$|\psi\rangle = \cos(\theta/2)|0\rangle + e^{i\varphi}\sin(\theta/2)|1\rangle$$

φ ↓ azimuthal angle

↑ polar angle

- Note that since a global phase of the state $e^{i\alpha}$ does not affect any measurement outcome, i.e.,

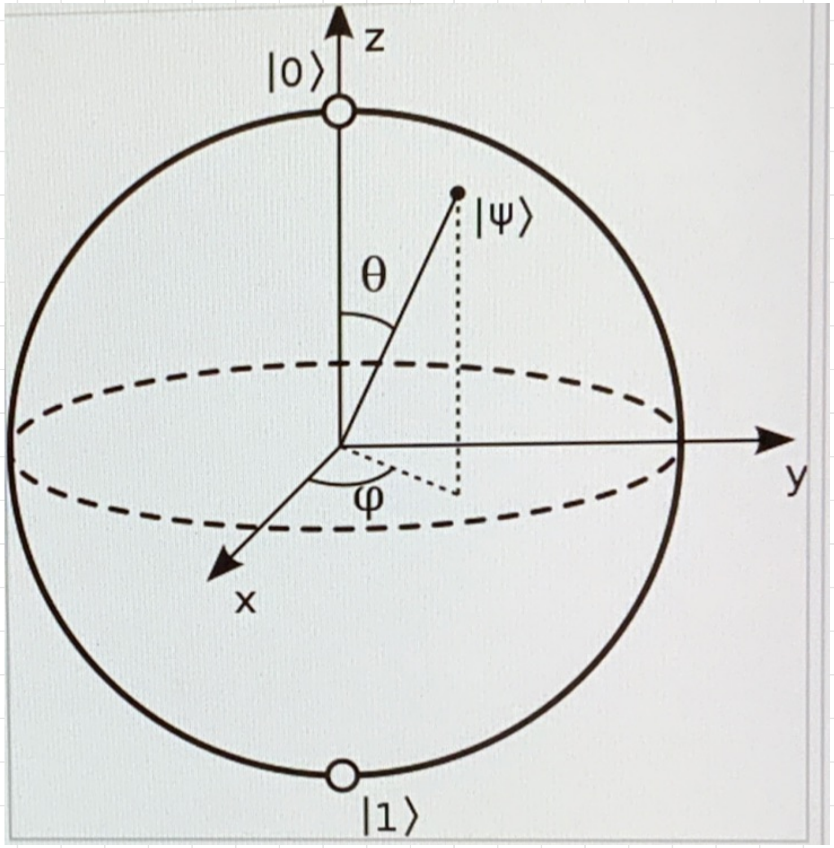
$$\langle\psi|\hat{A}|\psi\rangle = \langle\psi|\hat{A}e^{-i\alpha}e^{+i\alpha}|\psi\rangle = \langle\psi|e^{-i\alpha}\hat{A}e^{i\alpha}|\psi\rangle$$

$$= \langle e^{i\alpha}\psi|\hat{A}e^{i\alpha}|\psi\rangle,$$

we can always choose $c_0 \in \mathbb{R}$ in $|\psi\rangle = c_0|0\rangle + c_1|1\rangle$)

- Thus for each state there are unique $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi)$ which correspond to a point on a unit sphere as shown here ↗

Bloch sphere representation



ex.

$\hat{U}(t)$ are rotations of the Bloch vector

Lecture 5

ILOs

1. Apply tensor product to construct a quantum register of N qubits
2. Identify the constituents of a quantum algorithm
3. Apply the commutator to identify conserved quantities

Tunable Hamiltonian for quantum gates

- ~~Let~~ $\mathrm{span}\{|0\rangle, |1\rangle\} = \mathcal{H}_2$ and assume that control over the Hamiltonian s.t. $\hat{H} = \varepsilon_0 \vec{a}(t) \cdot \hat{\vec{\sigma}}$, where $\vec{a} \in \mathbb{R}^3, \|\vec{a}\|=1$

  ε_0 / Joule $\in \mathbb{R}$

- thus any unitary evolution $\hat{U} = \cos(\theta)\hat{1} + \vec{b}\cdot\hat{\vec{\sigma}}\sin(\theta)$ (which is referred to as a single qubit gate) can be implemented, for example, by a control sequence

$$\vec{a}(t) = \begin{cases} 0, & t < 0 \\ \varepsilon_0 \vec{b}, & 0 \leq t \leq 2\theta\hbar/\varepsilon_0 \\ 0, & t > 2\theta\hbar/\varepsilon_0 \end{cases}$$

