Taloustieteen matemaattiset menetelmät
31C01100
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Problem Set 8: Solutions

1. **Solution**

(a) By separating variables,

$$ydy = tdt.$$

Integrating both sides

$$\int ydy = \int tdt$$

and then evaluating the integrals yields

$$y^2 = t^2 + 2C.$$

At the initial condition, it must hold that

$$1 = 2 + 2C,$$

from which we easily get $C = -\frac{1}{2}$. Thus the unique solution of the IVP is

$$y(t) = \sqrt{t^2 - 1}.$$

Note that $y(t) = -\sqrt{t^2 - 1}$ is not a solution, as it does not satisfy the initial condition $y(\sqrt{2}) = 1$.

(b) By separating variables,

$$y^2 dy = (t + 1)dt.$$

Integrating both sides

$$\int y^2 dy = \int (t + 1)dt$$

and then evaluating the integrals yields

$$y^3 = \frac{3}{2}t^2 + 3t + 3C.$$

At the initial condition, it must hold that

$$1 = \frac{3}{2} + 3 + 3C,$$

from which we obtain $C = -\frac{7}{6}$. Thus the unique solution of the IVP is

$$y(t) = \sqrt[3]{\frac{3}{2}t^2 + 3t - \frac{7}{2}}.$$

(c) By separating variables,

$$\frac{1}{y^3}dy = \frac{1}{t^3}dt.$$

Integrating both sides and then evaluating the integrals yields

$$-\frac{1}{2}y^{-2} = -\frac{1}{2}t^{-2} + C,$$

so

$$y^{-2} = t^{-2} - 2C.$$

At the initial condition $y(1) = 1$ it must hold that

$$1 = 1 - 2C,$$

from which we obtain $C = 0$. Thus the unique solution of the IVP is $y(t) = t$.

(d) By separating variables,

$$y^3 dy = t^3 dt.$$

Integrating both sides and then evaluating the integrals yields

$$\frac{1}{4}y^4 = \frac{1}{4}t^4 + C,$$

so

$$y^4 = t^4 + 4C.$$

At the initial condition $y(1) = 1$ it must hold that

$$1 = 1 + 4C,$$

from which we obtain $C = 0$. Thus the unique solution of the IVP is $y(t) = t$.

2. **Solution**

(a) For an equilibrium point it holds that $\dot{y} = y^2 - y = y(y-1) = 0$. Thus, the equilibrium points are $y_1^* = 0$ and $y_2^* = 1$. The derivative of $f(y)$ is

$$f'(y) = \frac{(2y-1)(y^2+1) - (y^2-y)\cdot 2y}{(y^2+1)^2} = \frac{y^2+2y-1}{(y^2+1)^2}.$$

At $y_1^* = 0$, $f'(0) = -1 < 0$, so $y_1^* = 0$ is asymptotically stable. At $y_2^* = 1$, $f'(1) = \frac{1}{2} > 0$, so it is unstable.

(b) Since $e^y \neq 0$, it must be that $\sin y = 0$ when $\dot{y} = 0$. We know that $\sin y = 0$ if $y = ... - 2\pi, -\pi, 0, \pi, 2\pi....$ Thus, the equilibrium points are $y^* = \pm n\pi$, where n is some integer. The derivative of $f(y)$ is

$$f'(y) = e^y \cdot \sin y + e^y \cdot \cos y = e^y(\sin y + \cos y).$$

Let m be some odd integer. By inserting the equilibrium points to the derivative we get

$$f'(m\pi) = e^{m\pi}(\sin m\pi + \cos m\pi) = e^{m\pi} \cdot \cos m\pi = -e^{m\pi} < 0,$$

$$f'(2m\pi) = e^{2m\pi}(\sin 2m\pi + \cos 2m\pi) = e^{2m\pi} \cdot \cos 2m\pi = e^{2m\pi} > 0,$$

$$f'(0) = e^0(\sin 0 + \cos 0) = e^0 \cdot \cos 0 = e^0 = 1 > 0.$$

We see that when m is some odd integer, $m\pi$ is asymptotically stable and the two other equilibrium points, $2m\pi$ and 0, are unstable.

(c) The only equilibrium point is $y^* = 0$. The derivative of $f(y) = \frac{y}{y^2+1}$ at $y^* = 0$ is $f'(0) = 1 > 0$. Hence the equilibrium is unstable.

(d) There are two equilibrium points, $y_1^* = 0$ and $y_2^* = 1$. The derivative of $f(y) = y^2 - y^3$ at $y_1^* = 0$ is $f'(0) = 0$. However, when y is sufficiently close to zero, $f(y)$ is always positive. Hence we can conclude that $y_1^* = 0$ is unstable. The derivative of $f(y)$ at $y_2^* = 1$ is $f'(1) = -1 < 0$, hence y_2^* is locally asymptotically stable.

