Aalto University
School of Science

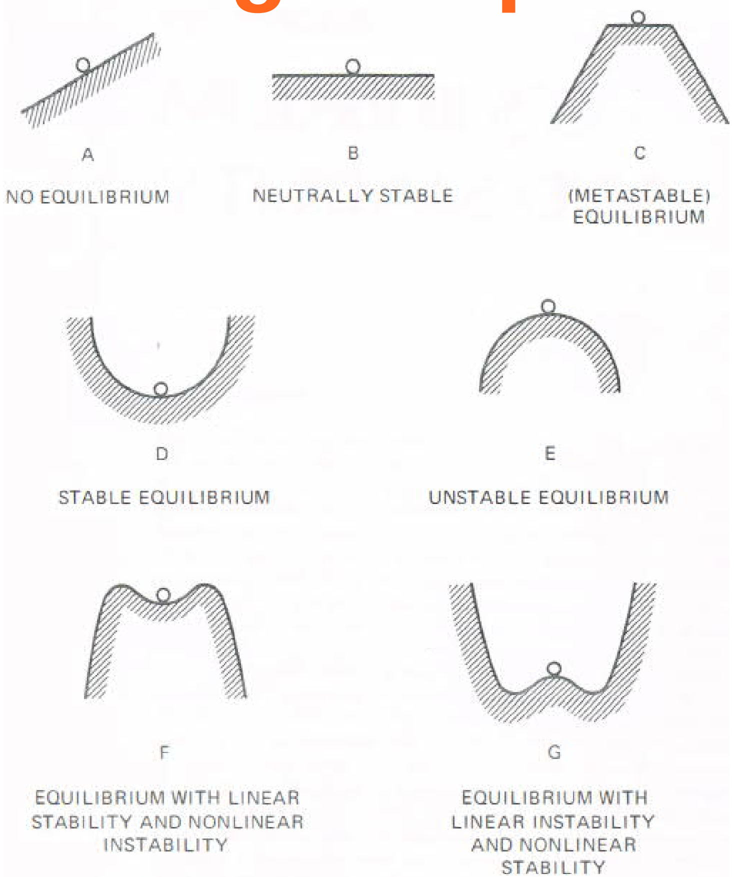
Lecture 10:
Plasma equilibrium & (in-)stability

Today's menu

- Equilibrium and force balance
- Plasma beta
- Z-pinch
- Bennett relation
- Screw-pinch
- Magnetic safety factor and shear
- Grad-Shafranov equation
- Eigenvalue problem for instabilities
- Energy principle for instabilities

The various ways of being in equilibium

Qualitatively different equilibria depending on how likely you are to stay in it – with small or large perturbations



Equilibrium and force balance

Equilibrium ➔ no acceleration: $\frac{\partial}{\partial t} = 0$

Analyze the simpliest magnetic equilibrium: $\boldsymbol{E} = 0, \boldsymbol{v} \approx 0$, isothermal

➔ $0 = -\nabla p + \boldsymbol{j} \times \boldsymbol{B}$

And we get the *force balance* between kinetic and magnetic forces:

$$\nabla p = \boldsymbol{j} \times \boldsymbol{B}$$

Additional information: $\boldsymbol{j} \perp \nabla p \perp \boldsymbol{B}$

i.e., both the confining magnetic field and current are *perpendicular* to the pressure gradient that they are holding up.

Confining current

Note: the force balance gives the relationships for $\nabla p, \boldsymbol{j}, \boldsymbol{B}$.

What is the current needed to hold up ∇p in given magnetic field $\boldsymbol{B}$?

$$\boldsymbol{j} = \boldsymbol{j}_\perp = \frac{\boldsymbol{B}\times\nabla p}{B^2} = (T_e + T_i)\,\frac{\boldsymbol{B}\times\nabla n}{B^2}$$

… and we have re-discovered the *diamagnetic current* !