- There are many other ways of course
  - Note there is also a way to use

$$\hat{H} = -\frac{\varepsilon}{2}\hat{\sigma}_z$$

and apply a field $\hat{H}_{ex}(t) = \frac{\Omega}{2}\sin(\omega t + \varphi)$ $\leftarrow \boxed{\varepsilon/\hbar}$

- That will result in so called Rabi oscillations to be discussed later

Single-qubit gates: examples

- The NOT gate corresponds to $\hat{\sigma}_x = |0\rangle\langle 1| + |1\rangle\langle 0| \hat{=} \begin{array}{ccc} |0\rangle & |1\rangle & \\ \end{array}$ $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{array}{c}|0\rangle \\ |1\rangle\end{array}$; $\sigma_x = X$
- Hadamard gate corresponds to

$$\hat{H}_g = \hat{\sigma}_z + \hat{\sigma}_x \hat{=} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\frac{1}{2} = H_g$$

- Phase flip corresponds to $\hat{\sigma}_z = |0\rangle\langle 0| - |1\rangle\langle 1|$

$$\hat{=} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_z = Z$$

Ctm

- Find $\hat{a}(t)$ implementing these
- $\hat{H}_g \hat{\sigma}_x \hat{H}_g = \hat{\sigma}_z$
- $\hat{H}^{+} = \hat{H} = \hat{H}^{-1}$

Qubit measurement

- Let $|\psi\rangle \in \mathcal{H}_2$ be quantum state.

  thus $|\psi\rangle = c_0|0\rangle + c_1|1\rangle$, where $c_0, c_1 \in \mathbb{C}$ s.t. $|c_0|^2 + |c_1|^2 = 1$

- Thus the measurement probabilities are given by

$$P_0 = |\langle 0|\psi\rangle|^2 = |c_0|^2$$

$$P_1 = |\langle 1|\psi\rangle|^2 = |c_1|^2 = 1 - |c_0|^2$$

- After applying a quantum gate $\hat{U}$ on $|\psi\rangle$ the probabilities are given by

$$P_0 = |\langle 0|\hat{U}|\psi\rangle|^2 = \langle\psi|\hat{U}^{+}|0\rangle\langle 0|\hat{U}|\psi\rangle$$

$$= |\langle\tilde{0}|\psi\rangle|^2, \text{ where } |\tilde{0}\rangle = \hat{U}^{+}|0\rangle$$

Similarly for $P_1 = |\langle 1|\psi\rangle|^2 = |\langle\tilde{1}|\psi\rangle|^2$, where $|\tilde{1}\rangle = \hat{U}^{+}|1\rangle$

2-qubit system

- The Hilbert space $\mathcal{H}_4 = \mathcal{H}_2^{(1)} \otimes \mathcal{H}_2^{(2)}$ is 4-dimensional

tensor product (or Kronecker product), that is, is a bilinear composition as the two vector spaces (with minimal constraints)

- Thus single-qubit operators are given by

$$\hat{A}_1 \otimes \hat{I} \quad \text{and} \quad \hat{I} \otimes \hat{A}_2 , \quad \text{where } \hat{A}_1 \in \mathcal{L}(\mathcal{H}_2^{(1)})$$
$$\hat{A}_2 \in \mathcal{L}(\mathcal{H}_2^{(2)})$$

- Let $\hat{A} \otimes \hat{B} = \hat{C} \in \mathcal{L}(\mathcal{H}_4)$
  $\hat{D} \otimes \hat{E} = \hat{F} \in \mathcal{L}(\mathcal{H}_4)$

$$\Rightarrow \quad \hat{C}\hat{F} = (\hat{A} \otimes \hat{B})(\hat{D} \otimes \hat{F}) = (\hat{A}\hat{D}) \otimes (\hat{B}\hat{F})$$

- We construct the basis for the two-qubit Hilbert space $\mathcal{H}_4$ as

$$|00\rangle \equiv |0\rangle \otimes |0\rangle$$
$$|01\rangle \equiv |0\rangle \otimes |1\rangle$$
$$|10\rangle \equiv |1\rangle \otimes |0\rangle$$
$$|11\rangle \equiv |1\rangle \otimes |1\rangle$$

, where $\{|0\rangle, |1\rangle\}$ is orthonormal basis set $\mathcal{H}_2^{(1)}$ and $\mathcal{H}_2^{(2)}$ correspondingly (have left out superscripts (1) and (2))

