PHYS-C0252 - Quantum Mechanics

Exercise set 3

Due date : November 13, 2020 before 10.00

1. Consider the temporal evolution of quantum states $\{|\psi\rangle\}$ determined by a Hamiltonian $\hat{H}(t)$. We define the evolution operator $\hat{U}(t, t_0)$ as

$$\hat{U}(t, t_0)|\psi(t_0)\rangle = |\psi(t)\rangle, \text{ for all } t > t_0 \in \mathbb{R} \text{ and } |\psi(t_0)\rangle \in \mathcal{H}$$

(a) Prove the following identities:

i. $\hat{U}(t_0, t_0) = \hat{I}$, for all $t_0 \in \mathbb{R}$.

ii. $\hat{U}^\dagger(t, t_0)\hat{U}(t, t_0) = \hat{I}$, for all $t > t_0 \in \mathbb{R}$. Hint: Use the result of i. as an initial condition and calculate $\partial_t[\hat{U}^\dagger(t, t_0)\hat{U}(t, t_0)]$ using the Schrödinger equation equivalent for $\hat{U}(t, t_0)$. Note that we do not assume the Hamiltonian to be independent of time.

iii. $\hat{U}(t_2, t_0) = \hat{U}(t_2, t_1)\hat{U}(t_1, t_0)$, for all $t_0 < t_1 < t_2 \in \mathbb{R}$.

(b) Construct the evolution operator $\hat{U}(t, t_0)$ for a Hamiltonian that depends on time and show that it reduces to $\hat{U}(t, t_0) = e^{\frac{-i}{\hbar}(t-t_0)\hat{H}}$ for a temporally independent Hamiltonian. Hint: Consider that the temporal dependence of the Hamiltonian is piece-wise constant in time, use the form of the corresponding evolution operators and a generalization of point iii. to many intervals. Finally take a limit of the length of the intervals where the Hamiltonian is constant to zero. You may leave your result in this form.

2. Let $\vec{a}$ be any real-valued three-dimensional unit vector and $\theta \in \mathbb{R}$. Prove that

$$\mathrm{e}^{\mathrm{i}\theta(\vec{a}\cdot\vec{\sigma})} = I\cos(\theta) + \mathrm{i}(\vec{a}\cdot\vec{\sigma})\sin(\theta),$$

where $\vec{a}\cdot\vec{\sigma} = a_x\sigma_x + a_y\sigma_y + a_z\sigma_z$, $\{a_k\}$ are the Cartesian components of $\vec{a}$, and $\{\sigma_k\}$ are the Pauli spin-1/2 matrices. Note: You may either calculate this with matrices or operators. Try both if you have time. Hint: Use the definition of the exponential function as its Taylor series, but before a brute-force calculation, try to find a trick how the calculation of the different powers of the operators in the series expansion is very easy. If you cannot find the trick, brute force is okay as well.

3. Any pure qubit state $|\psi\rangle$ can be written in the following way

$$|\psi\rangle = \cos(\theta/2)|0\rangle + e^{i\phi}\sin(\theta/2)|1\rangle,$$

where $0 \leq \theta \leq \pi$ and $0 \leq \phi < 2\pi$ are parameters that equal the polar and the azimuthal angle, respectively, of the Bloch sphere representation of the state, i.e., the Bloch sphere representation of $|\psi\rangle$ is given by the Bloch vector $\vec{\psi} = (\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta)$.

(a) Determine the Bloch sphere angles θ and ϕ, and write down the corresponding Bloch vectors for the following quantum states:

i. The states $|0\rangle$ and $|1\rangle$. Are the angles unique for these states? Are the corresponding Bloch vectors unique?

ii. The eigenstates of the operator $\hat{\sigma}_x$.

iii. The eigenstates of the operator $\hat{\sigma}_y$.

(b) Show that the Bloch sphere representation of the operator $\mathrm{e}^{\mathrm{i}\theta(\vec{a}\cdot\hat{\vec{\sigma}})/2}$ is a classical rotation of the Bloch vector about the axis $\vec{a}$ for an angle θ. Here, $\vec{a}\cdot\hat{\vec{\sigma}} = a_x\hat{\sigma}_x + a_y\hat{\sigma}_y + a_z\hat{\sigma}_z$, $\{a_k\}$ are the Cartesian components of the unit vector $\vec{a}$, and $\{\hat{\sigma}_k\}$ are the Pauli spin-1/2 operators.

4. Consider a particle in one physical dimension in a quantum state represented in the position basis at $t = 0$ as

$$\psi(x, t = 0) = \begin{cases} A(a^2 - x^2), & \text{if } -a \leq\ x \leq a; \\ 0, & \text{otherwise.} \end{cases}$$

(a) Normalize $\psi(x, t = 0)$ to find a suitable scalar A.

(b) Find the expectation values of $\hat{x}$ and $\hat{p}_x$ at time $t = 0$?

(c) Find $\langle\hat{x}^2\rangle$ and $\langle\hat{p}_x^2\rangle$ at time $t = 0$?

(d) Calculate $\Delta\hat{x} = \sqrt{\langle\hat{x}^2\rangle - \langle\hat{x}\rangle^2}$ and $\Delta\hat{p}_x = \sqrt{\langle\hat{p_x}^2\rangle - \langle\hat{p_x}\rangle^2}$.

(e) Show that $\Delta\hat{x}\Delta\hat{p}_x$ is in agreement with Heisenberg's uncertainty relation.