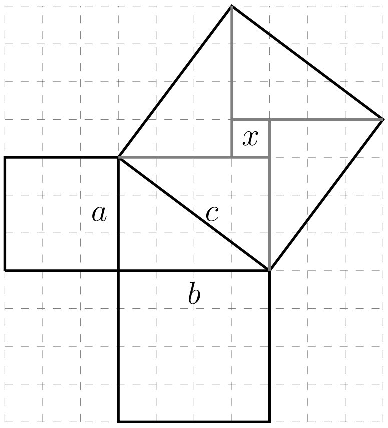


$c^2 = x^2 + 4\frac{ab}{2} = ...$

Figure 1: Geometry of Pythagoras' equation $a^2 + b^2 = c^2$, with $x = |a - b|$.

MS-A000* Matrix algebra
(4.10.2020 Ville Turunen, Aalto University)

Welcome to the lecture course on linear algebra! We shall treat vectors and matrices in Euclidean spaces, the spectral theory (eigenvalue decompositions) of normal matrices, finally reaching the so-called SVD (Singular Value Decomposition). Such vector issues will be found everywhere in advanced mathematics and its applications in engineering and sciences.

Feedback is welcome: please contact `ville.turunen@aalto.fi`

0. It's all about geometry and algebra

Vectors are best understood when illustrating the algebraic manipulations with schematic pictures. For the best learning outcomes, the reader of these notes should draw his/her own mathematical pictures next to the calculations. For example, see the picture visualizing Pythagoras' equation. There, by the areas of right-angled triangles and squares,

$$c^2 = |a - b|^2 + 4\frac{ab}{2} = (a^2 + b^2 - 2ab) + 2ab = a^2 + b^2.$$

Thus we get *Pythagoras' equation*

$$a^2 + b^2 = c^2. \tag{1}$$

Geometry of vectors is based on this! Actually, we may say without exaggeration that high-dimensional (even infinite-dimensional) calculations can often be correctly illustrated by just two-dimensional pictures!

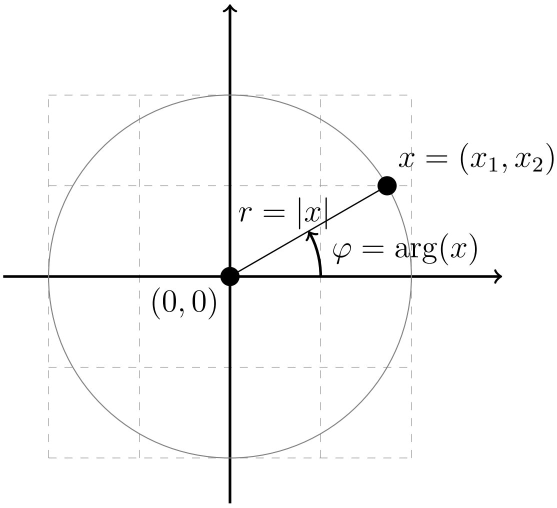


Figure 2: Polar coordinates (r, φ) for point $x = (x_1, x_2) \in \mathbb{R}^2$.

1. Real plane and complex numbers

Definition. Let $\mathbb{R}$ be the *real line*. *Real plane* $\mathbb{R}^2$ consists of *points*

$$x = (x_1, x_2),$$

where $x_1, x_2 \in \mathbb{R}$ are the *(Cartesian) coordinates* of $x \in \mathbb{R}^2$.

Definition. The *polar coordinates*

$$(r, \varphi) = (|x|, \arg(x))$$

of point $x = (x_1, x_2) \in \mathbb{R}^2$ satisfy

$$\begin{aligned} r &:= \sqrt{x_1^2 + x_2^2}; \\ \tan(\varphi) &= \frac{x_2}{x_1} \qquad \text{if} \quad x_1 \neq 0. \end{aligned}$$

Here $r \geq 0$ is distance of x from origin, and the argument $\varphi = \arg(x) \in \mathbb{R}$ is the angle between positive horizontal axis and interval from O to x.

Remark: the argument φ is identified with $\varphi + 2\pi$ here.