- Thus for $|\psi\rangle \in \mathcal{H}_4$ and $\hat{C} = \hat{A} \otimes \hat{B} \in \mathcal{L}(\mathcal{H}_4)$

$$|\psi\rangle = \sum_{k=0}^{3} c_k |k\rangle ,$$ where $|k\rangle \equiv |k_1 k_2\rangle$, where $k_1 k_2$ is binary representation of k

$$= c_0|00\rangle + c_1|01\rangle + c_2|10\rangle + c_3|11\rangle$$

$$= c_0|0\rangle + c_1|1\rangle + c_2|2\rangle + c_3|3\rangle$$

and

$$\hat{C}|\psi\rangle = \hat{C}\sum_{k=0}^{3} c_k|k\rangle = \sum_{k=0}^{3} c_k \hat{C}|k\rangle$$

$$= \sum_{k=0}^{3} c_k \, \hat{A} \otimes \hat{B} \, \overbrace{|k\rangle}^{\in \mathcal{H}_4}$$

$$= \sum_{k=0}^{3} c_k \, \hat{A}\overbrace{|k_1\rangle}^{\in \mathcal{H}_2^{(1)}} \otimes \hat{B}\overbrace{|k_2\rangle}^{\in \mathcal{H}_2^{(2)}}$$

Examples of two-qubit gates

Controlled NOT gate where qubit 1 is the control qubit and qubit 2 is the target qubit corresponds to

$$\hat{C}_{\text{NOT}}^{(1,2)} = |0\rangle\langle 0| \otimes \hat{I} + |1\rangle\langle 1| \otimes \hat{\sigma}_x$$

$$\hat{=} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & \sigma_x \end{pmatrix}$$

ex.
- construct the above matrix representations
- CNOT for qubit 1 as target

n-qubit system

Typical definition
$N \equiv 2^n = \dim\{\mathcal{H}_{2^n}\}$

- $\mathcal{H}_{2^n} = \mathcal{H}_2^{(1)} \otimes \mathcal{H}_2^{(2)} \otimes \cdots \otimes \mathcal{H}_2^{(n)}$
- $|\psi\rangle = \sum_{k=0}^{2^n-1} c_k|k\rangle = c_0|\underbrace{00\ldots0}_{n \text{ zeroes}}\rangle + c_1|\underbrace{0\ldots0}_{n-1 \text{ zeroes}}1\rangle + \ldots$
- $\underbrace{\hat{I} \otimes \cdots \otimes \hat{I}}_{m-1} \otimes \hat{A} \otimes \underbrace{\hat{I} \otimes \cdots \otimes \hat{I}}_{n-m}$ is a single-qubit operator for qubit m

Quantum algorithms for n qubits

1. Initialize qubits to $|0\rangle$ (not necessarily all qubits)
2. Apply a desired n-qubit gate U (can be constructed from single and two-qubit gates)
3. Measure qubits (not necessarily all qubits)
4. Use measurement data and go to 1, unless algorithm finished

* In the simplest case one goes only once through 1.→4. and initializes and measures all qubits in 1. and 3., respectively.

ex. Deutch algorithm

Entanglement for two qubits

- A quantum state of two qubits is defined entangled iff it cannot be represented as a product of two single-qubit states
- Thus $\forall |\psi\rangle \in \mathcal{H}_4$ that are not entangled $\exists\ |\psi_1\rangle \in \mathcal{H}_2^{(1)}$ and $|\psi_2\rangle \in \mathcal{H}_2^{(2)}$

  s.t. $|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$
- Examples of so-called maximally entangled states:

Bell states $|\Phi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)$

$$|\Psi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle)$$

Commuting operators

$\hat{A} = \hat{A}^{\dagger}$, $\hat{B} = \hat{B}^{\dagger}$,

- Let $\hat{A}, \hat{B} \in \mathcal{L}(\mathcal{H})$, and $[\hat{A}, \hat{B}] = 0$

  thus it can be shown that

  $\exists$ complete eigenbasis of $\hat{A}$, that is also an eigenbasis of $\hat{B}$.

