Aalto University
School of Electrical
Engineering

Safety and Constrained Optimal Control

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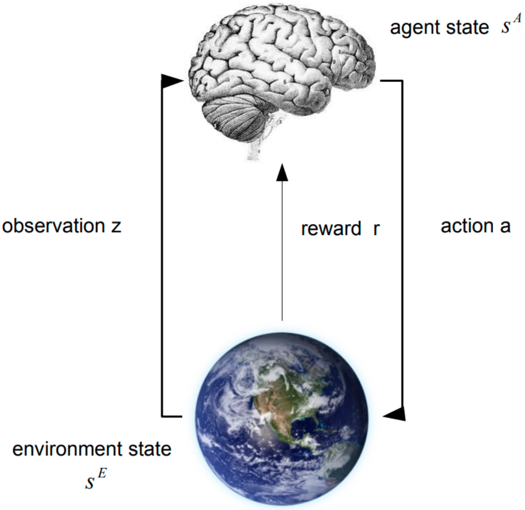
November 3, 2020

Reinforcement Learning



Safety in Reinforcement Learning

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Safety in Reinforcement Learning

- How would you define ***safety*** in RL?

- Safety in RL is an active research topic!
- The agent is trained to *maximize the expected return* in a given task ***while not taking any action*** that ***gives damage*** to the environment or itself during learning and/or deployment.

Safety in Reinforcement Learning

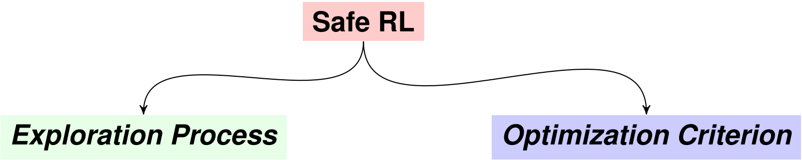
Safe RL

Safety in Reinforcement Learning

Safe RL

→ ***Exploration Process***

Safety in Reinforcement Learning



Safety in Reinforcement Learning

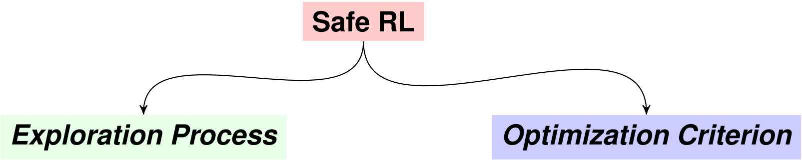
Safe RL

Exploration Process

- Risk-directed Exploration

Optimization Criterion

Safety in Reinforcement Learning



- Risk-directed Exploration
- Utilization of External Knowledge

Safety in Reinforcement Learning

Safe RL

Exploration Process

- Risk-directed Exploration
- Utilization of External Knowledge

Optimization Criterion

- Constrained Criterion

Safety in Reinforcement Learning

Safe RL

Exploration Process

- Risk-directed Exploration
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Optimization Criterion

- Constrained Criterion
- Worst Case Criterion

Safety in Reinforcement Learning

Safe RL

Exploration Process

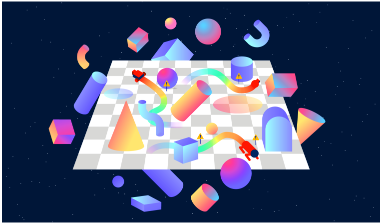
- Risk-directed Exploration
- Utilization of External Knowledge

Optimization Criterion

- Constrained Criterion
- Worst Case Criterion
- Risk-Sensitive Criterion

Safe Exploration

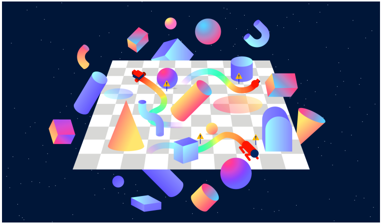
OpenAI Safety-Gym



OpenAI Safety-Gym: A. Ray, J. Achiam, and D. Amodei, “Benchmarking Safe Exploration in Deep Reinforcement Learning,” 2019, https://cdn.openai.com/safexp-short.pdf.