Example. If $x = (\sqrt{3}, 1)$, then $x = (2\cos(\pi/6), 2\sin(\pi/6))$, since

$$\begin{cases} \|x\| = \sqrt{\sqrt{3}^2 + 1^2} = 2, \\ \arg(x) = \arctan(1/\sqrt{3}) = \pi/6. \end{cases}$$

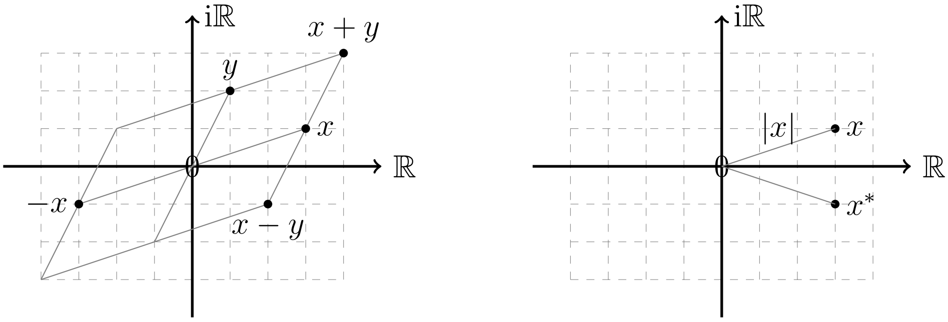


Figure 3: Arithmetic with complex numbers.

Complex numbers

Definition. Points $x = (x_1, x_2) \in \mathbb{R}^2$ can be thought as *complex numbers*

$$x = x_1 + \mathrm{i}x_2 \in \mathbb{C},$$

where $\mathrm{Re}(x) := x_1 \in \mathbb{R}$ is the *real part* and $\mathrm{Im}(x) := x_2 \in \mathbb{R}$ is the *imaginary part*, and $\mathrm{i} := 0 + \mathrm{i}1$ is the *imaginary unit*, and $t \in \mathbb{R}$ is identified with

$$t + \mathrm{i}0 \in \mathbb{C}.$$

If $x, y \in \mathbb{C}$, there are familiar formulas:

$$\begin{aligned} x + y &:= (x_1 + y_1) + \mathrm{i}(x_2 + y_2), \\ x - y &:= (x_1 - y_1) + \mathrm{i}(x_2 - y_2), \\ -x &:= (-x_1) + \mathrm{i}(-x_2). \end{aligned}$$

Complex conjugate of $x = x_1 + \mathrm{i}x_2 \in \mathbb{C}$ is

$$\overline{x} = x^* := x_1 - \mathrm{i}x_2 \quad \in \quad \mathbb{C}.$$

Absolute value of $x \in \mathbb{C}$ is (thinking of Pythagoras...)

$$|x| := \sqrt{x_1^2 + x_2^2}.$$

Example. If $x = 3 + \mathrm{i}$ and $y = 1 + 2\mathrm{i}$ then

$$\begin{aligned} x + y &= 4 + 3\mathrm{i}, \\ x - y &= 2 - \mathrm{i}, \\ x^* &= 3 - \mathrm{i}, \\ |x| &= \sqrt{3^2 + 1^2} = \sqrt{10}. \end{aligned}$$

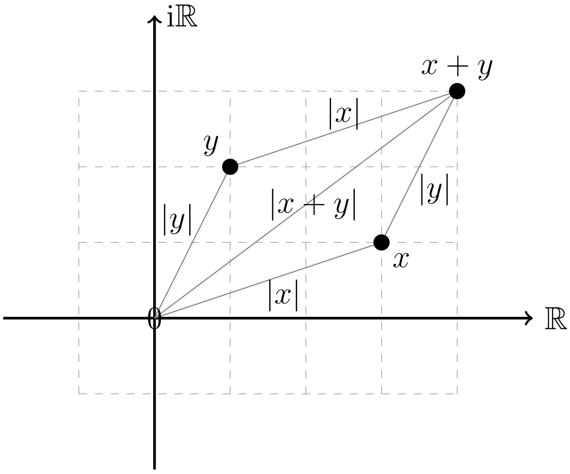


Figure 4: Triangle Inequality of complex numbers.