3. **Solution**

(a) The characteristic equation is $r^2 - 3 = 0$, with roots $r_1 = -\sqrt{3}$ and $r_2 = \sqrt{3}$. The general solution is

$$y(t) = C_1 e^{-\sqrt{3}t} + C_2 e^{\sqrt{3}t}.$$

(b) The characteristic equation is $r^2 + 4r + 8 = 0$, with roots $r_1 = -2 + 2i$ and $r_2 = -2 - 2i$. The general solution is

$$y(t) = e^{-2t}\left(C_1 \cos 2t + C_2 \sin 2t\right).$$

(c) The characteristic equation is $3r^2 + 8r = 0$, with roots $r_1 = 0$ and $r_2 = -\frac{8}{3}$. The general solution is

$$y(t) = C_1 + C_2 e^{-\frac{8}{3}t}.$$

(d) The characteristic equation is $4r^2 + 4r + 1 = 0$, whose only root $r = -\frac{1}{2}$ has multiplicity 2. The general solution is

$$y(t) = (C_1 + C_2 t)e^{-\frac{1}{2}t}.$$

4. **Solution**

The candidate solution $y(t) = u(t)e^{rt}$ is such that:

$$y = ue^{rt} \tag{1}$$
$$\dot{y} = \dot{u}e^{rt} + rue^{rt} = \dot{u}e^{rt} - \frac{b}{2}ue^{rt} \tag{2}$$
$$\ddot{y} = \ddot{u}e^{rt} + 2r\dot{u}e^{rt} + r^2ue^{rt} = \ddot{u}e^{rt} - b\dot{u}e^{rt} + \frac{b^2}{4}ue^{rt}. \tag{3}$$

Inserting (1)–(3) into the differential equation and rearranging yields

$$\ddot{u}e^{rt} + u(t)e^{rt}\left[-\frac{b^2}{4} + c\right] = 0. \tag{4}$$

Since $\frac{1}{4}b^2 = c$ by assumption, and since e^{rt} is always strictly positive, we have that equation (4) holds if and only if $\ddot{u} = 0$ for all t. This means that $\dot{u}$ must be constant and, consequently, u must be some affine function $u(t) = C_1 + C_2t$. Thus, $y(t) = u(t)e^{rt} = (C_1 + C_2t)e^{rt}$.

5. **Solution**

(a) The system of differential equations can be written in matrix form:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -12 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

The characteristic polynomial, where $A = \begin{pmatrix} 2 & 1 \\ -12 & -5 \end{pmatrix}$, is

$$\begin{aligned} \det(A - rI) &= \begin{vmatrix} 2-r & 1 \\ -12 & -5-r \end{vmatrix} = (2-r)(-5-r) - (-12) \\ &= r^2 + 3r + 2 = (r+2)(r+1) = 0. \end{aligned}$$

Thus, the eigenvalues are $r_1 = -1$ and $r_2 = -2$. For eigenvalues it holds that $(A - r_iI)v_i = 0$, where v_i is the eigenvector corresponding the eigenvalue r_i and $i = 1, 2$. Thus,

$$\begin{pmatrix} 2-(-1) & 1 \\ -12 & -5-(-1) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ -12 & -4 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

So the first eigenvector is $v_{r_1} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$. The second eigenvector can be solved in a similar way, and it is $v_{r_2} = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$.

Thus, the general solution is

$$\begin{pmatrix} x \\ y \end{pmatrix} = C_1 e^{-t} \begin{pmatrix} -1 \\ 3 \end{pmatrix} + C_2 e^{-2t} \begin{pmatrix} 1 \\ -4 \end{pmatrix}.$$

(b)

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -12 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

The eigenvalues and the eigenvectors can be solved as in (a): $r_1 = 0$ and $r_2 = 7$, and the corresponding eigenvectors are $v_{r_1} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $v_{r_1} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$. The general solution is

$$\begin{pmatrix} x \\ y \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 e^{7t} \begin{pmatrix} 3 \\ -1 \end{pmatrix}.$$

(c)

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

The eigenvalues and the eigenvectors can be solved as in (a): $r_1 = 5$ and $r_2 = -2$, and the corresponding eigenvectors are $v_{r_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $v_{r_1} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$. The general solution is

$$\begin{pmatrix} x \\ y \end{pmatrix} = C_1 e^{5t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{-2t} \begin{pmatrix} 4 \\ -3 \end{pmatrix}.$$