Safe Exploration

OpenAI Safety-Gym



Some Methods

- Constrained Policy Optimization
- Proximal Policy Optimization
- Trust Region Policy Optimization
- PPO Lagrangian
- TRPO Lagrangian

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Safety in Reinforcement Learning

Safe RL

Exploration Process

- Risk-directed Exploration
- Utilization of External Knowledge

Optimization Criterion

- **Constrained Criterion**
- Worst Case Criterion
- Risk-Sensitive Criterion

Constrained Optimal Control

Constrained Optimization

$$\min_{x \in \mathbb{R}^n} f(x) \quad \text{subject to} \begin{cases} c_i(x) = 0, & i \in \mathcal{E} \quad \textit{Equality Constraints} \\ c_i(x) \geq 0, & i \in \mathcal{I} \quad \textit{Inquality Constraints} \end{cases}$$

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Feasible Set:

$$\Omega = \{x \mid c_i(x) = 0, i \in \mathcal{E} \quad \textbf{and} \quad c_i(x) \geq 0, i \in \mathcal{I}\}$$

$$\implies \min_{x \in \Omega} f(x)$$

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Active Set:

$$\mathcal{A}(x) = \mathcal{E} \cup \{i \in \mathcal{I} \mid c_i(x) = 0\}$$

At a feasible point x, the inequality constraint $i \in \mathcal{I}$ is said to be **active** if $c_i(x) = 0$ and **inactive** if the strict inequality $c_i(x) > 0$ is satisfied.

Constrained Optimization

A Single Equality Constraint

$$\min_{x_1, x_2} x_1 + x_2 \quad \text{s.t.} \quad x_1^2 + x_2^2 - 2 = 0$$

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$$f(x) = x_1 + x_2$$

$$c_1(x) = x_1^2 + x_2^2 - 2$$

$$\mathcal{I} = \emptyset, \quad \mathcal{E} = \{1\}$$

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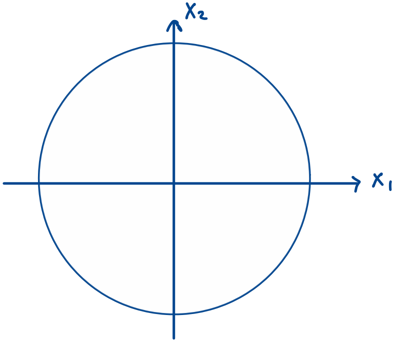

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Q: What is feasible set?

A: *Feasible set for this problem is a circle of radius* $\sqrt{2}$ *centered at origin. (Just boundary, not interior)*

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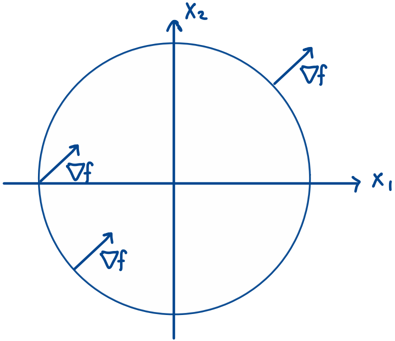

$$\nabla f = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

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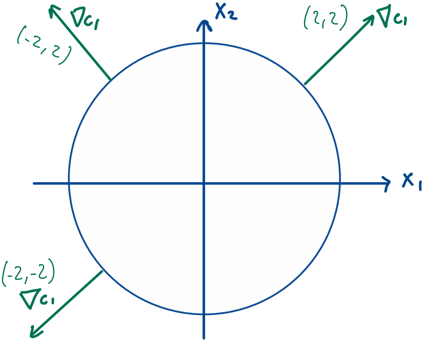

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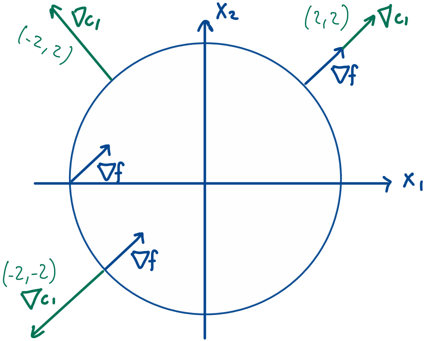

