Lecture notes based on an article in the Springer Encyclopedia of Complexity and System Science with Kent B. Lauritsen

Branching Processes and Complexity

Mikko Alava

Aalto University

Outline of Contents

Glossary

Markov process

A process characterized by a set of probabilities to go from a certain state at time t to another state at time $t+1$. These transition probabilities are independent of the history of the process and only depend on a fixed probability assigned to the transition.

Critical properties and scaling

The behavior of equilibrium and many non-equilibrium systems in steady states contain critical points where the systems display scale invariance and the correlation functions exhibit an algebraic behavior characterized by so-called critical exponents. A characteristics of this type of behavior is the lack of finite length and time scales (also reminiscent of fractals). The behavior near the critical points can be described by scaling functions that are universal and that do not depend on the detailed microscopic dynamics.

Avalanches

When a system is perturbed in such a way that a disturbance propagates throughout the system one speaks on an avalanche. The local avalanche dynamics may either conserve energy (particles) or dissipate energy. The avalanche may also loose energy when it reaches the system boundary. In the neighborhood of a critical point the avalanche distribution is described by a power-law distribution.

Self-organized criticality (SOC)
SOC is the surprising "critical" state in which many systems from physics to biology to social ones find themselves. In physics jargon, they exhibit scale-invariance, which means that the dynamics - consisting of avalanches - has no typical scale in time or space. The really necessary ingredient is that there is a hidden, fine-tuned balance between how such systems are driven to create the dynamic response, and how they dissipate the input ("energy") to still remain in balance.

1 Definition & Introduction

Consider the fate of a human population on a small, isolated island. It consists of a certain number of individuals, and the most obvious question, of importance in particular for the inhabitants of the island, is whether this number will go to zero. Humans die and reproduce in steps of one, and therefore one can try to analyze this fate mathematically by writing down what is called master equations, to describe the dynamics as a "branching process" (BP). The branching here means that if at time $t = 0$ there are N humans, at the next step $t = 1$ there can be $N - 1$ (or $N + 1$ or $N + 2$ if the only change from $t = 0$ was that a pair of twins was born). The outcome will depend in the simplest case on a "branching number", or the number of offspring λ that a human being will have [1, 2, 3, 4].

If the offspring created are too few, then the population will decay, or reach an "absorbing state" out of which it will never escape. Likewise if they are many (the Malthusian case in reality, perhaps), exponential growth in time will ensue in the simplest case. In between, there is an interesting twist: a phase transition that separates these two outcomes at a critical value of λ_c. As is typical of such transitions in statistical physics one runs into scalefree behaviour. The lifetime of the population suddenly has no typical scale, and its total size will be a stochastic quantity, described by a probability distribution that again has no typical scale exactly at λ_c.

The example of a small island also illustrates the many different twists that one can find in branching processes. The population can be "spatially dispersed" such that the individuals are separated by distance. There are in fact two interacting populations, called "male" and "female", and if the number of one of the populations becomes zero the other one will die out soon as well. The people on the island eat, and there is thus a hidden variable in the dynamics, the availability of food. This causes a history effect which makes the dynamics of human population what is called "non-Markovian". Imagine as above, that we look at the number of persons on the island at discrete times. A Markovian process is such that the probabilities to go from a state (say of) N to state $N+\delta$ depends only on the fixed probability assigned to the "transition" $N \rightarrow N + \delta N$. Clearly, any relatively faithful description of the death and birth rates of human beings has to consider the average state of nourishment, or whether there is enough food for reproduction.

Branching processes are often perfect models of complex systems, or in other words exhibit deep complexity themselves. Consider the following example of a one-dimensional model of activated random walkers [5]. Take a line of sites $x_i, i = 1 \ldots L$. Fill the sites randomly to a certain density $n = N/L$, where N is the pre-set number of individuals performing the random walk. Now, let us apply the simple rule, that if there are two or more walkers at the same x_j, two of them get "activated" and hop to $j-1$ or $j+1$, again at random. In other words, this version of drunken bar-hoppers problem has the twist that they do not like each other.

If the system is "periodic" or $i = 1$ is connected to $i = L$, then the dynamics is controlled by the density n. For a critical value n_c (estimated by numerical simulations to be about 0.9488... [6]) a phase transition takes place, such that for $n < n_c$ the asymptotic state is the "absorbing one", where all the walkers are immobilized since $N_i = 1$ or 0. In the opposite case for $n > n_c$ there is an active phase such that (in the infinite L-limit) the activity persists forever. This particular model is unquestionably non-Markovian if one only considers the number of active walkers or their density ρ. One needs to know the full state of N_i to be able to write down exact probabilities for how ρ changes in a discrete step of time.

The most interesting things happen if one opens up the one-dimensional lattice by adapting two new rules. If a walker walks out (to $i = 0$ or $i = L+1$), it disappears. Second, if there are no active ones ($\rho = 0$), one adds one new walker randomly to the system. Now the activity $\rho(t)$ is always at a marginal value, and the long-term average of n becomes a (possibly L-dependent) constant such that the first statement is true. With these rules, the model of activated random walkers is also known as the Manna model after the Indian physicist [5], and it exhibits the phenomenon dubbed "Self-Organized

Criticality" (SOC) [7]. Figure 1 shows an example of the dynamics by using what is called an "activity plot", where those locations x_i are marked both in space and time which happen to contain just-activated walkers. One can now apply several kinds of measures to the system, but the figure already hints about the reasons why these simple models have found much interest. The structure of the activity is a self-affine fractal.

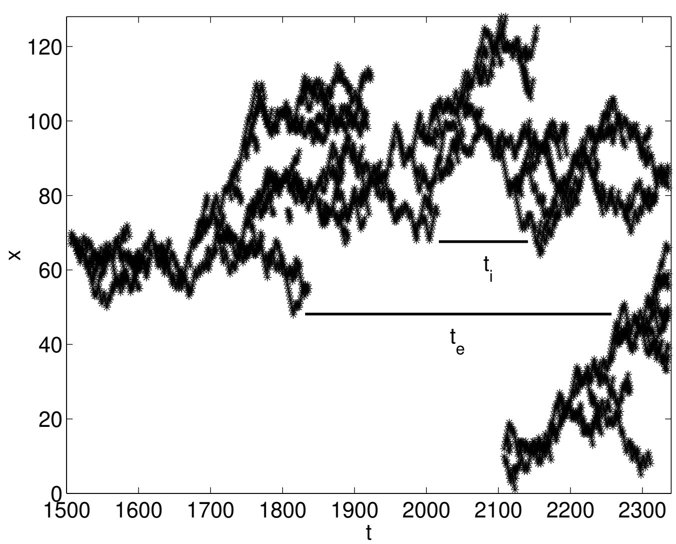

Figure 1: We follow the "activity" in a one-dimensional system of random activated walkers. The walkers stroll around the x-axis, and the pattern becomes in fact scale-invariant. This system is such that some of the walkers disappear (by escaping through open boundaries) and to maintain a constant density new ones are added. One question one may ask is what is the waiting time, that another (or the same) walker gets activated at the same location after a time of inactivity. As one can see from the figure, this can be a result of old activity getting back, or a new "avalanche" starting from the addition of an outsider (courtesy of Lasse Laurson).