Typical of plasma physics: the same observed phenomenon can be obtained both from the particle picture and fluid picture – with different interpretation:

- ***Particle picture:*** with $\nabla n \neq 0$ the gyro motions do not cancel out
- ***Fluid picture:*** ∇p generates $\boldsymbol{j}_\perp$ so that the $\boldsymbol{j}_\perp \times \boldsymbol{B}$ exactly balances the kinetic pressure on each fluid element

Magnetic (counter-) forces

But the current and magnetic field are also related by Maxwell's equations:

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{j}$$

➔ $\nabla p = \frac{1}{\mu_0}(\nabla \times \boldsymbol{B}) \times \boldsymbol{B}$ ➔

$$\nabla p = -\nabla(B^2/2\mu_0) + (B^2/\mu_0)\boldsymbol{\kappa}$$

where $\boldsymbol{\kappa}$ is the *field line curvature*, $\boldsymbol{\kappa} = \frac{\boldsymbol{B}}{B} \cdot \nabla(\frac{\boldsymbol{B}}{B})$, with $|\boldsymbol{\kappa}| = 1/R_c$.

So the magnetic field exerts force to plasma in two ways:

- If the plasma tries to compress the field lines ➔ restoring force via magnetic pressure: $\frac{B^2}{2\mu_0}$
- If the plasma tries to bend the field lines ➔ restoring force via field line tension: $(B^2/\mu_0)\boldsymbol{\kappa}$

Balancing the pressures

The field line tension that works to straighten out the field lines becomes important with *instabilities*, where the plasma tries to get out of control by (un-)bending field lines.

For ’straight’ plasmas the equilibrium condition becomes

$$\nabla\left(p + \frac{B^2}{2\mu_0}\right) = 0$$

$$\Rightarrow p + \frac{B^2}{2\mu_0} = constant.$$

➔ In equilibrium plasmas the sum of kinetic and magnetic pressures is constant!

Plasma beta

So if we want to have a pressure gradient (= plasma confinement), the magnetic field strength has to diminish as we go inward!

How does that happen???

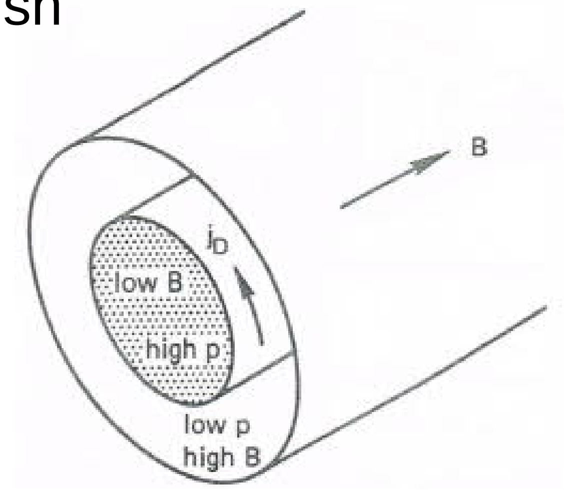
Via diamagnetic effect. ☺

The strength of the diamagnetic effect is given by a parameter called the *plasma beta*:

$$\beta \equiv \frac{\sum n_j T_j}{B^2/2\mu_0}$$

If β is NOT small, we cannot assume constant B.

Later: β is also a measure for the performance of B field.



Simplest case: axial field

So how would a real equilibrium look like?

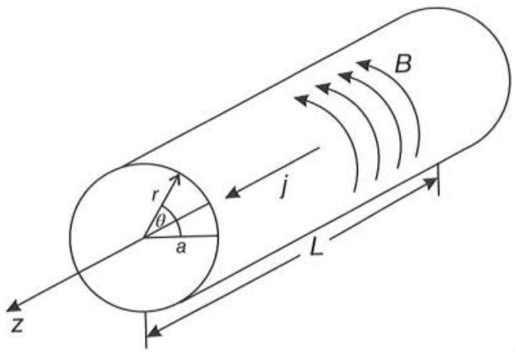
Again, start with the simpliest geometry: *linear* device

But axial field is clearly no good due to unavoidable end losses

➔ *let's start pinching …*

The z-pinch

A different magnetic bottle



Since axial field is out of question, let's get going with axial *current*:

$$j = j_z \rightarrow B = B_\theta$$

Note: even though we now deal with cylindrical symmetry, we shall use the *toroidal* nomenclature:

Polar/azimuthal (angle, field) ⇔ poloidal (angle, field)

Reason: in the first approximation, many phenomena in large-aspect ratio tokamaks are analyzed in the limit A –> ∞, and then torus –> cylinder: $R\varphi \rightarrow z$

Magnetic field in z-pinch

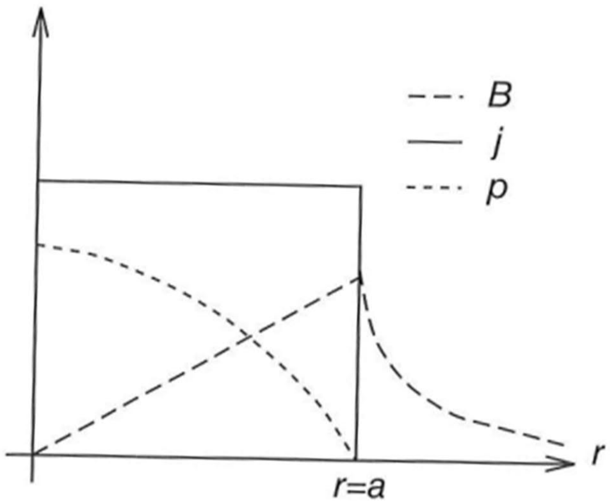
Ampere's law in z-pinch: $\mu_0 j = \frac{1}{r}\frac{d}{dr} r B_\theta \quad \left(= \frac{1}{r} B_\theta + \frac{dB_\theta}{dr} \right)$

Assume uniform current density, $j_z = const = j_0$, $dS = r d\theta dr = 2\pi r dr$

- $r > a$: $I_p(r) \equiv I_p(a) = j_0 \pi a^2$ ➔ $B_\theta = \frac{\mu_0 I_p}{2\pi r}$
- $r < a$: $\frac{d}{dr} r B_\theta = \mu_0 j_0 r$ ➔ $r B_\theta = \frac{1}{2}\mu_0 j_0 r^2 + C$; B.C.@ $r = 0 \rightarrow C = 0$

$$B_\theta = \frac{\mu_0 I_p}{2\pi r}, \qquad when\, r > a$$
$$B_\theta = \frac{\mu_0 I_p}{2\pi a^2} r, \qquad when\, r < a$$

Pressure profile in z-pinch



Force balance: $\frac{dp}{dr} = -j_z B_\theta = -\frac{1}{\mu_0 r} B_\theta^2 - \frac{1}{2\mu_0}\frac{dB_\theta^2}{dr}$

$r < a$: $B_\theta = \frac{\mu_0 I_p}{2\pi a^2} r \;\rightarrow\; \frac{dB_\theta^2}{dr} = \left(\frac{\mu_0 I_p}{2\pi a^2}\right)^2 2r$

➔ $\frac{dp}{dr} = -\left(\frac{\mu_0 I_p}{2\pi a^2}\right)^2 \left[\frac{r}{\mu_0} + \frac{r}{\mu_0}\right] = -\left(\frac{\mu_0 I_p}{2\pi a^2}\right)^2 \frac{2r}{\mu_0}$

➔ $p(r) = -\left(\frac{\mu_0 I_p}{2\pi a^2}\right)^2 \frac{r^2}{\mu_0} + const.$; B.C: $p(r = a) = 0 \rightarrow const = \frac{\mu_0 I_p^2}{(2\pi a)^2}$

$$p = \frac{\mu_0 I_p^2}{(2\pi a)^2}\left(1 - \left(\frac{r}{a}\right)^2\right)$$

Plasma beta in z-pinch

Let us calculate the *volume-averaged* pressure:

$$< p > = \frac{1}{V} \int p dV$$

$V = \pi a^2 L$, $dV = 2\pi r dr dz$ ➔ $< p > = \frac{2}{a^2} \int_0^a p(r) r dr$ ➔

$$< p > = \frac{\mu_0 I_p^2}{4\pi^2 a^2} \frac{1}{2} \equiv \frac{B_\theta^2(r = a)}{2\mu_0}$$

➔ For z-pinch $\beta = \frac{<p>}{B_\theta^2/(2\mu_0)} = 1$!!!