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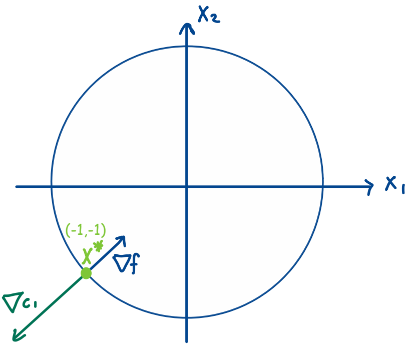

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Q: What is the solution x^*?

A: $x^* = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$

$$\nabla f(x^*) // \nabla c_1(x^*)$$

$$\nabla f(x^*) = \lambda_1^* \nabla c_1(x^*) \quad \lambda_1^* = -1/2$$

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A Single Equality Constraint

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Let's introduce **Lagrangian function**

$$\mathcal{L}(x, \lambda_1) = f(x) - \lambda_1 c_1(x)$$

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$$\nabla_x \mathcal{L}(x, \lambda_1) = \nabla f(x) - \lambda_1 \triangle c_1(x)$$

$$1 - 2\lambda_1^* x_1 = 0 \quad \text{and} \quad 1 - 2\lambda_1^* x_2 = 0$$

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Let's check our solution $x^* = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \lambda_1^* = -1/2$

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A Single Equality Constraint

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$$1 - 2(-1/2)(-1) = 0 \quad \text{and} \quad 1 - 2(-1/2)(-1) = 0$$ ✔

Constrained Optimization

A Single Equality Constraint

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$$1 - 2\lambda_1^* x_1 = 0 \quad \text{and} \quad 1 - 2\lambda_1^* x_2 = 0$$

Q: What about $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \lambda_1 = 1/2$?

Constrained Optimization

A Single Equality Constraint

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Let's introduce **Lagrangian function**

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At solution x^*, there is a scalar λ_1^* such that $\nabla_x \mathcal{L}(x^*, \lambda_1^*) = 0$

This condition is **necessary** but **not sufficient**.

Constrained Optimization

A Single Inequality Constraint

$$\min_{x_1, x_2} x_1 + x_2 \quad \text{s.t.} \quad 2 - x_1^2 - x_2^2 \geq 0$$

$$f(x) = x_1 + x_2$$

$$c_1(x) = 2 - x_1^2 - x_2^2$$

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Constrained Optimization

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Q: What is feasible set?

Constrained Optimization

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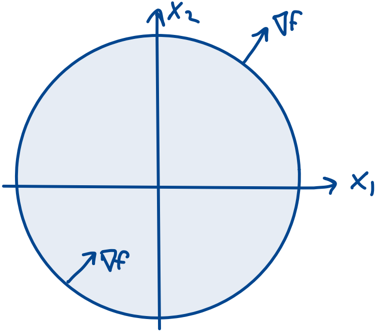

Q: What is feasible set?

A: *Now, feasible set consists of the circle and its interior!*

Constrained Optimization

A Single Inequality Constraint

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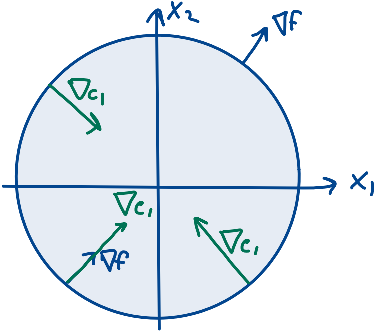

$$f(x) = x_1 + x_2$$

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$$\mathcal{I} = \{1\}, \quad \mathcal{E} = \emptyset$$

$$\nabla f = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \nabla c_1 = \begin{bmatrix} -2x_1 \\ -2x_2 \end{bmatrix}$$

Constraint normal ∇c_1 points toward the interior of the feasible region at each point on the boundary of the circle.

