Taloustieteen matemaattiset menetelmät
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Problem Set 6: Solutions

1. **Solution**

First of all, a solution exists by Weierstrass's Theorem. The Kuhn-Tucker Lagrangian is

$$\tilde{L} = x^a y^{1-a} - \mu \left(p_x x + p_y y - w\right).$$

The first order conditions are

$$ax^{a-1}y^{1-a} - \mu p_x \leq 0 \tag{1}$$
$$(1-a)x^a y^{-a} - \mu p_y \leq 0 \tag{2}$$
$$x\left(ax^{a-1}y^{1-a} - \mu p_x\right) = 0 \tag{3}$$
$$y\left((1-a)x^a y^{-a} - \mu p_y\right) = 0 \tag{4}$$
$$p_x x + p_y y \leq w \tag{5}$$
$$\mu\left(p_x x + p_y y - w\right) = 0. \tag{6}$$

Notice that a solution must be such that $x > 0$ and $y > 0$. If not, total utility is zero. But then it would be feasible to attain strictly positive utility by choosing positive quantities of both commodities while satisfying the budget constraint.

Since we must have $x > 0$ and $y > 0$, (3) and (4) imply that both (1) and (2) hold with equality and, consequently, $\mu > 0$, which in turn implies that the budget constraint $g(x, y) = p_x x + p_y y \leq w$ is binding via (6). Combining (1) and (2), we get $y = \frac{p_x}{p_y}\frac{(1-a)}{a}x$. Combining the latter expression with the budget constraint we obtain $x = \frac{aw}{p_x}$ and $y = \frac{(1-a)w}{p_y}$, which is the unique solution. NDCQ is satisfied, as $\frac{\partial g}{\partial x} = p_x > 0$ and $\frac{\partial g}{\partial y} = p_y > 0$.

2. **Solution**

(a) The first order conditions are

$$\frac{\partial f}{\partial x} = r^2 - 2x = 0$$

$$\frac{\partial f}{\partial y} = 3s^2 - 16y = 0$$

The Hessian is

$$H = \begin{pmatrix} -2 & 0 \\ 0 & -16 \end{pmatrix}$$

and its leading principal minors are $\det(A_1) = -2 < 0$ and $\det(A_2) = 32 > 0$, so the matrix is always negative definite. Therefore, given parameters r and s, the objective function is concave (in x and y). Thus a solution exists. From the first onder conditions we get the unique critical point

$$(x, y) = \left(\frac{r^2}{2}, \frac{3}{16}s^2\right),$$

which is the global maximizer.

(b) We can write the solution as $x^*(r, s) = \frac{r^2}{2}$ and $y^*(r, s) = \frac{3}{16}s^2$. The value function is

$$V(r, s) = f(x^*(r, s), y^*(r, s); r, s) = \frac{r^4}{4} + \frac{9}{32}s^4.$$

Then it is easy to check that:

$$\begin{aligned}
&\frac{dV}{dr}(r, s) = r^3 \\
&\frac{\partial f}{\partial r}(x^*(r, s), y^*(r, s); r, s) = 2rx^*(r, s) = r^3 \\
&\frac{dV}{ds}(r, s) = \frac{9s^3}{8} \\
&\frac{\partial f}{\partial s}(x^*(r, s), y^*(r, s); r, s) = 6sy^*(r, s) = \frac{9s^3}{8}.
\end{aligned}$$

3. **Solution**

(a) The Lagrangian is

$$L = x + 2z - \lambda_1 (x + y + z - 1) - \lambda_2 \left(x^2 + y^2 + z - \frac{7}{4}\right).$$

The rank of the Jacobian of the two constraint functions is equal to 2 unless $x = y = \frac{1}{2}$, in which case the rank is equal to 1. However, if $x = y = \frac{1}{2}$, then from the first constraint we get $z = 0$. Now $g_2(x, y, z) = \frac{1}{2} \neq \frac{7}{4}$, so there is no point in the constraint set in which $x = y = \frac{1}{2}$. Therefore, the NDCQ is satisfied.