➔ z-pinch utilizes the poloidal magnetic field with 100% efficiency.

Bennett relation

The relation $< p > = \frac{\mu_0}{(2\pi a)^2}\frac{1}{2}I_p^2$ is called the *Bennett relation.*

Physics of the Bennett relation:

the good performance comes with a price …

- If the total current I_p and averaged pressure $< p >$ are fixed, the plasma can exist *only at a single radius value a!*

➔ if you heat the plasma (= increase $< p >$), the plasma will pinch !

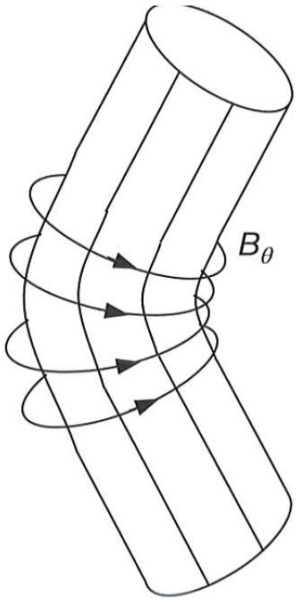
Isn't a small plasma a good thing?

Big magnets are expensive…

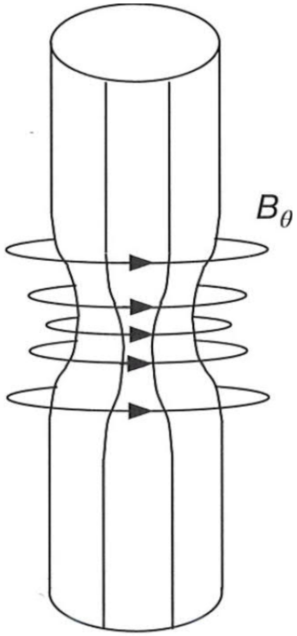
Pinching in imperfect world = first glimpse at instabilities …

Any small perturbation can make the plasma in z-pinch unstable.

Kink instability:
If the cylinder is ever so slightly bent, the magnetic pressure is smaller at the kinking part ➔ perturbation grows.



Sausage instability:
If the contraction of plasma is not homogeneous, the pressure at 'waist line' is stronger, pinching it further ➔ perturbation grows.



Now is the time to revive the so-far-neglected term in our force balance

$$\nabla p = -\nabla(B^2/2\mu_0) + (B^2/\mu_0)\boldsymbol{\kappa}$$

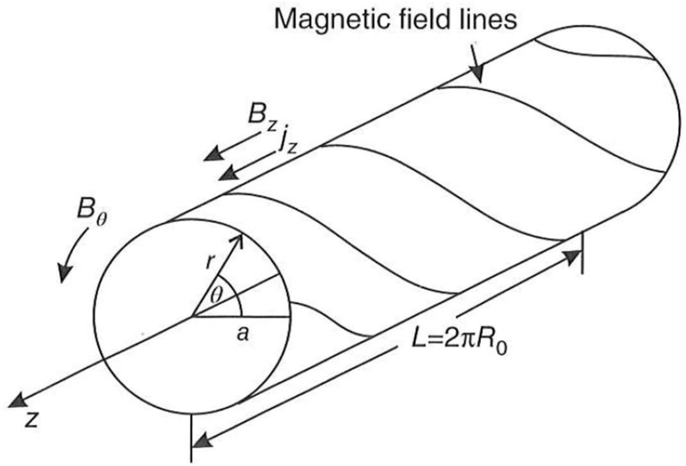
➔ if we introduce an axial field *in addition* to the poloidal field, this axial field will make the cylinder stiff = ensure stability of the z-pinch plasma.

To have a substantial restoring force on field lines, the *stabilizing* axial field has to be larger than the *confining* poloidal field.