The main enthusiasm about SOC comes from the avalanches. These are in other words the bursts of activity that separate quiescent times (when $\rho = 0$). The silence is broken by the addition of a particle or a walker, and it creates an integrated quantity (volume) of activity, $s = \int_0^T \rho(t)dt$, where for $0 < t < T$, $\rho > 0$ and $\rho = 0$ at the endpoints t and T. The original boost to SOC took place after Per Bak, Chao Tang, and Kay Wiesenfeld published in 1987 a highly influential paper in the premium physics journal Physical Review Letters, and introduced what is called the Bak-Tang-Wiesenfeld (BTW) sandpile model - of which the Manna one is a relative [7]. The BTW and Manna models and many others exibit the important property that the avalanche sizes s have *scale-free* probability distribution, which is usually written as

$$P(s) \sim s^{-\tau_s} f_s(s/L^{D_s}), \tag{1}$$

here all the subscripts refer to the fact that we look at avalanches. τ_s and D_s define the avalanche exponent and the cut-off exponent, respectively. f_s is a cut-off function that together with D_s includes the fact that the avalanches are restricted somehow by the system size (in the one-dimensional Manna model, if s becomes too large many walkers are lost, and first n drops and then ρ goes to zero). Similar statements can be made about the avalanche durations (T), area or support A, and so forth [8, 9, 10, 11, 12].

The discovery of simple-to-define models, yet of very complex behavior has been a welcome gift since there are many, many phenomena that exhibit apparently scalefree statistics and/or bursty or intermittent dynamics similar to SOC models. These come from natural sciences - but not only, since economy and sociology are also giving rise to cases that need explanations. One particular field where these ideas have found much interest is the physics of materials ranging from understanding earthquakes to looking at the behavior of vortices in superconductors. The Gutenberg-Richter law of earthquake magnitudes is a power-law, and one can measure similar statistics from fracture experiments even on normal paper [13].

Whenever comparing with real, empirical phenomena models based on branching processes give a paradigm on two levels. In the case of SOC this is given by a combination of "internal dynamics" and "an ensemble". Of these, the first means e.g. that activated walkers or particles of type A are moved around with certain rules, and of course there is an enormous variety of possible models. E.g. in the Manna-model this is obvious if one splits the walkers into categories A (active) and B (passive). Then, there is the question of how the balance (assuming this *is* true) is maintained. The SOC models do this by a combination of dissipation (by e.g. particles dropping off the edge of a system) and drive (by addition of B's), where the rates are chosen actually pretty carefully.

For the theory of complex systems branching processes thus give rise to two questions: what classes of models are there? What kinds of truly different ensembles are there? The theory of reaction-diffusion systems has tried to answer to the first question since the 70's, and the developments reflect the idea of universality in statistical physics. There, this means that the behavior of systems at "critical points" - such as defined by the λ_c and n_c from above - follows from the dimension at hand and the "universality class" at hand. The activated walkers' model, it has recently been established, belongs to the "fixed energy sandpile" one, and is closely related to other seemingly far-fetched ones such as the depinning/pinning of domain walls in magnets (see e.g. [14]). The second question can be stated in two ways: forgetting about the detailed model at hand, when can one expect complex behavior such as power-law distributions, avalanches etc.? Second, and more technically, can one derive the exponents such as τ_s and D_s from those of the same model at its "usual" phase transition?

The theory of branching processes provide many answers to these questions, and in particular help to illustrate the influence of boundary conditions and modes of driving on the expected behavior. Thus, one gets a clear idea of the kind of complexity one can expect to see in the many different kinds of systems where avalanches and intermittency and scaling is observed.

2 Branching Processes

The mathematical branching process is defined for a set of objects that do not interact. At each iteration, each object can give rise to new objects with some probability p (or in general it can be a set of probabilities). By continuing this iterative process the objects will form what is referred to as a cluster or an avalanche. We can now ask questions of the following type: Will the process continue for ever? Will the process maybe die out and stop after a finite number of iterations? What will the average lifetime be? What is the average size of the clusters? And what is (the asymptotic form of) the probability that the process is active after a certain number of iterations, etc.

2.1 BP definition

We will consider a simple discrete BP denoted the Galton-Watson process. For other types of BPs (including processes in continuous time) we refer to the book by Harris [1]. We will denote the number of objects at generation

n as: $z_0, z_1, z_2, ..., z_n, \ldots$. The index n is referred to as time and time $t = 0$ corresponds to the 0th generation where we take $z_0 = 1$, i.e., the process starts from one object. We will assume that the transition from generation n to $n+1$ is given by a probability law that is independent of n, i.e., it is assumed that it forms a Markov process. And finally it will be assumed that different objects do not interact with one another.

The probability measure for the process is characterized by probabilities as follows: p_k is the probability that an object in the nth generation will have k offsprings in the $n+1$th generation. We assume that p_k is independent of n. The probabilities p_k thus reads $p_k = \mathrm{Prob}(z_1 = k)$ and fulfills:

$$\sum_k p_k = 1. \quad (2)$$

Figure 2 shows an example of the tree structure for a branching process and the resulting avalanche.

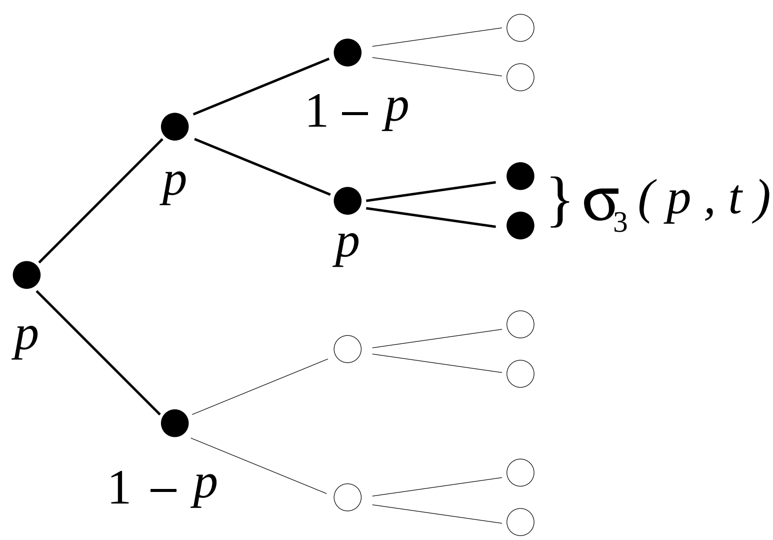

Figure 2: Schematic drawing of an avalanche in a system with a maximum of $n = 3$ avalanche generations corresponding to $N = 2^{n+1} - 1 = 15$ sites. Each black site relaxes with probability p to two new black sites and with probability $1-p$ to two white sites (i.e., $p_0 = 1-p$, $p_1 = 0$, $p_2 = p$). The black sites are part of an avalanche of size $s = 7$, whereas the active sites at the boundary yield a boundary avalanche of size $\sigma_3(p,t) = 2$.

