31E99906 Microeconomic policy

Lecture 11: Uncertainty

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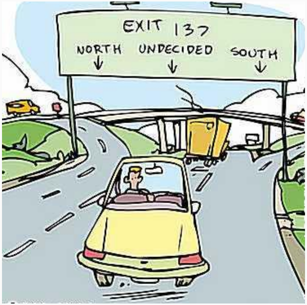
Fall 2020

Choice under uncertainty



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Purpose

We want to develop a framework for thinking choices under uncertainty. Helps us to

- make investment decisions, both public and private
- conduct sensitivity analysis
- discuss risk and the sources of caution in decision making: WTP pay for risk elimination
- identify the value of acquiring information

What is uncertainty? Risk?

Classical:

- Probability of an event = long-run frequency of occurrence of event in a sequence of independent experiments)
- Coin toss results in Heads
- analysis consists of thinking uncertainty as contingencies with specific probabilities of occurrence. Risk=uncertainty.

more subjective:

- What is the probability that a pin ends up the sharp side up?
- analysis consists of thinking uncertainty as contingencies with probabilities of occurrence that must be updated with experience. Probabilities subjective=uncertainty

Think about policy making – which approach to uncertainty is more relevant?

Simple gamble

Let G denote a gamble
Each gamble results in one outcome from a finite set of possible outcomes $A = \{a_1, a_2, \ldots, a_n\}$

- outcomes are deterministic objects: e.g., number hospitalized patients, accidents, income levels, etc.
- A simple gamble is a probability distribution on A
- $p_i = \Pr\{a = a_i\}$ for $i \in \{1, 2, ..., n\}$
- $p_i \geq 0$ for all i and $\sum_i p_i = 1$

Expected utility

- Let $u(a_i)$, $i = 1, 2, .., n$ be the utility when outcome is i
- The expected utility from gamble G is then

$$U(G) = \sum_{i=1}^{n} p_i u\left(a_i\right) \tag{1}$$

  - U denotes total utility from gamble G, u is the utility from an outcome
  - evaluating utility of each possible outcome, taking weighted average of those utilities, weights given by probabilities

G_1 could be such that outcome $i = 1$ arises for sure, with probability one. G_2 could be another gamble such that $i = 2$ arises with probability one, and so on. What is the expected utility of $p_1 G_1 + ... + p_n G_n$? It is $U(p_1 G_1 + ... + p_n G_n) = \sum_{i=1}^{n} p_i u\left(a_i\right)$. The key property of expected utility is that it is linear in probabilities.

Why important?

Consider the following gambles:

- receive income w for sure:

$$U(G') = u(w)$$

- zero mean risk added to sure income w:

$$U(G'') = Prob_{win} \cdot u(w + x) + Prob_{lose} \cdot u(w - x)$$

  where $0 = Prob_{win} \cdot x - Prob_{lose} \cdot x$

- Most decision makers would not like to face such fair risk. Often people do not even like actuarially favorable risk, $0 < Prob_{win} \cdot x - Prob_{lose} \cdot x$, and prefer G' over G''.

The theory helps us to separate the effect of (i) risk and (ii) risk attitudes on choices, and to answer the question how much you are willing to pay to eliminate the risk?

How much are you willing to pay to eliminate the risk?

Continue with the previous page example and set $w = 1$. Consider a gamble where the decision maker receives 1 with probality 1/2 and loses 1 with probability 1/2. What is the sure amount of income that you would be willing to accept instead of entering the gamble? This CERTAINTY EQUIVALENT (CE) for the gamble defines also RISK PREMIUM (RP) as the amount of money that the decision maker is willing to pay to escape the risk.

1. calculate expected utility from the gamble, U(G)
2. find CE: $u(CE) = U(G)$
3. RP is the difference between the expected income and CE.