… the field lines are now helical and we get a configuration called ..

The screw-pinch

The ’straight tokamak’ …



- Drive an *axial current* by, e.g., axial electric field

➔ poloidal field B_θ

- Wind coils poloidally around the plasma

➔ axial magnetic field $B_{z0} \approx constant$

But that is not all:

A *pinching* plasma implies radial motion ($\boldsymbol{v}_r$)

➔ $\boldsymbol{v}_r \times B_z \rightarrow j_\theta$ → additional axial field: $\frac{dB_{z1}}{dr} = \mu_0 j_\theta$

Physical interpretation:

- Incompressibility of the axial magnetic field (ideal MHD)

Get 'real' …

A uniform current profile is not very realistic (HW)

➔ let's take a simple form where the current profile peaks at the center:

$$j_z = j_0 \left(1 - \frac{r^2}{a^2}\right)^{\alpha}$$

Then the *plasma current* inside a radius r becomes (HW)

$$I_p(r) = j_0 \frac{\pi a^2}{\alpha + 1} \left\{1 - \left[\left(1 - \frac{r^2}{a^2}\right)^{\alpha+1}\right]\right\}$$

And the *confining* poloidal magnetic field is (HW)

$$B_\theta(r) = \frac{\pi a^2}{\alpha + 1} \frac{\mu_0 j_0}{2\pi r} \left\{1 - \left[\left(1 - \frac{r^2}{a^2}\right)^{\alpha+1}\right]\right\}$$

Force balance & plasma beta are screwed

The force balance thus becomes

$$\frac{dp}{dr} = -\frac{1}{\mu_0 r} B_\theta^2 - \frac{1}{2\mu_0} \frac{d}{dr}\left(B_\theta^2 + B_z^2\right)$$

where $B_z^2 = B_{z0}^2 + B_{z1}^2$.

While B_{z0} and B_θ are externally imposed, B_{z1} is determined by the plasma.

➔ additional degree of freedom

➔ equilibrium configuration can be found for any minor radius a !

But there is a price to pay:

Now the magnetic pressure has also a contribution from the axial field B_{z0} that does not contribute to confinement ➔ $\beta < 1$.

Helical field lines & magnetic safety factor

Suddenly the field lines are *screwed* = *helical*

The field line pitch is given by the so-called *safety factor,*

$$q \equiv \frac{\#\, of\ toroidal\ turns}{\#\, of\ poloidal\ turns}$$

. ➔ safety factor = ratio of the toroidal to poloidal angle along the field line.

Along the field line: $\frac{B_\theta}{B_z} = \frac{r\Delta\theta}{R\Delta\varphi} \rightarrow q = \frac{\Delta\varphi}{\Delta\theta} = \frac{r}{R}\frac{\mathrm{B_z}}{\mathrm{B}_\theta}$; *(remember* $2\pi R \leftrightarrow L$*)*

The safety factor is not usually constant across the plasma

➔ magnetic field lines are *sheared* radially.

The shear s can be calculated from the safety factor q: $s = \frac{r}{q}\frac{dq}{dr}$

What is so safe about the safety factor?

Remember:

the axial field was needed to *stabilize* the plasma against any bending.

Clearly the safety factor increases with increasing axial field.

High enough q thus keeps the plasma *safe* against such instability.

For instance, to stabilize the kink instability (in a tokamak) we need $q > 1$.

Typically $q(r = 0) \sim 1$, $q(r = a) \sim 3$ in a 'large' aspect ratio tokamak, $R/a = 3$

➔ $B_{tor} \sim 10 B_{pol}$

➔ tokamak β's are only a few % : $\beta = \frac{<p>}{B_{tot}^2/(2\mu_0)} \approx \frac{<p>}{B_{tor}^2/(2\mu_0)} \approx \frac{1}{100}\beta_{pol}$

Toroidal configurations

Toroidal plasma & flux surfaces

We already slipped into toroidal geometry – and there is no return:

By introducing the axial field to z-pinch we also re-introduced end losses. ☹

➔ let's eliminate the ends by going to torus!