One can define the probability generating function $f(s)$ associated to the transition probabilities:

$$f(s) = \sum_k p_k s^k. \tag{3}$$

The first and second moments of the number z_1 are denoted as:

$$m = E z_1, \qquad \sigma^2 = \mathrm{Var}\, z_1. \tag{4}$$

By taking derivatives of the generating function at $s = 1$ it follows that:

$$E z_n = m^n, \qquad \mathrm{Var}\, z_n = n\sigma^2. \tag{5}$$

For the BP defined in Fig. 2, it follows that $m = 2p$ and $\sigma^2 = 4p(1-p)$.

An important quantity is the probability for extinction q. It is obtained as follows:

$$q = P(z_n \to 0) = P(z_n = 0 \text{ for some } n) = \lim_n P(z_n = 0). \tag{6}$$

It can be shown that for $m \leq 1 : q = 1$; and for $m > 1$: there exists a solution that fulfills $q = f(q)$, where $0 < q < 1$ [1]. It is possible to show that $\lim_n P(z_n = k) = 0$, for $k = 1, 2, 3, ...$, and that $z_n \to 0$ with probability q, and $z_n \to \infty$, with probability $1 - q$. Thus, the sequence $\{z_n\}$ does not remain positive and bounded [1].

The quantity $1 - q$ is similar to an order parameter for systems in equilibrium and its behavior is schematically shown in Fig. 3. The behavior around the value $m_c = 1$, the so-called critical value for the BP (see below), can in analogy to second order phase transitions in equilibrium systems be described by a critical exponent β defined as follows ($\beta = 1$, cf. [1]):

$$\mathrm{Prob(survival)} = \begin{cases} 0, & m \leq m_c \\ (m - m_c)^\beta, & m > m_c. \end{cases} \tag{7}$$

2.2 Avalanches and Critical Point

We will next consider the clusters, or avalanches, in more detail and obtain the asymptotic form of the probability distributions. The size of the cluster is given by the sum $z = z_1 + z_2 + z_3 + ...$ One can also consider other types of clusters, e.g., the activity σ of the boundary (of a finite tree) and define the boundary avalanche (cf. Fig. 2).

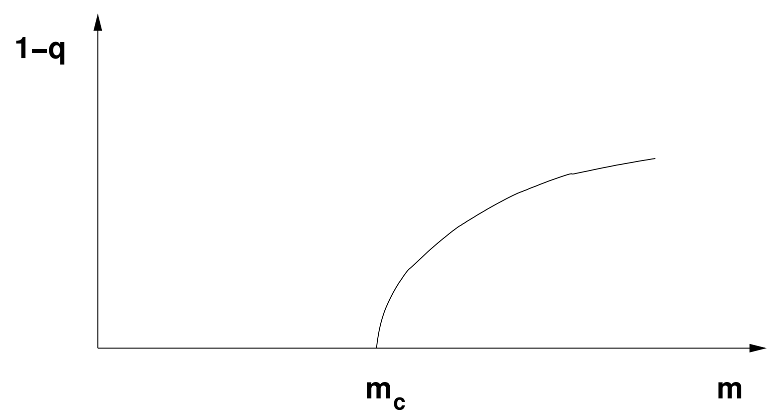

Figure 3: Schematic drawing of the behavior of the probability of extinction, q, of the branching process. The quantity $1-q$ is similar to the order parameter for systems in equilibrium at their critical points and the quantity m_c is referred to as the critical point for the BP.

For concreteness, we consider the BP defined in Fig. 2. The quantities $P_n(s,p)$ and $Q_n(\sigma,p)$ denote the probabilities of having an avalanche of size s and boundary size σ in a system with n generations. The corresponding generating functions are defined by [1]

$$f_n(x,p) \equiv \sum_s P_n(s,p)x^s, \tag{8}$$

$$g_n(x,p) \equiv \sum_\sigma Q_n(\sigma,p)x^\sigma. \tag{9}$$

Due to the hierarchical structure of the branching process, it is possible to write down recursion relations for $P_n(s,p)$ and $Q_n(\sigma,p)$:

$$f_{n+1}(x,p) = x\left[(1-p) + pf_n^2(x,p)\right], \tag{10}$$

$$g_{n+1}(x,p) = (1-p) + pg_n^2(x,p), \tag{11}$$

where $f_0(x,p) = g_0(x,p) = x$.

The avalanche distribution $D(s)$ is determined by $P_n(s,p)$ by using the recursion relation (10). The solution of Eq. (10) in the limit $n \gg 1$ is given by

$$f(x,p) = \frac{1-\sqrt{1-4x^2p(1-p)}}{2xp}. \tag{12}$$

By expanding Eq. (12) as a series in x, and comparing with the definition (8), we obtain for sizes such that $1 \ll s \lesssim n$

$$P_n(s,p) = \frac{\sqrt{2(1-p)/\pi p}}{s^{3/2}} \exp\left(-s/s_c(p)\right). \tag{13}$$

The cutoff $s_c(p)$ is given by $s_c(p) = -2/\ln 4p(1-p)$. As $p \to 1/2$, $s_c(p) \to \infty$, thus showing explicitly that the critical value for the branching process is $p_c = 1/2$ (i.e., $m_c = 1$), and that the mean-field avalanche exponent, cf. Eq. (1), for the critical branching process is $\tau = 3/2$.

The expression (13) is only valid for avalanches which do not feel the finite size of the system. For avalanches with $n \lesssim s \lesssim N$, it is possible to solve the recursion relation (10), and then obtain $P_n(s,p)$ for $p \geq p_c$ by the use of a Tauberian theorem [15, 16, 17]. By carrying out such an analysis one obtains after some algebra $P_n(s,p) \approx A(p) \exp\left(-s/s_0(p)\right)$, with functions $A(p)$ and $s_0(p)$ which can not be determined analytically. Nevertheless, we see that for any p the probabilities $P_n(s,p)$ will decay exponentially. One can also calculate the asymptotic form of $Q_n(\sigma,p)$ for $1 \ll \sigma \lesssim n$ and $p \geq p_c$ by the use of a Tauberian theorem [15, 16, 17]. We will return to study this in the next section where we will also define and investigate the distribution of the time to extinction.

