PHYS-C0252 - Quantum Mechanics

Exercise set 5

Due date : November 27, 2020 before 10.00

1. (a) Derive momentum space representation of the position operator. (Hint: Use the commutation relation $[\hat{x}, \hat{p}] = \mathrm{i}\hbar$ and calculate $p\langle p|\hat{x}|\psi\rangle$ for any arbitrary $|\psi\rangle$)
   (b) Calculate the commutator $[\hat{x}, \hat{p}]$ in the momentum representation and verify that it is equal to $\mathrm{i}\hbar$.

2. Consider the time-independent Schrödinger's equation in position representation for a free particle ($V(x) = 0$).
$$-\frac{\hbar^2}{2m}\frac{d\psi(x)}{dx^2} = E\psi(x)$$
   (a) Find a general solution for $\psi(x)$.

   (b) Show that
$$\psi(x,t) = Ae^{ik(x-\frac{\hbar k}{2m}t)} + Be^{-ik(x+\frac{\hbar k}{2m}t)},$$
   where A, B are constants and $k = \sqrt{2mE}/\hbar$.
   (c) The first term in $\psi(x,t)$ represents a wave traveling to the right, and the second term represents a wave (of the same energy) going to the left. Since they only differ by the sign in front of k, we can write the wave function as
$$\psi_k(x,t) = Ae^{i(kx-\frac{\hbar k^2}{2m}t)}$$
   and k runs negative to cover the case of wave travelling to the left. Find the velocity of this wave and compare with the velocity of classical free particle. Hint : Velocity of the wave $v_{\text{quantum}} = \frac{\hbar|k|}{2m}$.
   (d) The wave function $\psi_k(x,t)$ is not normalizable but the general solution to the time-dependent Schrödinger equation is still a linear combination of separable solutions (it's an integral over the continuous variable k, instead of a sum over the discrete index n). We call it as a wave packet:
$$\psi(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\phi(k)e^{i(kx-\frac{\hbar k^2}{2m}t)}dk,$$
   where $\phi(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\psi(x,0)e^{-ikx}dx$. Show that the new $\psi(x,t)$ is normalizable and find the velocity of the wave packet.

3. An electron is moving freely inside a one-dimensional infinite potential box with walls at $x = 0$ and $x = a$. If the electron is initially in the ground state ($n = 1$) of the box and if we suddenly quadruple the size of the box (i.e.,the right-hand side wall is moved instantaneously from $x = a$ to $x = 4a$), calculate the probability of finding the electron in:
   (a) The ground state of the new box.
   (b) The first excited state of the new box.

4. Consider a particle of mass m in the one-dimensional potential energy field

$$V(x) = \begin{cases} 0, & \text{if } -\infty < x < -a; \\ -V_0, & \text{if } -a < x < +a; \\ 0, & \text{if } +a < x < -\infty. \end{cases}$$

The potential is symmetric about $x = 0$, there are two types of energy eigenfunctions. There are symmetric eigenfunctions which obey

$$\psi(x) = \psi(-x)$$

and anti-symmetric eigenfunctions which obey

$$\psi(x) = -\psi(-x).$$

(a) Show, by considering the energy eigenvalue equation in the three regions of x, that a symmetric eigenfunction with energy $E = -\hbar^2\alpha^2/2m$ has the form:

$$\psi(x) = \begin{cases} Ae^{+\alpha x}, & \text{if } -\infty < x < -a; \\ C\cos(k_0 x), & \text{if } -a < x < +a; \\ Ae^{-\alpha x}, & \text{if } +a < x < -\infty. \end{cases}$$

where A and C are constants and $k_0 = \sqrt{2m(E + V_0)/\hbar^2}$.

(b) Show that $\alpha = k_0 \tan(k_0 a)$.

Hint: use the continuity of $\psi(x)$ and $d\psi(x)/dx$ at the edges of the potential .

(c) By seeking a graphical solutions of the equations

$$\alpha = k_0 \tan(k_0 a) \text{ and } \alpha^2 + k_0^2 = w^2,$$

where $w = \sqrt{2mV_0/\hbar^2}$, show that there is one bound state if,

$$0 < w < \frac{\pi}{2a},$$

and two bound states if

$$\frac{\pi}{2a} < w < \frac{3\pi}{2a}.$$

(d) Find bound state energies for the anti-symmetric eigenfunction.