- Especially is $[\hat{A}, \hat{H}(t)] = 0$ $\forall t$ the eigenvalues of $\hat{A}$ are referred to as conserved quantities since we have

$$\hat{A}|\psi(t)\rangle = \hat{A}\hat{U}(t)|\psi(0)\rangle = \hat{U}(t)\hat{A}|\psi(0)\rangle$$

$$= \lambda \hat{U}(t)|\psi(0)\rangle = \lambda|\psi(t)\rangle,$$

Proved according to excercise 3.1.(b)

Easy for temporally constant Hamiltonians $\hat{A}e^{\alpha\hat{H}} = \hat{A}\sum\frac{(\alpha\hat{H})^n}{n!} = e^{\alpha\hat{H}}\hat{A}$

where we have assumed that $\hat{A}|\psi(0)\rangle = \lambda|\psi(0)\rangle$, i.e., we start from an eigenstate of $\hat{A}$

Lecture 6 (last from Mihko)

ILOs

1. Identify Heisenberg's uncertainty relation.
2. Apply creation and annihilation operators for a harmonic oscillator.
3. Apply canonical commutation relations

Uncertainty relations

- Heisenberg's uncertainty relation states that

$$\boxed{\Delta q \Delta p \geq \frac{\hbar}{2}},$$

where $\Delta A = \langle \hat{A}^2 \rangle - \langle \hat{A} \rangle^2$

$[\hat{q}, \hat{p}] = i\hbar$ since $\hat{q}$ and $\hat{p}$ are a canonical conjugate pair

Warning: does not strictly speaking apply if an operator is bounded

- Robertson uncertainty relation

$$\boxed{\Delta A \Delta B \geq \frac{1}{2} \left| \langle [\hat{A}, \hat{B}] \rangle \right|},$$

when $\hat{A}, \hat{B} \in \mathcal{L}(\mathcal{H})$ unbounded, $\hat{A} = \hat{A}^\dagger$, $\hat{B} = \hat{B}^\dagger$,

and $\langle \cdot \rangle = \langle \psi | \cdot | \psi \rangle$

- Let us define $|f\rangle = (\hat{A} - \langle \hat{A} \rangle)|\psi\rangle$
and $|g\rangle = (\hat{B} - \langle \hat{B} \rangle|\psi\rangle$

$$\Delta A^2 = \langle \psi | (\hat{A} - \langle A \rangle)^2 | \psi \rangle$$

$$= \underbrace{\langle \psi | (\hat{A} - \langle \hat{A} \rangle \hat{I})}_{((\hat{A} - \langle \hat{A} \rangle \hat{I})|\psi\rangle)^\dagger} (\hat{A} - \langle \hat{A} \rangle \hat{I}) | \psi \rangle$$

$$= \langle f | f \rangle = \| |f\rangle \|^2$$

$$\Delta B^2 = \langle g | g \rangle = \| |g\rangle \|^2$$

$$\boxed{\begin{aligned} |z|^2 &= (\mathrm{Re}\, z)^2 + (\mathrm{Im}\, z)^2 \\ &\geq (\mathrm{Im}\, z)^2 \\ &= \left(\frac{z - z^*}{2i} \right)^2 \end{aligned}}$$

Thus Cachy inequality implies

$$|\langle f | g \rangle| \leq \| |f\rangle \| \, \| |g\rangle \|$$

$$\Rightarrow \Delta A^2 \Delta B^2 \geq |\underbrace{\langle f | g \rangle}^{=: z \in \mathbb{C}}|^2$$

$$= |\langle \psi | (\hat{A} - \langle \hat{A} \rangle \hat{I})(\hat{B} - \langle \hat{B} \rangle \hat{I}) | \psi \rangle|^2$$

$$\geq | \langle \psi | (\hat{A} - \langle \hat{A} \rangle \hat{I})(\hat{B} - \langle \hat{B} \rangle \hat{I} | \psi \rangle - \langle \psi | (\hat{B} - \langle \hat{B} \rangle \hat{I})(\hat{A} - \langle \hat{A} \rangle \hat{I} | \psi \rangle |^2 / 4$$

$$= |\langle \psi | [\hat{A} - \langle \hat{A} \rangle \hat{I}, \hat{B} - \langle \hat{B} \rangle \hat{I}] | \psi \rangle|^2 / 4$$

$$= |\langle [\hat{A}, \hat{B}] \rangle|^2 / 4 \qquad \square$$

One-dimensional quantum harmonic oscillator

As we derived earlier

$$\hat{H} = \frac{\hat{p}^2}{2m} + \underbrace{\frac{mg}{2l}}_{=: \omega^2} \hat{q}^2 = \frac{\hat{p}^2}{2m} + \frac{1}{2} m \omega^2 \hat{q}^2,$$

where $[\hat{q}, \hat{p}] = i\hbar$, $\hat{q} = \hat{q}^\dagger$, $\hat{p} = \hat{p}^\dagger$