3 Self-Organized Branching Processes

We now return to discussing the link between self-organized criticality as mentioned in the introduction and branching processes. The simplest theoretical approach to SOC is mean-field theory [18], which allows for a qualitative description of the behavior of the SOC state. Mean-field exponents for SOC models have been obtained by various approaches [18, 19, 20, 21] and it turns out that their values (e.g., $\tau = 3/2$) are the same for all the models considered thus far. This fact can easily be understood since the spreading of an avalanche in mean-field theory can be described by a front consisting of non-interacting particles that can either trigger subsequent activity or die out. This kind of process is reminiscent of a branching process. The connection between branching processes and SOC has been investigated, and it has been discussed that the mean-field behavior of sandpile models can be described by a critical branching process [22, 23, 24].

For a branching process to be critical one must fine tune a control parameter to a critical value. This, by definition, cannot be the case for a SOC system, where the critical state is approached dynamically without the need to fine tune any parameter. In the so-called self-organized branching-process (SOBP), the coupling of the local dynamical rules to a global condition drives the system into a state that is indeed described by a critical branching process [25]. It turns out that the mean-field theory of SOC models can be exactly mapped to the SOBP model.

In the mean-field description of the sandpile model ($d \to \infty$) one neglects correlations, which implies that avalanches do not form loops and hence spread as a branching process. In the SOBP model, an avalanche starts with a single active site, which then relaxes with probability p, leading to two new active sites. With probability $1-p$ the initial site does not relax and the avalanche stops. If the avalanche does not stop, one repeats the procedure for the new active sites until no active site remains. The parameter p is the probability that a site relaxes when it is triggered by an external input. For the SOBP branching process, there is a critical value, $p_c = 1/2$, such that for $p > p_c$ the probability to have an infinite avalanche is non-zero, while for $p < p_c$ all avalanches are finite. Thus, $p = p_c$ corresponds to the critical case, where avalanches are power law distributed.

In this description, however, the boundary conditions are not taken into account—even though they are crucial for the self-organization process. The boundary conditions can be introduced in the problem in a natural way by allowing for no more than n generations for each avalanche. Schematically, we can view the evolution of a single avalanche of size s as taking place on a tree of size $N = 2^{n+1} - 1$ (see Fig. 2). If the avalanche reaches the boundary of the tree, one counts the number of active sites σ_n (which in the sandpile language corresponds to the energy leaving the system), and we expect that p decreases for the next avalanche. If, on the other hand, the avalanche stops before reaching the boundary, then p will slightly increase. The number of generations n can be thought of as some measure of the linear dimension of the system.

The above avalanche scenario is described by the following dynamical equation for $p(t)$:

$$p(t+1) = p(t) + \frac{1 - \sigma_n(p,t)}{N}, \tag{14}$$

where σ_n, the size of an avalanche reaching the boundary, fluctuates in time and hence acts as a stochastic driving force. If $\sigma_n = 0$, then p increases (because some energy has been put into the system without any output), whereas if $\sigma_n > 0$ then p decreases (due to energy leaving the system). Equation (14) describes the global dynamics of the SOBP, as opposed to the local dynamics which is given by the branching process. One can study the model for a fixed value of n, and then take the limit $n \to \infty$. In this way, we perform the long-time limit before the "thermodynamic" limit, which corresponds exactly to what is done in sandpile simulations.

We will now show that the SOBP model provides a mean-field theory of self-organized critical systems. Consider for simplicity the sandpile model of activated random walkers from the Introduction [5]: When a particle is added to a site z_i, the site will relax if $z_i = 1$. In the limit $d \to \infty$, the avalanche will never visit the same site more than once. Accordingly, each site in the avalanche will relax with the same probability $p = P(z = 1)$. Eventually, the avalanche will stop, and $\sigma \geq 0$ particles will leave the system. Thus, the total number of particles $M(t)$ evolves according to

$$M(t+1) = M(t) + 1 - \sigma. \tag{15}$$

The dynamical equation (14) for the SOBP model is recovered by noting that $M(t) = NP(z = 1) = Np$. By taking the continuum time limit of Eq. (14), it is possible to obtain the following expression:

$$\frac{dp}{dt} = \frac{1 - (2p)^n}{N} + \frac{\eta(p,t)}{N}, \tag{16}$$

where $\eta = \langle \sigma_n \rangle - \sigma_n = (2p)^n - \sigma_n(p,t)$ describes the fluctuations in the steady state. Without the last term, Eq. (16) has a fixed point ($dp/dt = 0$) for $p = p_c = 1/2$. On linearizing Eq. (16), one sees that the fixed point is attractive, which demonstrates the self organization of the SOBP model since the noise η/N will have vanishingly small effect in the thermodynamic limit [25].

Fig. 4 shows the value of p as a function of time. Independent of the initial conditions, one finds that after a transient $p(t)$ reaches the self-organized state described by the critical value $p_c = 1/2$ and fluctuates around it with short-range correlations (of the order of one time unit). By computing the variance of $p(t)$, one finds that the fluctuations can be very well described by a Gaussian distribution, $\phi(p)$ [25]. In the limit $N \to \infty$, the distribution $\phi(p)$ approaches a delta function, $\delta(p - p_c)$.

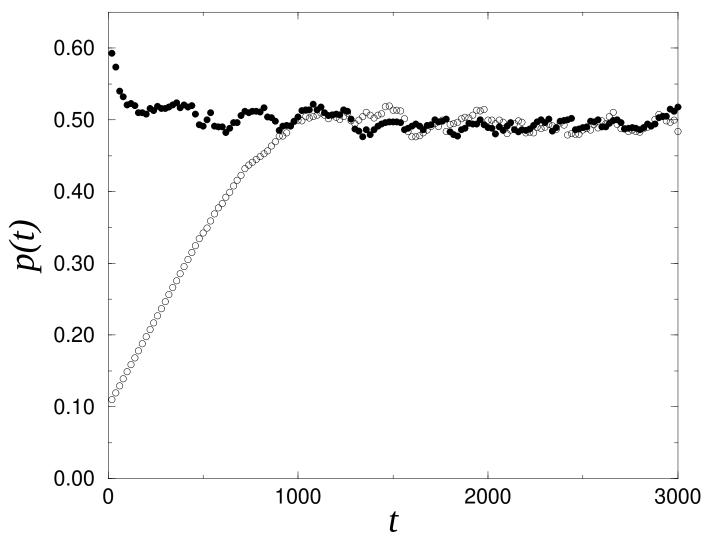

Figure 4: The value of p as a function of time for a system with $n = 10$ generations. The two curves refer to two different initial conditions, above p_c ($\bullet$) and below p_c ($\circ$). After a transient, the control parameter $p(t)$ reaches its critical value p_c and fluctuates around it with short-range correlations.