The first order conditions are

$$1 - \lambda_1 - 2\lambda_2 x = 0 \tag{7}$$

$$-\lambda_1 - 2\lambda_2 y = 0 \tag{8}$$

$$2 - \lambda_1 - \lambda_2 = 0 \tag{9}$$

$$x + y + z = 1 \tag{10}$$

$$x^2 + y^2 + z = \frac{7}{4}. \tag{11}$$

From (9) we obtain $\lambda_2 = 2 - \lambda_1$, which inserted into (8) gives $\lambda_1(2y-1) = 4y$. This equation implies that $y \neq \frac{1}{2}$, so $\lambda_1 = \frac{4y}{2y-1}$. Inserting this into (7) with $\lambda_2 = 2 - \lambda_1$ eventually yields $y = 2x - \frac{1}{2}$. Inserting the last expression into the two constraints yields $3x + z = \frac{3}{2}$ and $5x^2 - 2x + z = \frac{3}{2}$. These two equations combined give $z = \frac{3}{2} - 3x$ and $5x(x-1) = 0$. Thus, $x = 0$ or $x = 1$. If $x = 0$, we obtain $y = -\frac{1}{2}$, $z = \frac{3}{2}$, and $\lambda_1 = \lambda_2 = 1$. If $x = 1$, we get $y = \frac{3}{2}$, $z = -\frac{3}{2}$, $\lambda_1 = 3$, and $\lambda_2 = -1$. Evaluating f at these two critical points we get $f(0, -1/2, 3/2) = 3$ and $f(1, 3/2, -3/2) = -2$. Thus the only candidate for a solution is $(0, -1/2, 3/2)$. Given $\lambda_1 = \lambda_2 = 1$, f is linear (and thus also concave), $\lambda_1 g_1$ is linear (and thus also convex), and $\lambda_2 g_2$ is convex. Thefore the Lagrangian is a concave function in (x, y, z). So we can conclude that $(0, -1/2, 3/2)$ is the solution to this maximization problem.

(b) Let $c_1 = 1$, $c_2 = \frac{7}{4} = 1.75$, $dc_1 = -0.02$, and $dc_2 = 0.05$. In addition, let $V(c_1, c_2)$ be the problem's value function. By the envelope theorem, $\frac{\partial V}{\partial c_1}(c_1, c_2) = \lambda_1(c_1, c_2)$ and $\frac{\partial V}{\partial c_2}(c_1, c_2) = \lambda_2(c_1, c_2)$. Using the total differential, we have

$$\begin{aligned} dV &= \frac{\partial V}{\partial c_1}(1, 1.75)dc_1 + \frac{\partial V}{\partial c_2}(1, 1.75)dc_2 \\ &= \lambda_1(1, 1.75) \times (-0.02) + \lambda_2(1, 1.75) \times 0.05 \\ &= 1 \times (-0.02) + 1 \times 0.05 \\ &= 0.03. \end{aligned}$$

4. **Solution**

(a) The Lagrangian is

$$L = ax + by + cz - \lambda(\alpha x^2 + \beta y^2 + \gamma z^2 - L).$$

The first order conditions are

$$\begin{aligned} a - 2\lambda\alpha x &= 0 \\ b - 2\lambda\beta y &= 0 \\ c - 2\lambda\gamma z &= 0 \\ \lambda(\alpha x^2 + \beta y^2 + \gamma z^2 - L) &= 0 \\ \lambda &\geq 0 \end{aligned}$$

Because a, b and c are all positive, also λ, x, y, and z must all be positive (otherwise the FOCs are not satisfied). Thus, $\lambda > 0$ and the constraint binds. From the first three conditions we get $\lambda = \frac{a}{2\alpha x} = \frac{b}{2\beta y} = \frac{c}{2\gamma z}$. Also, $x^* = \frac{a}{2\alpha\lambda}$, $y^* = \frac{b}{2\beta\lambda}$, and $z^* = \frac{c}{2\gamma\lambda}$.

Substituting x^*, y^*, and z^* into the constraint yields

$$\alpha\left(\frac{a}{2\lambda\alpha}\right)^2 + \beta\left(\frac{b}{2\lambda\beta}\right)^2 + \gamma\left(\frac{c}{2\lambda\gamma}\right)^2 = L$$

and by solving this for λ we get

$$\lambda = \frac{1}{2}L^{-\frac{1}{2}}\sqrt{\frac{a^2}{\alpha}+\frac{b^2}{\beta}+\frac{c^2}{\gamma}}.$$

Because f is linear (and therefore also concave) and λg is convex, the Lagrangian is concave. Thus, $(x^*, y^*, z^*) = (\frac{a}{2\lambda\alpha}, \frac{b}{2\lambda\beta}, \frac{c}{2\lambda\gamma})$ is the solution for the maximization problem.

(b) The value function $V(L)$ is

$$V(L) = ax^* + by^* + cz^* = \frac{a^2}{2\lambda\alpha}+\frac{b^2}{2\lambda\beta}+\frac{c^2}{2\lambda\gamma}.$$

Inserting $\lambda = \frac{1}{2}L^{-\frac{1}{2}}\sqrt{\frac{a^2}{\alpha}+\frac{b^2}{\beta}+\frac{c^2}{\gamma}}$ into the value function yields

$$V(L) = \sqrt{L}\sqrt{\frac{a^2}{\alpha}+\frac{b^2}{\beta}+\frac{c^2}{\gamma}}.$$

Now we can verify that

$$\frac{dV}{dL}(L) = L^{-\frac{1}{2}}\sqrt{\frac{a^2}{\alpha}+\frac{b^2}{\beta}+\frac{c^2}{\gamma}} = \lambda(L).$$