We illustrate using three attitudes towards risk: (i) $u(w) = w$, (ii) $u(w) = w^2$, (iii) $u(w) = w^{1/2}$

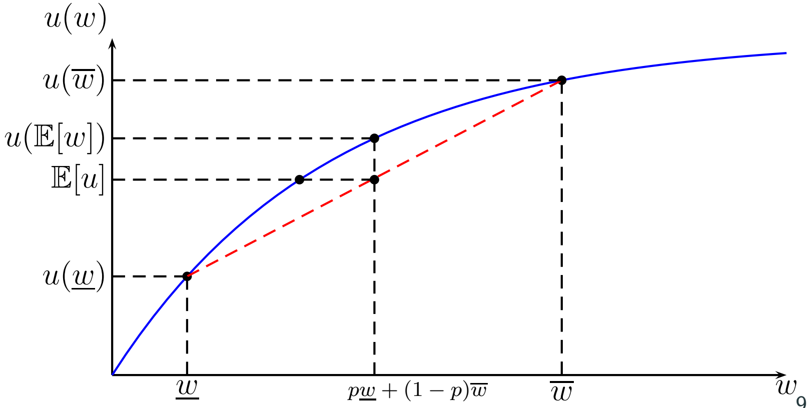
How much are you willing to pay to eliminate the risk?

- $\mathrm{u(w)} = \mathrm{w}$:
  $U(G) = \frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 2 = 1 \Rightarrow u(CE) = 1, CE = 1$
- $\mathrm{u(w)} = \mathrm{w}^2$:
  $U(G) = \frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 2^2 = 2 \Rightarrow u(CE) = 2, CE = 1.41$
- $\mathrm{u(w)} = \mathrm{w}^{1/2}$:
  $U(G) = \frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 2^{1/2} = .71 \Rightarrow u(CE) = .71, CE = .5$

- $\mathrm{u(w)} = \mathrm{w}$: risk neutral individual, CE=expected income
- $\mathrm{u(w)} = \mathrm{w}^2$: risk loving, CE exceeds expected income
- $\mathrm{u(w)} = \mathrm{w}^{1/2}$: risk averse, CE less than expected income

Illustration of risk aversion

Prior to knowing the outcome, how much are you willing to pay for a policy that eliminates the risk? Where is CE in this figure?



Trading risk: differences in capacity to bear risks

Consider 50% risk of loosing 100, and two individuals with wealth levels 500 and 10000. Both have utility function $u = ln(w)$.

- Individual 1 has expected utility of

  $$.5u(500 - 100) + .5u(500) = 6.103$$

  CE is $exp(6.103) = 447.214$ so this individual is willing to give away 2.786 units of money to avoid the risk (450-447.214).

- Individual 2 has expected utility of

  $$.5u(10000 - 100) + .5u(10000) = 9.205$$

  CE is $exp(9.205) = 9949.874$ so this individual is willing to give away only .126 units of money to avoid the risk.

Can you see how the individuals can trade to share the risk?

Is it crazy to love risk?

Not necessarily. Think of some desirable payoff $a > 0$ and its utility $u(a)$. Uncertainty about the timing of the payoff: either $t = 5$ weeks or $t = 15$ weeks. Let discount factor be $\beta = \frac{1}{1+r}$ with interest rate $r > 0$.

- What is the gamble?
- What is the expected utility?
- The consumer rather takes the risky waiting time than the average but sure waiting time – Why?

Trading risk: pooling risks that can be diversified

- The key feature of diversification: part of individuals' risks must independent. The risk of my housing burning down is independent of the same event in your case. If the probability is p for each house to burn down, by the law of large numbers, the share of houses burning down becomes predictable in a large population.
- We have seen in the case of health insurance (first reading) that risk averse individuals individuals are willing to buy full insurance, the risk can be fully diversified, and the price is actuarially fair. Let us see the meaning of this precisely. Agent has initial wealth w but runs a risk of losing $d > 0$ with probability p. Insurance costs q euros per one euro loss covered. When α units of coverage bought, expected wealth is

$$p(w - d - \alpha q + \alpha) + (1 - p)(w - \alpha q) = w - pd + \alpha(p - q)$$

Trading risk: pooling risks that can be diversified

- To illustrate: assume wealth 500, loss 100 with 50% risk, coverage bought 100, and the price is 60 cents per one euro loss,

  $$w - pd + \alpha(p - q) = 500 - 50 + 100(50 - 60) = 350$$

  Is this plan actuarially fair?
- The consumer optimally chooses coverage to maximize the expected utility:

  $$Max_{\alpha \geq 0}\{p \cdot u(w - d - \alpha q + \alpha) + (1 - p) \cdot u(w - \alpha q)\}$$

Optimal $\alpha^* > 0$ must satisfy:

$$-(1-p)qu'(w-\alpha q) + p(1-q)u'(w-d-\alpha q+\alpha) = 0$$
$$\Rightarrow \frac{p \cdot u'(w-d-\alpha q+\alpha)}{(1-p) \cdot u'(w-\alpha q)} = \frac{q}{1-q}$$

- If $q = p$, the price of insurance is actuarially fair and, then $\alpha^* = d$. That is, the agent insures fully: the marginal utilities are the same with and without the loss.
- If $q > p$, the agent will bear some risk. Why is it that the insurance market does not often diversify all risks?