➔ each field line traces one concentric toroidal surface.

Recall:

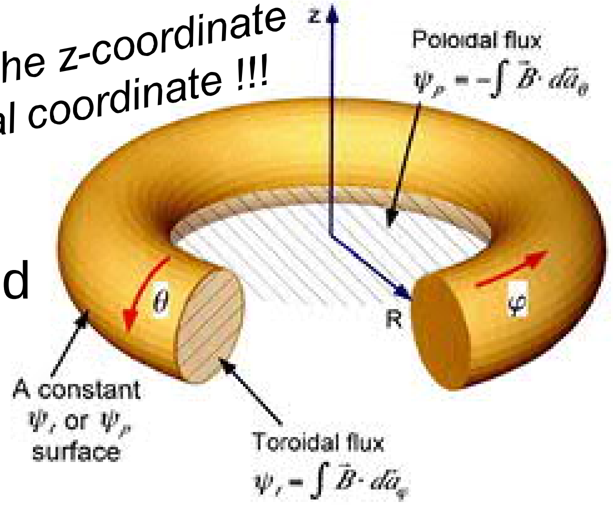
$$\boldsymbol{B} \cdot \nabla p = \boldsymbol{B} \cdot \boldsymbol{j} \times \boldsymbol{B} = 0 = \boldsymbol{j} \cdot \nabla p$$

➔ pressure gradient can exist only *perpendicular* to these surfaces.

The surfaces are called *flux surfaces* because they are defined by …

Flux integrals

Please note that the z-coordinate now is the vertical coordinate !!!



Toroidal magnetic flux:

- Integrate toroidal field through a vertical surface spanned by one of the concentric plasma surfaces
  - ➔ flux surface label that steadily increases from magnetic axis

Poloidal magnetic flux:

- Integrate poloidal field through a horizontal surface that increases in size from edge towards magnetic axis
  - ➔ flux surface label that steadily decreases from magnetic axis

Toroidal current: similarly $I_{tor} = \int \boldsymbol{j} \cdot d\boldsymbol{a}_\varphi = \frac{1}{\mu_0} \oint \boldsymbol{B} \cdot d\boldsymbol{l}_\varphi$

Poloidal current: similarly $I_{pol} = \int \boldsymbol{j} \cdot d\boldsymbol{a}_\theta = \frac{1}{\mu_0} \oint \boldsymbol{B} \cdot d\boldsymbol{l}_\theta$

Either of the magnetic fluxes can be used as a generalized radial coordinate for plasmas with arbitrary cross section. ☺

Magnetic field in terms of …

$$\Psi_{pol} = 2\pi \int_0^R B_z(R', z) R' dR'$$

Please note that the z-coordinate is now the vertical coordinate !!!

$$\Rightarrow B_z(R, z) = \frac{1}{2\pi R} \frac{\partial \Psi_{pol}(R,z)}{\partial R}$$

Total magnetic field has to be divergence free: $\nabla \cdot \boldsymbol{B} = 0$

$$\Rightarrow B_R(R, z) = -\frac{1}{2\pi R} \frac{\partial \Psi_{pol}(R,z)}{\partial z}$$

➔ the poloidal magnetic field can be expressed as

$$\Rightarrow \boldsymbol{B}_{pol} = B_R \nabla R + B_z \nabla z = \frac{1}{2\pi R}\left(-\frac{\partial \Psi_{pol}}{\partial z} \nabla R + \frac{\partial \Psi_{pol}}{\partial R} \nabla z\right) \equiv \frac{1}{2\pi} \nabla \Psi_{pol} \times \nabla \varphi$$

(In cylindrical coordinates: $R\nabla\varphi = \nabla z \times \nabla R$)

… the flux functions

How about the toroidal field?