3.1 Avalanche distributions

Figure 5 shows the avalanche size distribution $D(s)$ for different values of the number of generations n. One notices that there is a scaling region ($D(s) \sim s^{-\tau}$ with $\tau = 3/2$), whose size increases with n, and characterized by an exponential cutoff. This power-law scaling is a signature of the mean-field criticality of the SOBP model. The distribution of active sites at the boundary, $D(\sigma)$, for different values of the number of generations falls off exponentially [25].

In the limit where $n \gg 1$ one can obtain various analytical results and e.g. calculate the avalanche distribution $D(s)$ for the SOBP model. In addition, one can obtain results for finite, but large, values of n. The distribution $D(s)$ can be calculated as the average value of $P_n(s,p)$ with respect to the

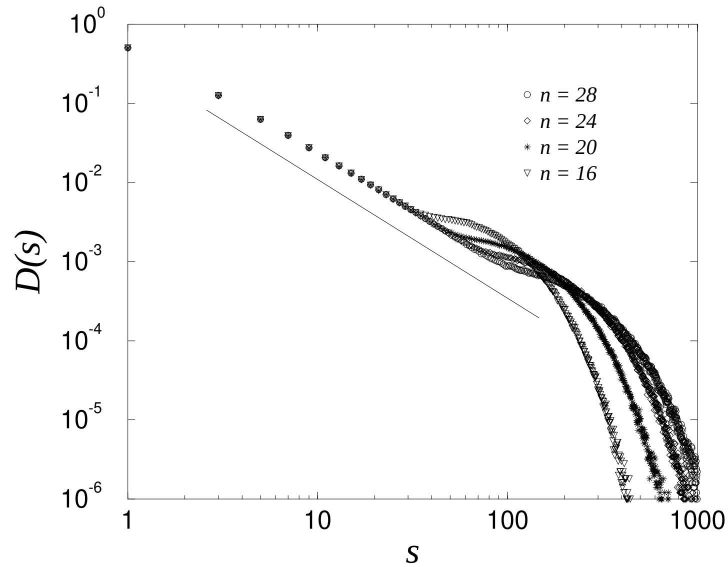

Figure 5: Log-log plot of the avalanche distribution $D(s)$ for different system sizes. The number of generations n increases from left to right. A line with slope $\tau = 3/2$ is plotted for reference, and it describes the behavior of the data for intermediate s values, cf. Eq. (18). For large s, the distributions fall off exponentially.

probability density $\phi(p)$, i.e., according to the formula

$$D(s) = \int_0^1 dp\, \phi(p)\, P_n(s,p). \tag{17}$$

The simulation results in Fig. 4 yields that $\phi(p)$ for $N \gg 1$ approaches the delta function $\delta(p - p_c)$. Thus, from Eqs. (13) and (17) we obtain the power-law behavior

$$D(s) = \sqrt{\frac{2}{\pi}}\, s^{-\tau}, \tag{18}$$

where $\tau = 3/2$, and for $s \gtrsim n$ we obtain an exponential cutoff $\exp(-s/s_0(p_c))$. These results are in complete agreement with the numerical results shown in Fig. 5. The deviations from the power-law behavior (18) are due to the fact that Eq. (13) is only valid for $1 \ll s \lesssim n$. One can also calculate the asymptotic form of $Q_n(\sigma, p)$ for $1 \ll \sigma \lesssim n$ and $p \geq p_c$ by the use of a Tauberian theorem [15, 16, 17]; the result show that the boundary avalanche distribution is

$$D(\sigma) = \int_0^1 dp\, \phi(p)\, Q_n(\sigma, p) = \frac{8}{n^2} \exp\left(-2\sigma/n\right), \tag{19}$$

which agrees with simulation results for $n \gg 1$, cf. [25].

The avalanche lifetime distribution $L(t) \sim t^{-y}$ yields the probability to have an avalanche that lasts for a time t. For a system with m generations one obtains $L(m) \sim m^{-2}$ [1]. Identifying the number of generations m of an avalanche with the time t, we thus obtain the mean-field value $y = 2$, in agreement with simulations of the SOBP model [25]. In summary, the self-organized branching process captures the physical features of the self-organization mechanism in sandpile models. By explicitly incorporating the boundary conditions it follows that the dynamics drives the system into a stationary state, which in the thermodynamic limit corresponds to the critical branching process.

4 Scaling and Dissipation

Sometimes it can be difficult to determine whether the cutoff in the scaling is due to finite-size effects or due to the fact that the system is not *at* but rather only *close to* the critical point. In this respect, it is important to test the robustness of critical behavior by understanding which perturbations destroy the critical properties. It has been shown numerically [26, 27, 28] that the breaking of the conservation of particle numbers leads to a characteristic size in the avalanche distributions. We will now allow for dissipation in branching processes and show how the system self-organizes into a sub-critical state. In other words, the degree of nonconservation is a relevant parameter in the renormalization group sense [29].

Consider again the two-state model introduced by Manna [5]. Some degree of nonconservation can be introduced in the model by allowing for energy dissipation in a relaxation event. In a continuous energy model this can be done by transferring to the neighboring sites only a fraction $(1 - \epsilon)$ of the energy lost by the relaxing site [26]. In a discrete energy model, such as the Manna two-state model, one can introduce dissipation as the probability ϵ

that the two particles transferred by the relaxing site are annihilated [27]. For $\epsilon = 0$ one recovers the original two-state model.

Numerical simulations [26, 27] show that different ways of considering dissipation lead to the same effect: a characteristic length is introduced into the system and the criticality is lost. As a result, the avalanche size distribution decays not as a pure power law but rather as

$$D(s) \sim s^{-\tau}\, h_s(s/s_c). \tag{20}$$

Here $h_s(x)$ is a cutoff function and the cutoff size scales as

$$s_c \sim \epsilon^{-\varphi}. \tag{21}$$

The size s is defined as the number of sites that relax in an avalanche. We define the avalanche lifetime T as the number of steps comprising an avalanche. The corresponding distribution decays as

$$D(T) \sim T^{-y}\, h_T(T/T_c), \tag{22}$$

where $h_T(x)$ is another cutoff function and T_c is a cutoff that scales as

$$T_c \sim \epsilon^{-\psi}. \tag{23}$$

The cutoff or "scaling" functions $h_s(x)$ and $h_T(x)$ fall off exponentially for $x \gg 1$.