Choosing risk exposure

Here we want to show that risk aversion does mean full avoidance of risks. Assume initial wealth $w > 0$ to be allocated between a risk-free and risky asset. Let $(w - \alpha, \alpha)$ denote the portfolio where α is the amount invested in the risky asset. Risk-free rate of return is $r > 0$ and risky rate of return is a random variable $\tilde{r}$. Utility function depending on final wealth $u(w)$ is differentiable and strictly concave.

Value after the realization

$$(w - \alpha)(1 + r) + \alpha(1 + \tilde{r}) = w_0 + \alpha\tilde{x}$$

where $w_0 = w(1 + r)$, $\tilde{x} = \tilde{r} - r$ (excess return). Expected utility is

$$Eu(w_0 + \alpha\tilde{x})$$

Optimal portfolio follows simply from

$$\max_{\alpha} V(\alpha) = Eu(w_0 + \alpha\tilde{x})$$

First-order condition

$$V'(\alpha^*) = E\tilde{x}u'(w_0 + \alpha^*\tilde{x}) = 0$$

Note that if $E\tilde{x} > 0$

$$V'(0) = E\tilde{x}u'(w_0) > 0.$$

- Risk-averse investor invests some fraction in the risky asset as long as the the excess return is positive

Basic concepts summary

It is easy to get lost. To be sure, consider consumer facing a wealth risk.

Add pure risk $\tilde{x}$ to sure wealth w_0 such that the final wealth is $w_0 + \tilde{x}$ with $E\tilde{x} = 0$. Then, for risk averse person

$$u(w_0) \geq Eu(w_0 + \tilde{x}).$$

How much would a risk-averse person pay to escape pure risk $\tilde{x}$? Find RP such that

$$Eu(w_0 + \tilde{x}) = u(w_0 - RP)$$

Value RP is the risk premium. Note that RP depends on wealth level w_0, risk attitude u, and risk $\tilde{x}$.

Using this we can compare risk attitudes

- Agent 1 is more risk averse than agent 2 if and only if $RP_1 \geq RP_2$ for all w_0 and all pure risks $\tilde{x}$.

Let us now consider risks that are not pure: define $\tilde{y} = \mu + \tilde{x}$ with $E\tilde{x} = 0$ and $\mu > 0$. There is thus some expected gain in this gamble. Certainty equivalent CE is the value of such gamble:

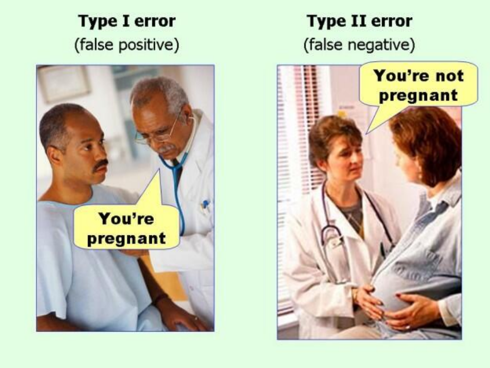
$$Eu(w_0 + \tilde{y}) = u(w_0 + CE),$$

that is, it is the sure amount CE added to your wealth that makes you indifferent taking that CE or the gamble. Comparing the definitions of RP and CE,

$$CE = \mu - RP,$$

thus the value of the gamble is the expected gain minus the risk premium.

The value of information



The value of more precise information

One action, three outcomes for costs. The expected cost:

$$v = \min_a \{p_1 C_1(a) + p_2 C_2(a) + p_3 C_3(a)\}$$

Suppose it is possible to conduct a study that tells surely whether "1" will happen.Two actions, three outcomes for costs:

$$V = P \min_a \{C_1(a)\} + (1 - P) \min_a \{pC_2(a) + (1 - p) C_3(a)\}$$

The action can be better targeted to the outcome if the decision maker has more information, for example, through testing of patients. The value of information is $V - v$, and thus determines the willingness to pay for the study. Note that if the test is imprecise, the value of information will be affected