Try writing using $A, B, C \in \mathbb{R}$

$$\hat{H} = (A\hat{q} - iB\hat{p})(A\hat{q} + iB\hat{p}) + C$$

$$= A^2 \hat{q}^2 + iAB\hat{q}\hat{p} - iB\hat{p}A\hat{q} + B^2\hat{p}^2 + C$$

$$= A^2\hat{q}^2 + B^2\hat{p}^2 + \underbrace{iAB\underbrace{[\hat{q}, \hat{p}]}_{i\hbar}}_{-\hbar BA} + C$$

choose

$A = \sqrt{\frac{1}{2} m\omega^2}$

$B = \sqrt{1/2m}$

$C = \hbar AB$

$= \frac{\hbar \omega}{2}$

$$= \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega^2 \hat{q}^2$$

$$= \left(\hat{q}\sqrt{\frac{m\omega^2}{2}} + i\hat{p}\sqrt{\frac{1}{2m}}\right)^\dagger \left(\hat{q}\sqrt{\frac{m\omega^2}{2}} + i\hat{p}\sqrt{\frac{1}{2m}}\right) + \frac{1}{2}\hbar\omega$$

$$= \hbar\omega\left[\underbrace{\sqrt{\frac{m\omega}{2\hbar}}\left(\hat{q} + \frac{i}{m\omega}\hat{p}\right)^\dagger}_{= \hat{a}^\dagger} \underbrace{\sqrt{\frac{m\omega}{2\hbar}}\left(\hat{q} + \frac{i}{m\omega}\hat{p}\right)}_{=: \hat{a}} + \frac{1}{2}\right]$$

Math

$(x-y)(x+y) = x^2 - y^2$, $x, y \in \mathbb{R}$

$$\underbrace{(x - iy)}_{z^*}\underbrace{(x + iy)}_{=: z} = x^2 + y^2 = |z|^2$$

$$\sqrt{\hbar\omega}\sqrt{\frac{m\omega}{2\hbar}} = \sqrt{\frac{\hbar m\omega^2}{2\hbar}} = \sqrt{\frac{m\omega^2}{2}} \quad ✓.$$

$$\sqrt{\hbar\omega}\sqrt{\frac{m\omega}{2\hbar}}\frac{1}{m\omega} = \sqrt{\frac{m\omega^2}{2}}\frac{1}{m\omega} = \sqrt{\frac{m\omega^2}{2m^2\omega^2}} = \sqrt{\frac{1}{2m}} \quad ✓.$$

$$\Rightarrow \hat{H} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right), \text{ where}$$

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{q} + \frac{i}{m\omega}\hat{p}\right)$$

$$\hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{q} - \frac{i}{m\omega}\hat{p}\right)$$

Let us calculate

$$[\hat{a}, \hat{a}^\dagger] = \frac{m\omega}{2\hbar}\left[\hat{q} + \frac{i}{m\omega}\hat{p}, \hat{q} - \frac{i}{m\omega}\hat{p}\right]$$

$$= \frac{m\omega}{2\hbar}\left(\left[\hat{q}, -\frac{i}{m\omega}\hat{p}\right] + \left[\frac{i}{m\omega}\hat{p}, \hat{q}\right]\right)$$

$$= \frac{i}{2\hbar}\left(-\underbrace{[\hat{q}, \hat{p}]}_{= i\hbar} + \underbrace{[\hat{p}, \hat{q}]}_{= -i\hbar}\right) = 1$$

- observations:

1. $\langle\psi|\hat{H}|\psi\rangle \geq 0$, for $\forall|\psi\rangle$

since $\langle\psi|\hbar\omega(\hat{a}^\dagger\hat{a}+\frac{1}{2})|\psi\rangle = \langle\hat{H}\rangle$

$$= \frac{\hbar\omega}{2} + \langle\psi|\hbar\omega\,\hat{a}^\dagger\hat{a}\,|\psi\rangle$$

$$= \hbar\omega\left(\frac{1}{2} + \|\hat{a}|\psi\rangle\|^2\right) \geq 0$$

Thus all eigenenergies are positive.