Use the definition of poloidal current: $I_{pol} = \frac{1}{\mu_0} B_{tor} \cdot 2\pi R$

➔ $B_{tor} = \frac{\mu_0 I_{pol}}{2\pi R}$ ➔ $\boldsymbol{B}_{tor} = \frac{\mu_0 I_{pol}}{2\pi} \nabla\varphi$

➔ Total magnetic field: $\boldsymbol{B}_{tot} = \frac{1}{2\pi}\left(\nabla\Psi_{pol} \times \nabla\varphi + \mu_0 I_{pol} \nabla\varphi\right)$

This is associated with the current density given by Ampere's law:

$\boldsymbol{j} = \frac{1}{\mu_0} \nabla \times \boldsymbol{B} =$ … which takes a little work to give ..

Grad-Shafranov equation

$$\boldsymbol{j} = \frac{1}{2\pi\mu_0}\left(\mu_0 \nabla I_{pol} \times \nabla\varphi - \Delta^* \Psi_{pol} \nabla\varphi\right)$$

Where $\Delta^* \Psi_{pol} \equiv R^2 \nabla \cdot \frac{\nabla \Psi_{\mathrm{pol}}}{R^2}$ is the so-called *Stokes operator.*

In cylindrical coordinates: $\Delta^* = R \frac{\partial}{\partial R} \frac{1}{R} \frac{\partial}{\partial R} + \frac{\partial^2}{\partial z^2}$

For those brave of heart, plug these expressions into the force balance ➔

$$\Delta^* \Psi_{pol} = -\mu_0 2\pi R j_\varphi = -\mu_0 (2\pi R)^2 p' - \mu_0^2 I'_{pol} I_{pol}$$

This is called the *Grad-Shafranov equation* and it gives the *equilibrium* (= flux surface structure Ψ_{pol}) dictated by the pressure profile and the currents.

Not a piece of cake: *non-linear elliptic PDE – remember:* $p = p(\Psi_{pol})$

How to determine the stability of our equilibria?

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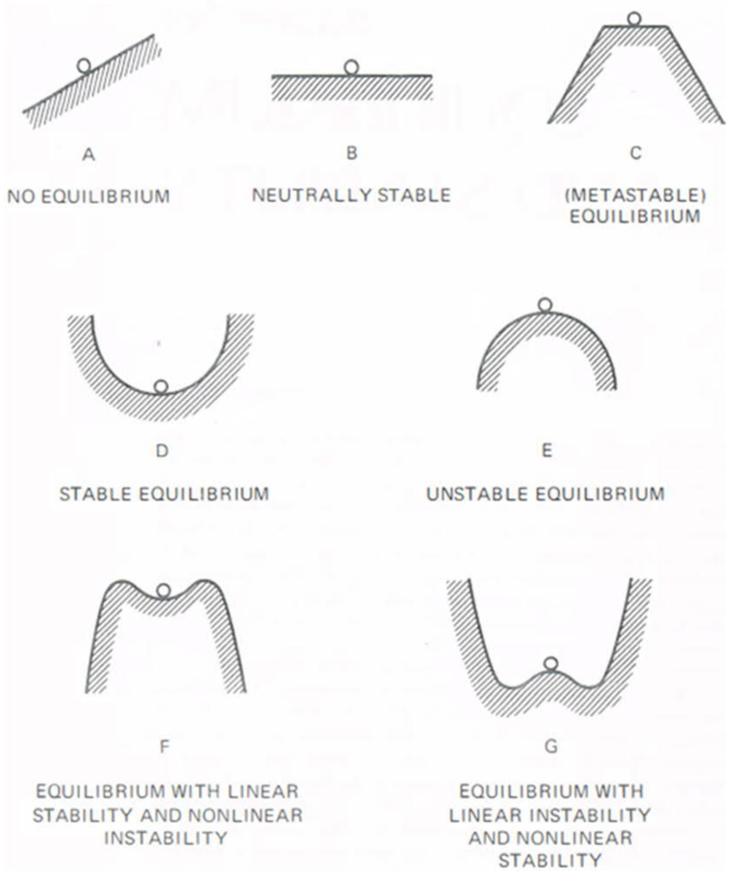
Stability of an equilibrium

Recall start of the lecture: equilibria can vary wrt their *stability properties.*

After solving for our equilibrium, how can we find whether it is stable or not?