Constrained Optimization

A Single Inequality Constraint

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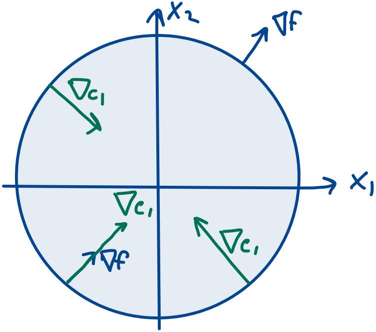

$$\nabla f = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \nabla c_1 = \begin{bmatrix} -2x_1 \\ -2x_2 \end{bmatrix}$$

Q: What is the solution x^*?

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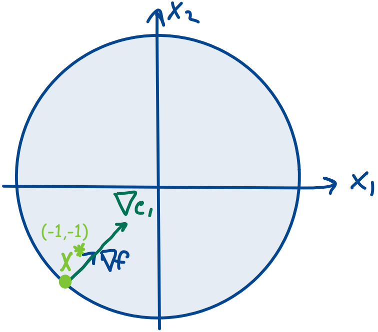

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A given feasible point **x** is **not optimal**, if we can find a small step **s** that **both**

- retains feasibility,
- decreases the objective function $f(x)$ to first order.

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A Single Inequality Constraint

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Approximate $c_1(x)$ to first order: $c_1(x+s) \approx c_1(x) + \nabla c_1(x)^\top s$

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If **s** retains feasibility $\implies c_1(x) + \nabla c_1(x)^\top s \geq 0$

Constrained Optimization

A Single Inequality Constraint

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Similarly, approximate $f(x)$ to first order: $f(x + s) \approx f(x) + \nabla f(x)^\top s$

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$$f(x) = x_1 + x_2$$

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Similarly, approximate $f(x)$ to first order: $f(x+s) \approx f(x) + \nabla f(x)^\top s$

$f(x)$ is decreasing $\implies f(x+s) - f(x) < 0$

Constrained Optimization

A Single Inequality Constraint

$$f(x) = x_1 + x_2$$

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A given feasible point **x** is **not optimal**, if we can find a small step **s** that **both**

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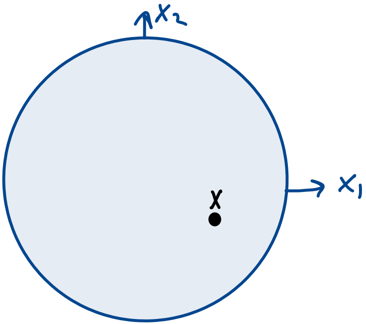

Q: How would you select s?

Remember the conditions:

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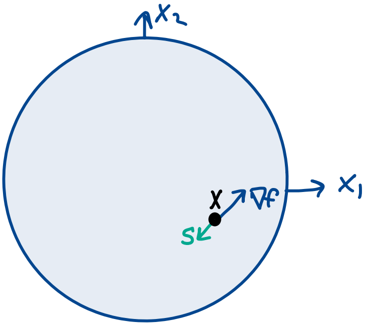

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s$= -\alpha \nabla f(x)$
for any positive scalar α
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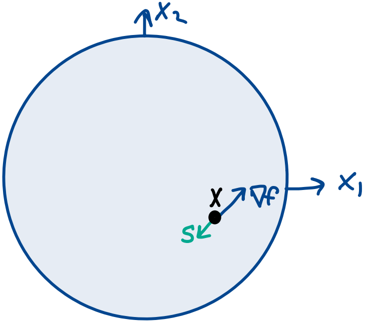

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However, no step **s** is given when $\nabla f(x) = 0$

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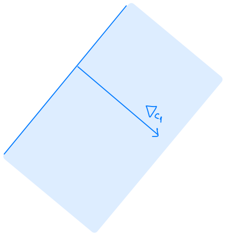
C1: $\nabla c_1(x)^\top s \geq 0 \rightarrow$ *Closed half-space*



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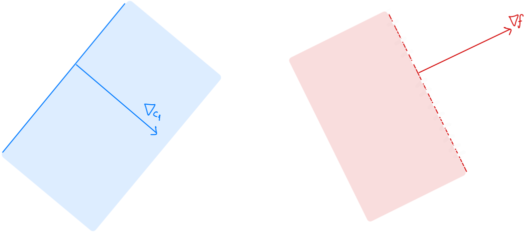

