PHYS-C0252 - Quantum Mechanics

Exercise set 6

Due date : December 4, 2020 before 10.00

1. Consider a pair of identical particles. We can specify a two -particle state by

$$|a, b\rangle = |a\rangle_1 \otimes |b\rangle_2$$

where a single-particle state such as $|a\rangle_1$ specifies the state of particle 1 and $|b\rangle_2$ specifies the state of particle 2. We introduce the exchange operator $\hat{P}_{12}$, which is defined by

$$\hat{P}_{12}|a, b\rangle = |b, a\rangle$$

The "exchanged" state must be the same physical state and therefore can differ from the initial state by at most an overall phase.
(a) Show that $P_{12}^2 = \hat{I}$
(b) Find eigenvalues of exchange operator $\hat{P}_{12}$
(c) Find the eigenstates of exchange operator $\hat{P}_{12}$ and show that these two eigenstates are symmetric and anti-symmetric.
(d) Show that two identical particles must be in either the symmetric state or in the anti-symmetric state, but they cannot be in a superposition of these states.
(Hint: Check the action of $\hat{P}_{12}$ on a superposition state)

2. Consider a charged particle in the one dimensional harmonic oscillator potential.

$$V(\hat{x}) = \frac{1}{2} m\omega^2 \hat{x}^2$$

Suppose we turn on a weak electric field (ε) , so that the potential energy is shifted by an amount $H' = -q\varepsilon x$
(a) Using the perturbation theory approximation, show that there is no first order change in the energy level and calculate the second order correction.
(b) The Schrödinger equation can be solved directly in this case, by a change of variable $x' = x - \frac{qE}{m\omega^2}$. Find the exact energies and show that they are consistent with the perturbation theory approximation.

3. Consider the three dimensional infinite cubical well

$$V(x) = \begin{cases} 0, & \text{if } 0 < x < a,\ 0 < y < a,\ 0 < z < a\ ; \\ \infty, & \text{otherwise.} \end{cases}$$

The stationary states are

$$\psi_{n_x, n_y n_z}(x, y, z) = \left(\frac{2}{a}\right)^{3/2} \sin\left(\frac{n_x \pi}{a} x\right) \sin\left(\frac{n_y \pi}{a} y\right)\ \sin\left(\frac{n_z \pi}{a} z\right),$$

where n_x, n_y and n_z are positive integers. The corresponding allowed energies are

$$E = \frac{\pi^2 \hbar^2}{2ma^2}(n_x^2 + n_y^2 + n_z^2).$$

The ground state $\psi_{1,1,1}$ is non degenerate with energy

$$E_0 = \frac{3\pi^2\hbar^2}{2ma^2}.$$

The first excited state is triple degenerate:

$$\psi_a = \psi_{112}, \quad \psi_b = \psi_{121}, \text{ and } \psi_c = \psi_{211}.$$

All share same energy

$$E_0 = \frac{3\pi^2\hbar^2}{ma^2}.$$

Suppose we introduce a perturbation

$$H' = \begin{cases} V_0, & \text{if } 0 < x < a/2,\, 0 < y < a/2,\, 0 < z < a\ ; \\ \infty, & \text{otherwise.} \end{cases}$$

(a) Show that the first-order correction to the ground state is $E_0^1 = V_0/4$.
(b) Find the first-order corrections to the first excited state using the degenerate perturbation theory.

4 Consider a quantum state characterized by the density matrix (operator)

$$\hat{\rho}(t) = |\psi\rangle\langle\psi|$$

The state vector $|\psi\rangle$ evolves in time according to the schödinger equation

$$i\hbar\frac{d|\psi(t)\rangle}{dt} = \hat{H}(t)|\psi(t)\rangle.$$

Show that time evolution of the density matrix follows the Von Neumann equation

$$\frac{d\hat{\rho}(t)}{dt} = \frac{-i}{\hbar}[\hat{H}(t), \hat{\rho}(t)].$$