To construct the mean-field theory one proceeds as follows [30]: When a particle is added to an arbitrary site, the site will relax if a particle was already present, which occurs with probability $p = P(z = 1)$, the probability that the site is occupied. If a relaxation occurs, the two particles are transferred with probability $1 - \epsilon$ to two of the infinitely many nearest neighbors, or they are dissipated with probability ϵ (see Fig. 6).

The avalanche process in the mean-field limit is a branching process. Moreover, the branching process can be described by the *effective* branching probability

$$\tilde{p} \equiv p(1 - \epsilon), \tag{24}$$

where $\tilde{p}$ is the probability to create two new active sites. We know that there is a critical value for $\tilde{p} = 1/2$, or

$$p = p_c \equiv \frac{1}{2(1 - \epsilon)}. \tag{25}$$

Thus, for $p > p_c$ the probability to have an infinite avalanche is non-zero, while for $p < p_c$ all avalanches are finite. The value $p = p_c$ corresponds to the critical case where avalanches are power law distributed.

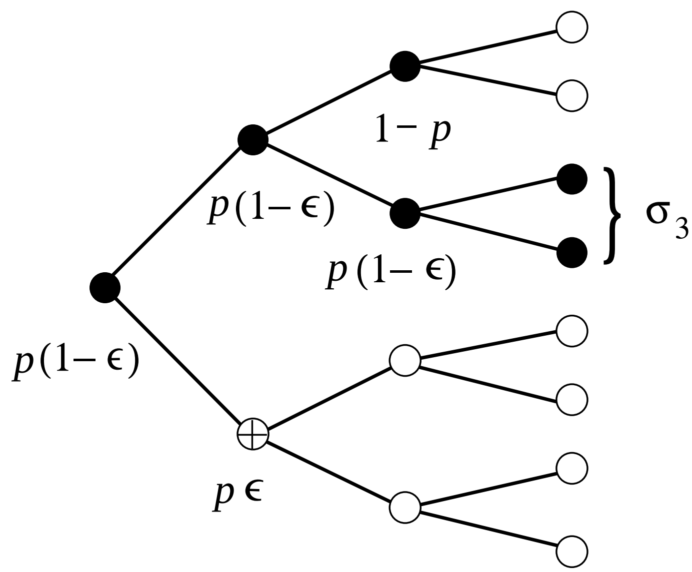

Figure 6: Schematic drawing of an avalanche in a system with a maximum of $n = 3$ avalanche generations corresponding to $N = 2^{n+1} - 1 = 15$ sites. Each black site (•) can relax in three different ways: (i) with probability $p(1-\epsilon)$ to two new black sites, (ii) with probability $1-p$ the avalanche stops, and (iii) with probability $p\epsilon$ two particles are dissipated at a black site, which then becomes a marked site ($\oplus$), and the avalanche stops. The black sites are part of an avalanche of size $s = 6$, whereas the active sites at the boundary yield $\sigma_3(p, t) = 2$. There was one dissipation event such that $\kappa = 2$.

4.1 The properties of the steady state

To address the self-organization, consider the evolution of the total number of particles $M(t)$ in the system after each avalanche:

$$M(t+1) = M(t) + 1 - \sigma(p,t) - \kappa(p,t). \tag{26}$$

Here σ is the number of particles that leave the system from the boundaries and κ is the number of particles lost by dissipation. Since (cf. Sec. 3) $M(t) = NP(z=1) = Np$, we obtain an evolution equation for the parameter p:

$$p(t+1) = p(t) + \frac{1 - \sigma(p,t) - \kappa(p,t)}{N}. \tag{27}$$

This equation reduces to the SOBP model for the case of no dissipation ($\kappa = 0$). In the continuum limit one obtains [30]

$$\frac{dp}{dt} = \frac{1}{N}\left[1 - (2p(1-\epsilon))^n - p\epsilon H(p(1-\epsilon))\right] + \frac{\eta(p,t)}{N}. \tag{28}$$

Here, we defined the function $H(p(1-\epsilon))$, that can be obtained analytically, and introduced the function $\eta(p,t)$ to describe the fluctuations around the average values of σ and κ. It can be shown numerically that the effect of this "noise" term is vanishingly small in the limit $N \to \infty$ [30].

Without the noise term one can study the fixed points of Eq. (28) and one finds that there is only one fixed point,

$$p^* = 1/2, \tag{29}$$

independent of the value of ϵ; the corrections to this value are of the order $O(1/N)$. By linearizing Eq. (28), it follows that the fixed point is attractive. This result implies that the SOBP model self-organizes into a state with $p = p^*$. In Fig. 7 is shown the value of p as a function of time for different values of the dissipation ϵ. We find that independent of the initial conditions after a transient $p(t)$ reaches the self-organized steady-state described by the fixed point value $p^* = 1/2$ and fluctuates around it with short-range correlations (of the order of one time unit). The fluctuations around the critical value decrease with the system size as $1/N$. It follows that in the limit $N \to \infty$ the distribution $\phi(p)$ of p approaches a delta function $\phi(p) \sim \delta(p - p^*)$.

By comparing the fixed point value (29) with the critical value (25), we obtain that in the presence of dissipation ($\epsilon > 0$) the self-organized steady-state of the system is *subcritical*. Fig. 8 is a schematic picture of the phase space of the model, including the line $p = p_c$ of critical behavior (25) and the line $p = p^*$ of fixed points (29). These two lines intersect only for $\epsilon = 0$.

4.2 Avalanche and lifetime distributions

In analogy to the results in Sec. 3, we obtain similar formulas for the avalanche size distributions but with $\tilde{p}$ replacing p. As a result we obtain the distribution

$$D(s) = \sqrt{\frac{2}{\pi}}\,\frac{1+\epsilon+\ldots}{s^{\tau}}\exp\left(-s/s_c(\epsilon)\right). \tag{30}$$

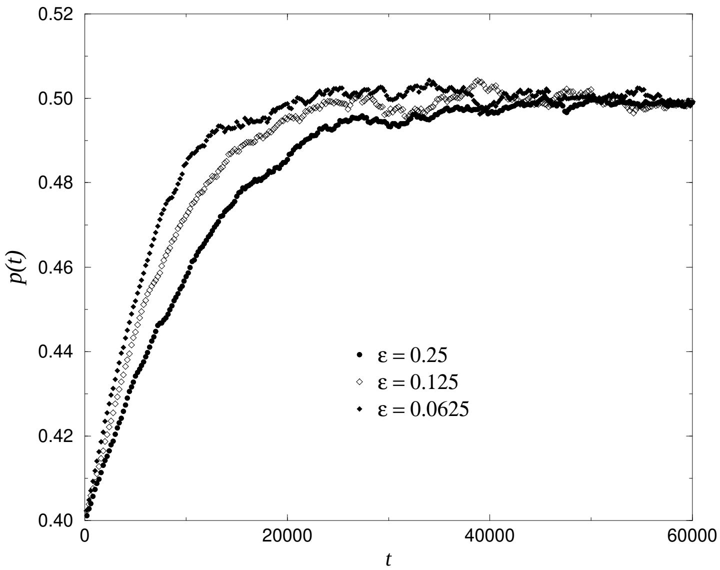

Figure 7: The value of the control parameter $p(t)$ as a function of time for a system with different levels of dissipation. After a transient, $p(t)$ reaches its fixed-point value $p^* = 1/2$ and fluctuates around it with short-range time correlations.