Triangle Inequality in $\mathbb{C}$

For $x, y \in \mathbb{C}$, there is the *triangle Inequality*

$$|x+y| \leq |x| + |y|. \tag{2}$$

Let's verify this. First, calculating

$$\begin{aligned}
& (|x| + |y|)^2 - |x+y|^2 \\
= \;& (|x| + |y|)^2 - |(x_1 + y_1) + \mathrm{i}(y_1 + y_2)|^2 \\
= \;& \left(|x|^2 + 2\,|x|\,|y| + |y|^2\right) - \left((x_1 + y_1)^2 + (x_2 + y_2)^2\right) \\
= \;& 2\,|x|\,|y| - 2\,(x_1\,y_1 + x_2\,y_2)\,,
\end{aligned}$$

we need to show that $|x|\,|y| \geq x_1\,y_1 + x_2\,y_2$. Here

$$\begin{aligned}
& |x|^2\,|y|^2 - (x_1\,y_1 + x_2\,y_2)^2 \\
= \;& (x_1^2 + x_2^2)(y_1^2 + y_2^2) - (x_1^2\,y_1^2 + 2x_1\,x_2\,y_1\,y_2 + x_2^2\,y_2^2) \\
= \;& x_1^2\,y_2^2 + x_2^2\,y_1^2 - 2\,x_1\,x_2\,y_1\,y_2 \\
= \;& (x_1\,y_2 - x_2\,y_1)^2 \quad \geq \quad 0.
\end{aligned}$$

This proved the Triangle Inequality (2).

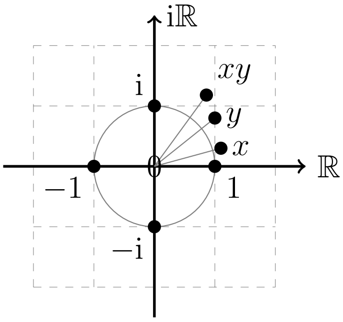


Figure 5: Product of complex numbers.

Product of complex numbers

Product $xy \in \mathbb{C}$ of $x, y \in \mathbb{C}$ has the natural polar coordinate definition:

$$\begin{cases} |xy| = |x||y|, \\ \arg(xy) = \arg(x) + \arg(y). \end{cases} \tag{3}$$

Equivalently, in the Cartesian coordinates

$$xy := (x_1 y_1 - x_2 y_2) + \mathrm{i}(x_1 y_2 + x_2 y_1) \in \mathbb{C}. \tag{4}$$

Especially,

$$\mathrm{i}^2 = (0 + 1\mathrm{i})(0 + 1\mathrm{i}) = (0 - 1) + \mathrm{i}(0 + 0) = -1.$$

Equivalently,

$$\begin{array}{rcccccl} |\mathrm{i}|^2 & = & |\mathrm{i}||\mathrm{i}| & = & 1, & & \text{and} \\ \arg(i^2) & = & 2\arg(\mathrm{i}) & = & 2\pi/2 & = & \pi. \end{array}$$

Example. If $x = \frac{4}{5} + \frac{3}{5}\mathrm{i}$, $y = \frac{12}{13} + \frac{5}{13}\mathrm{i}$, then

$$xy = \frac{(48 - 15) + \mathrm{i}(20 + 36)}{65} = \frac{33}{65} + \frac{56}{65}\mathrm{i}.$$

By the way, here $|x| = |y| = 1 = |xy|$.

Remark. Always

$$\begin{aligned} x^* x &= (x_1 - \mathrm{i}x_2)(x_1 + \mathrm{i}x_2) \\ &= x_1^2 + x_2^2 \\ &= |x|^2 \quad \geq \quad 0. \end{aligned}$$