2. Let $|\psi\rangle$ be an eigenstate of $\hat{H}$ s.t. $\hat{H}|\psi\rangle = \varepsilon|\psi\rangle$, Thus

$$\hat{H}\,\hat{a}\,|\psi\rangle = \hbar\omega(\underbrace{\hat{a}^\dagger\hat{a}}_{=\hat{a}\hat{a}^\dagger - 1}+\tfrac{1}{2})\hat{a}|\psi\rangle$$

$$= \hat{a}\left[\hbar\omega(\hat{a}^\dagger\hat{a}+\frac{1}{2}-1)\right]|\psi\rangle$$

$$= \hat{a}(\varepsilon-\hbar\omega)|\psi\rangle = (\varepsilon-\hbar\omega)\hat{a}|\psi\rangle$$

1. & 2. $\Rightarrow \exists\,|0\rangle \in \mathcal{H}$ s.t. $\hat{a}|0\rangle = 0$

- $|0\rangle$ is the ground state, i.e. the state with the lowest energy
- $\hat{H}|0\rangle = \hbar\omega(\hat{a}^\dagger\hat{a}+\frac{1}{2})|0\rangle = \frac{\hbar\omega}{2}|0\rangle$

$\Rightarrow \varepsilon_0 = 0$

$$\hat{H}\,\hat{a}^\dagger|\psi\rangle = (\varepsilon+\hbar\omega)|\psi\rangle,$$

where $\hat{H}|\psi\rangle = \varepsilon$. Q.

Thus spectrum of $\hat{H}$ is $\{\varepsilon_n\} = \{\hbar\omega(n+\frac{1}{2})\}$ with eigenstates $\{|n\rangle\}$.

that is $\hat{H}|n\rangle = \hbar\omega(n+\frac{1}{2})|n\rangle$

Symbolic operator differential

- Let $\hat{q}$ and $\hat{p}$ be a conjugate pair and $\hat{q}$ has a continuous spectrum.

Thus we have

$$[\hat{q}, \hat{p}] = i\hbar$$

and this satisfied by symbolically defining $\hat{p} = -i\hbar\partial_{\hat{q}}$, where

$\partial_{\hat{q}}$ means the we take symbolically a derivative w.r.t. $\hat{q}$.

example $\forall |\psi\rangle \in \mathcal{H}$

$$\partial_{\hat{q}} f(\hat{q}) |\psi\rangle$$

$$= \left[f'(\hat{q}) + f(\hat{q}) \partial_{\hat{q}} \right] |\psi\rangle,$$

where f is a continuosly differentiable function and f' means its derivative

Let us check the claim

$$[\hat{q}, \hat{p}] = \hat{q}\hat{p} - \hat{p}\hat{q}$$

$$= \hat{q}(-i\hbar\partial_{\hat{q}}) - (-i\hbar\partial_{\hat{q}})\hat{q}$$

$$= -i\hbar\, \hat{q}\partial_{\hat{q}} + i\hbar\, \partial_{\hat{q}}\hat{q}$$

$$= -i\hbar\, \hat{q}\partial_{\hat{q}} + i\hbar \underbrace{(\partial_{\hat{q}}\hat{q})}_{=\hat{I}} + i\hbar\, \hat{q}\partial_{\hat{q}}$$

$$= i\hbar \quad //.$$

added after lecture

Solving the ground state in position representation

We know that

$$0 = \langle x' | \overbrace{\hat{a}|0\rangle}^{=0} = \sqrt{\frac{m\omega}{2\hbar}} \overbrace{\left[\hat{x} + \frac{i}{m\omega}(-i\hbar\partial_{\hat{x}})\right]}^{\int d\tilde{x} |\tilde{x}\rangle\langle\tilde{x}|} |0\rangle$$

$$= \sqrt{\frac{m\omega}{2\hbar}} \int d\tilde{x} \underbrace{\langle x' | \tilde{x} \rangle}_{= \delta(\tilde{x} - x')} \left(\tilde{x} + \frac{\hbar}{m\omega} \partial_{\tilde{x}} \right) \underbrace{\psi_0(\tilde{x})}_{= \langle \tilde{x} | 0 \rangle}$$

$$\Rightarrow \left(x + \frac{\hbar}{m\omega} \partial_x \right) \psi_0(x) = 0$$

$$\Rightarrow \psi_0(x) = C e^{-\frac{x^2 m\omega^2}{2\hbar}}$$

↑ normalization coefficient $= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}$

$\psi_1 \propto \hat{a}^\dagger |0\rangle$ ...