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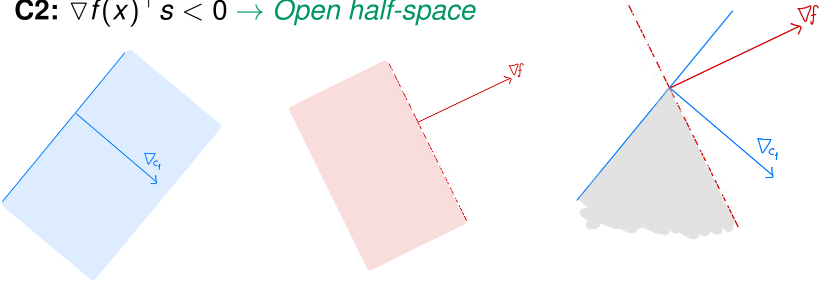

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If ∇f and ∇c_1 point in the opposite direction

$\nabla f = \lambda_1 \nabla c_1$ for some $\lambda_1 < 0$

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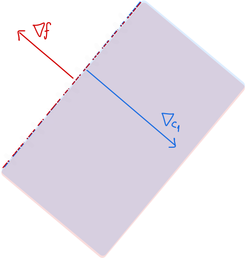
Intersection region is
entire open half-space!



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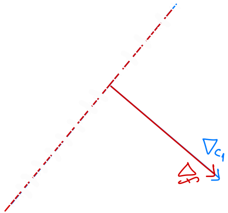
Intersection region is **empty!**



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Optimality Conditions for both Case 1 and Case 2:

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λ_1 can be strictly positive **only** when the corresponding c_1 is **active**.

Constrained Optimization

$$
\begin{aligned}
\nabla_x \mathcal{L}(x^*, \lambda^*) &= 0, \\
c_i(x^*) &= 0, \quad \text{for all } i \in \mathcal{E}, \\
c_i(x^*) &\geq 0, \quad \text{for all } i \in \mathcal{I}, \\
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Often known as the Karush-Kuhn-Tucker (**KKT**) conditions.

Constrained Optimization

Robotic Application: Safe Trajectory Optimization

$$\begin{aligned} \min_{\mathbf{u}_0,\ldots,\mathbf{u}_{N-1}} \quad & \ell_f(\mathbf{x}_N) + \sum_{k=0}^{N-1} \ell(\mathbf{x}_k, \mathbf{u}_k) \\ \text{subject to} \quad & \mathbf{x}_{k+1} = f(\mathbf{x}_k, \mathbf{u}_k), \\ & \boldsymbol{g}(\mathbf{x}_k, \mathbf{u}_k) \geq \mathbf{0}, \end{aligned}$$

For details: G. Alcan, and V. Kyrki, "Differential Dynamic Programming with Nonlinear Safety Constraints Under System Uncertainties", arXiv:2011.01051, 2020.

Constrained Optimization

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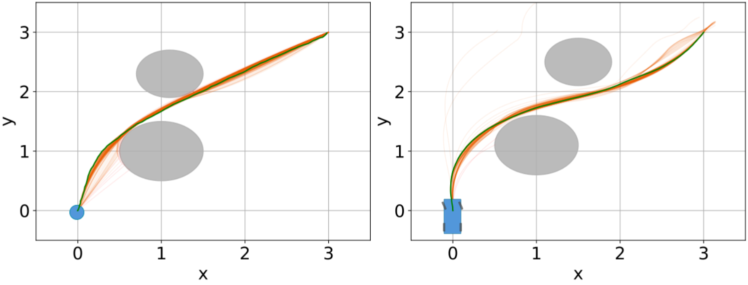

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- Safety in RL is an *active* and *popular* research area.
- **Definitions** and **methodologies** are subject to change depending on the applications and requirements.

- Adapting optimization procedure to safety requirements are often preferred, especially for a *known / partially known* transition dynamics and environment.
- This adaptation for constrained optimal control should be performed in such a way that **the KKT conditions must be satisfied**.