At least two methods:

1. As an eigenvalue problem
2. Via energy principle



Intuitive approach to stability

Let's once again perturb our *equilibrium* and make a *linear stability analysis* by writing all terms as $f = f_0 + f_1$ and keeping only terms up to first order.

Here our primary quantity is the *plasma displacement,* $\boldsymbol{\xi}$: $\mathbf{v}_1 = \frac{d\xi}{dt}$

This means that we have to *integrate in time* many of the MHD equations.

Starting with our standard, simple plasma, the linearized equations become:

Continuity: $\rho_1 = -\nabla \cdot (\rho_0 \boldsymbol{\xi})$

Equation of state: $p_1 = -p_0 \gamma \nabla \cdot \boldsymbol{\xi} - \boldsymbol{\xi} \cdot \nabla p_0$

Faraday + Ohm: $\boldsymbol{B}_1 = \nabla \times (\boldsymbol{\xi} \times \boldsymbol{B}_0)$

Instability as an eigenvalue problem

And last but not least (using also Ampere's law) …

The equation of motion:

$$\rho_0 \frac{\partial^2 \xi}{\partial t^2} = \frac{1}{\mu_0}[(\nabla \times \boldsymbol{B}_0) \times \boldsymbol{B}_1 + (\nabla \times \boldsymbol{B}_1) \times \boldsymbol{B_0}]\nabla(p_0 \gamma \nabla \cdot \boldsymbol{\xi} + \boldsymbol{\xi} \cdot \nabla p_0)$$

This can be expresses as $\rho_0 \frac{\partial^2 \xi}{\partial t^2} = \boldsymbol{F}(\boldsymbol{\xi})$

Now applying the Fourier decomposition gives $\omega^2 \rho_0 \boldsymbol{\xi} = \boldsymbol{F}(\boldsymbol{\xi})$

Which is an *eigenvalue problem* for ω^2 and gives the stability:

- $\omega^2 > 0 \rightarrow$ stable
- $\omega^2 < 0 \rightarrow$ unstable

➔ not only do we get the (in-)stability, but even the *growth rate*, $Im(\omega)$!

Energy principle in stability analysis

Unfortunately, the eigenvalue problems tend to be mathematically very complicated and can be solved only numerically.

However, if one is only interested whether a given equilibrium is stable or not, one can apply the *energy principle:*

Multiply the eigenvalue problem by $\boldsymbol{\xi}^*$, the complex-conjugate of $\boldsymbol{\xi}$, and integrate over the whole volume ➔

$$\omega^2 \int |\boldsymbol{\xi}|^2 dV = -\int \boldsymbol{\xi}^* \cdot \boldsymbol{F}(\boldsymbol{\xi}) dV$$

LHS: clearly the *kinetic energy* of the system, $K(\boldsymbol{\xi}, \boldsymbol{\xi}^*)$

RHS: the *work* done against the force $\boldsymbol{F}$ ➔ potential energy $\delta W(\boldsymbol{\xi}, \boldsymbol{\xi}^*)$

Understanding the energy principle

So we have a very simple-looking equation for ω^2: $\omega^2 = \frac{\delta W(\boldsymbol{\xi},\boldsymbol{\xi}^*)}{K(\boldsymbol{\xi},\boldsymbol{\xi}^*)}$

But we don't have to solve that to find the stability: $K(\boldsymbol{\xi},\boldsymbol{\xi}^*) > 0$ always ➔

Stability of the equilibrium is given by $\delta W(\boldsymbol{\xi},\boldsymbol{\xi}^*)$:

- $\delta \mathrm{W}(\boldsymbol{\xi},\boldsymbol{\xi}^*) > 0$ ➔ a stable equilibrium
- $\delta \mathrm{W}(\boldsymbol{\xi},\boldsymbol{\xi}^*) < 0$ ➔ unstable equilibrium.

Looks easy? Not necessarily:

- There is a lot of sophisticated math skipped here
- One has to come up with an appropriate test function $\boldsymbol{\xi}$
- δW has actually three terms: plasma+vacuum+surface ...