We can expand $s_c(\tilde{p}(\epsilon)) = -2/\ln 4\tilde{p}(1-\tilde{p})$ in ϵ with the result

$$s_c(\epsilon) \sim \frac{2}{\epsilon^{\varphi}}, \qquad \varphi = 2. \tag{31}$$

Furthermore, the mean-field exponent for the critical branching process is obtained setting $\epsilon = 0$, i.e.,

$$\tau = 3/2. \tag{32}$$

These results are in complete agreement with the SOBP model and the simulation of $D(s)$ for the SOBP model with dissipation (cf. Fig. 9). The deviations from the power-law behavior (30) are due to the fact that Eq. (13) is only valid for $1 \ll s \lesssim n$.

Next, consider the avalanche lifetime distribution $D(T)$ characterizing the probability to obtain an avalanche which spans m generations. It can be

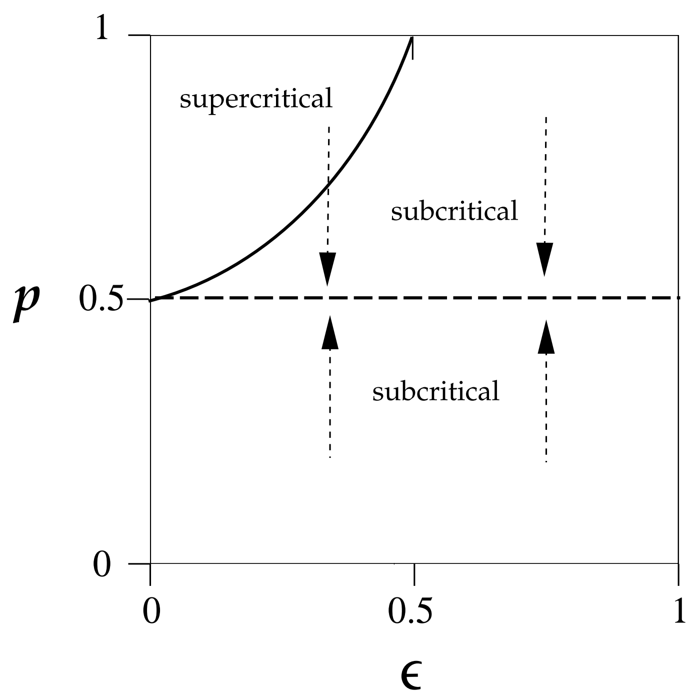

Figure 8: Phase diagram for the SOBP model with dissipation. The dashed line shows the fixed points $p^* = 1/2$ of the dynamics, with the flow being indicated by the arrows. The solid line shows the critical points, cf. Eq. (25).

shown that the result can be expressed in the scaling form [1, 30]

$$D(T) \sim T^{-y} \exp(-T/T_c), \tag{33}$$

where

$$T_c \sim \epsilon^{-\psi}, \qquad \psi = 1. \tag{34}$$

The lifetime exponent y was defined in Eq. (22), wherefrom we confirm the mean-field result

$$y = 2. \tag{35}$$

In Fig. 10, we show the data collapse produced by Eq. (33) for the lifetime distributions for different values of ϵ.

In summary, the effect of dissipation on the dynamics of the sandpile model in the mean-field limit ($d \to \infty$) is described by a branching process.

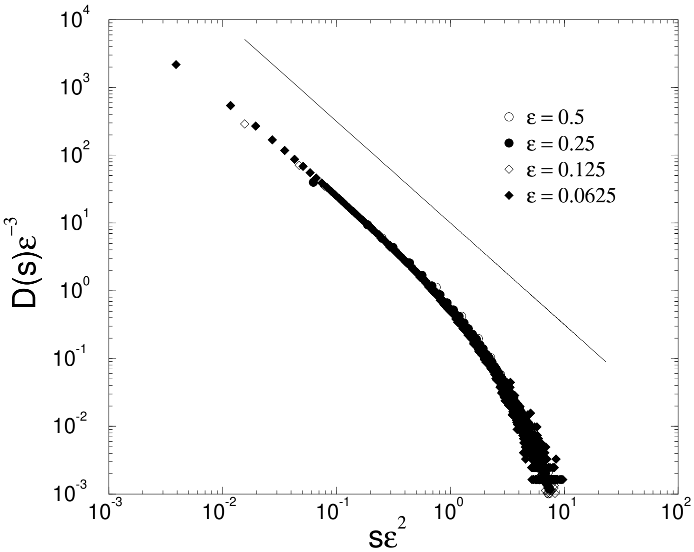

Figure 9: Log-log plot of the avalanche distribution $D(s)$ for different levels of dissipation. A line with slope $\tau = 3/2$ is plotted for reference, and it describes the behavior of the data for intermediate s values, cf. Eq. (30). For large s, the distributions fall off exponentially. The data collapse is produced according to Eq. (30).

The evolution equation for the branching probability has a single attractive fixed point which in the presence of dissipation is not a critical point. The level of dissipation ϵ therefore acts as a relevant parameter for the SOBP model. These results show, in the mean-field limit, that criticality in the sandpile model is lost when dissipation is present.

5 Final remarks

In these notes we have given some ideas of how to understand complexity via the tool of branching processes. The main issue has been that they are an excellent means of understanding "criticality" and "complexity" in many systems. Many other important fields where BP-based ideas find use are not overviewed, from biology and the dynamics of species and molecules to

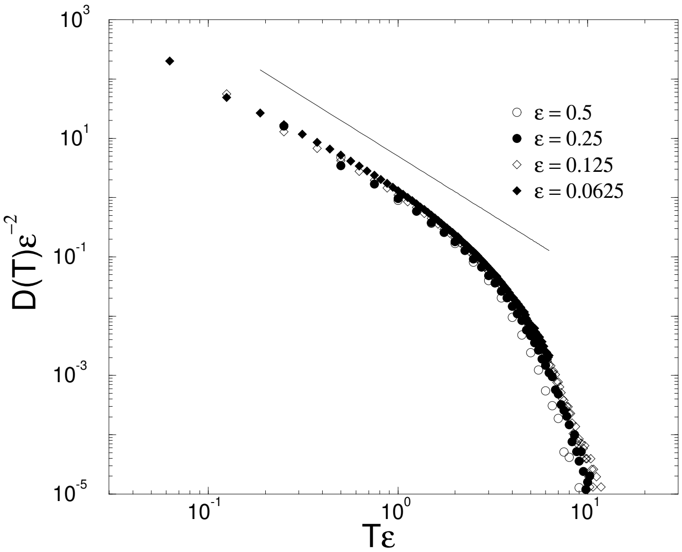

Figure 10: Log-log plot of the lifetime distribution $D(T)$ for different levels of dissipation. A line with slope $y = 2$ is plotted for reference. Note the initial deviations from the power law for $\epsilon = 0$ due to the strong corrections to scaling. The data collapse is produced according to Eq. (33).

geophysics and the spatial and temporal properties of say earthquakes. An example is the so-called ETAS model used for their modelling (see eg. [31]).

The inclusion of spatial effects and temporal memory dynamics is another interesting and important future avenue. Essentially, one searches for complicated variants of usual BP's to be able to model avalanching systems or cases where one wants to compute the typical time to reach an absorbing state, and the related distribution. Or, the question concerns the supercritical state ($\lambda > \lambda_c$) and the spreading from a seed. This can be complicated by the presence of non-Poissonian temporal statistics. Another exciting future task is the description of non-Markovian phenomena, as when for instance the avalanche shape is non-symmetrical [32, 33] This indicates that there is an underlying mechanism which needs to be incorporated into the BP model.

References

[] **Cited Primary Literature**

[1] Harris TE (1989) The Theory of Branching Processes. Dover, New York.

[2] Kimmel M, Axelrod DE (2002) Branching Processes in Biology. Springer, New York.

[3] Athreya KB, Ney PE (2004) Branching Processes. Dover Publications.

[4] Haccou P, Jagers P, Vatutin VA (2005) Branching Processes: Variation, Growth, and Extinction of Populations. Cambridge University Press.

[5] Manna SS (1991) J. Phys. A **24**, L363: In this two-state model, the energy takes the two stable values, $z_i = 0$ (empty) and $z_i = 1$ (particle). When $z_i \geq z_c$, with $z_c = 2$, the site relaxes by distributing two particles to two randomly chosen neighbors.

[6] Dickman R, Alava MJ, Munoz MA, Peltola J, Vespignani A, Zapperi S (2001) Phys. Rev. E64, 056104

[7] Bak P, Tang C, and Wiesenfeld K (1987) Phys. Rev. Lett. **59**, 381; (1988) Phys. Rev. A **38**, 364

[8] Vespignani A, Dickman R, Munoz MA, Zapperi S (2000) Phys. Rev. E 62, 4564

[9] Dickman R, Munoz MA, Vespignani A, Zapperi S (2000) Braz. J. Phys. 30, 27

[10] Alava M (2004) Self-Organized Criticality as a Phase Transition, 45pp, in: Advances in Condensed Matter and Statistical Physics, Eds. E. Korutcheva and R. Cuerno, Nova Publishers.

[11] Lubeck S (2004) Int. J. Mod. Phys. B18, 3977

[12] Jensen HJ (1998) Self-Organized Criticality. Cambridge University Press.

[13] Alava MJ, Nukala PKNN, Zapperi S (2006) Statistical models of fracture, Advances in Physics 55, 349-476

[14] Bonachela JA, Chate H, Dornic I, Munoz MA (2007) Phys. Rev. Lett. 98, 115702

[15] Feller W (1971) An Introduction to Probability Theory and its Applications, Vol. 2, 2nd ed. John-Wiley, New York.

[16] Asmussen S, Hering H (1883) Branching Processes. Birkhäuser, Boston.

[17] Weiss GH (1994) Aspects and Applications of the Random Walk. North-Holland, Amsterdam.

[18] Tang C, Bak P (1988) J. Stat. Phys. **51**, 797

[19] Dhar D, Majumdar SN (1990) J. Phys. A **23**, 4333

[20] Janowsky SA, Laberge CA (1993) J. Phys. A **26**, L973

[21] Flyvbjerg H, Sneppen K, Bak P (1993) Phys. Rev. Lett. **71**, 4087; de Boer J, Derrida B, Flyvbjerg H, Jackson AD, Wettig T (1994) Phys. Rev. Lett. **73**, 906

[22] Alstrøm P (1988) Phys. Rev. A **38**, 4905

[23] Christensen K, Olami Z (1993) Phys. Rev. E **48**, 3361

[24] García-Pelayo R (1994) Phys. Rev. E **49**, 4903

[25] Zapperi S, Lauritsen KB, Stanley HE (1995) Phys. Rev. Lett. 75, 4071

[26] Manna SS, Kiss LB, and Kertész J (1990) J. Stat. Phys. **61**, 923

[27] Tadić B, Nowak U, Usadel KD, Ramaswamy R, and Padlewski S (1992) Phys. Rev. A **45**, 8536

[28] Tadić B, Ramaswamy R (1996) Phys. Rev. E 54, 3157

[29] Vespignani A, Zapperi S, and Pietronero L (1995) Phys. Rev. E **51**, 1711

[30] Lauritsen KB, Zapperi S, Stanley HE (1996) Phys. Rev. E **54**, 2483

[31] Lippidello E, Godano C, De Arcangelis L (2007) Phys. Rev. Lett. **98**, 098501

[32] Zapperi S, Castellano C, Colaiori F, Durin G (2005) Nature Phys. 1, 46.

[33] Laurson L, Illa X, Santucci S, Tallakstad KT, Maloy KJ, Alava MJ (2013) Nature Comm. 4, 2927.

[] **Books and Reviews**

[34] Harris TE (1989) The Theory of Branching Processes. Dover, New York.

[35] Kimmel M, Axelrod DE (2002) Branching Processes in Biology. Springer, New York.

[36] Athreya KB, Ney PE (2004) Branching Processes. Dover Publications.

[37] Haccou P, Jagers P, Vatutin VA (2005) Branching Processes: Variation, Growth, and Extinction of Populations. Cambridge University Press.

[38] Jensen HJ (1998) Self-Organized Criticality. Cambridge University Press.

[39] Feller W (1971) An Introduction to Probability Theory and its Applications, Vol. 2, 2nd ed. John-Wiley, New York.

[40] Asmussen S, Hering H (1883) Branching Processes. Birkhäuser, Boston.

[41] Weiss GH (1994) Aspects and Applications of the Random Walk. North-Holland, Amsterdam.