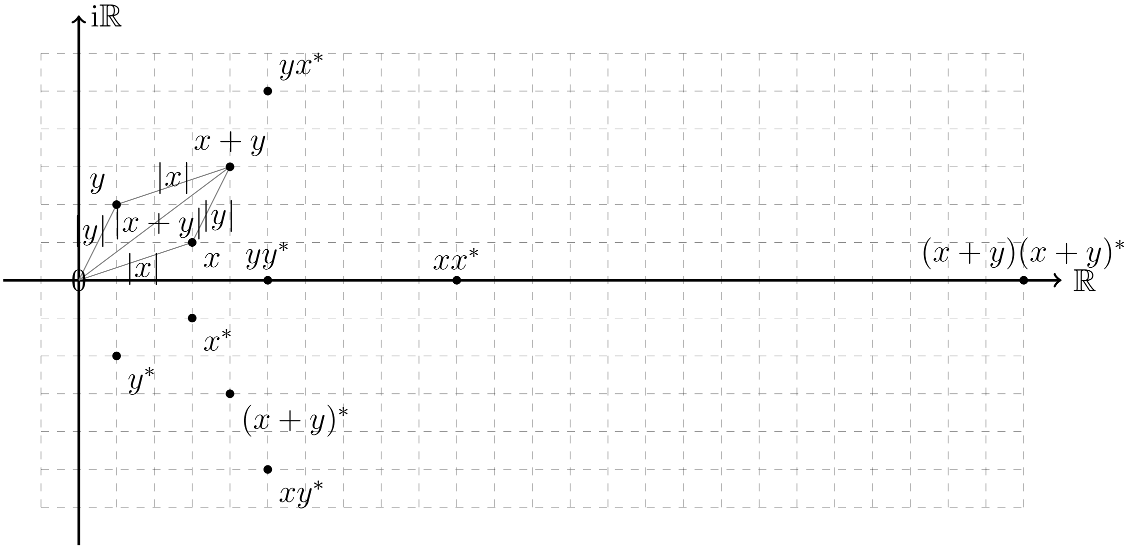


Figure 6: Triangle Inequality of complex numbers, again.

Another proof of Triangle Inequality

$$
\begin{aligned}
|x+y|^2 &= (x+y)(x+y)^* \\
&= (x+y)(x^*+y^*) \\
&= xx^* + yy^* + xy^* + yx^* \\
&= |x|^2 + |y|^2 + 2\,\mathrm{Re}(xy^*) \\
&\leq |x|^2 + |y|^2 + 2|xy^*| \\
&= |x|^2 + |y|^2 + 2|x||y| \\
&= (|x|+|y|)^2.
\end{aligned}
$$

Thus the Triangle Inequality follows by taking square roots:

$$|x+y| \leq |x| + |y|.$$

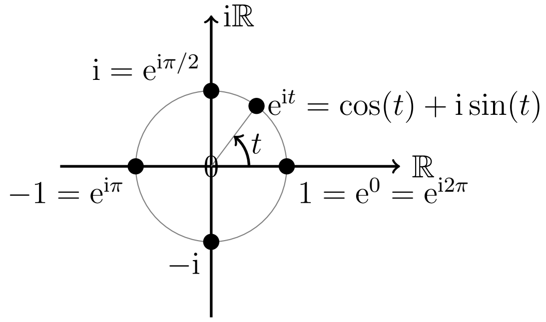


Figure 7: Euler's formula.

Division and powers of complex numbers

Division $\frac{x}{y} = x/y \in \mathbb{C}$ of $x, y \in \mathbb{C}$ (where $y \neq 0$) satisfies

$$\begin{cases} |x/y| = |x|/|y|, \\ \arg(x/y) = \arg(x) - \arg(y). \end{cases} \tag{5}$$

Notice that $\dfrac{x}{y} = \dfrac{xy^*}{yy^*} = \dfrac{xy^*}{|y|^2}$.

Example. If $x = \frac{4}{5} + \frac{3}{5}\mathrm{i}$ and $y = \frac{12}{13} + \frac{5}{13}\mathrm{i}$ then

$$\frac{x}{y} = \frac{xy^*}{|y|^2} \overset{|y|=1}{=} xy^* = \frac{(48+15) + \mathrm{i}(-20+36)}{65} = \frac{63}{65} + \frac{16}{65}\mathrm{i}.$$

Example. If $n \in \mathbb{Z}$ then $|z^n| = |z|^n$ and $\arg(z^n) = n\arg(z)$.
In case $z = \cos(t) + \mathrm{i}\sin(t)$, we get *de Moivre's formula:*

$$(\cos(t) + \mathrm{i}\sin(t))^n = \cos(nt) + \mathrm{i}\sin(nt). \tag{6}$$

It can also be shown that the following *Euler's formula*

$$\mathrm{e}^{\mathrm{i}t} = \cos(t) + \mathrm{i}\sin(t) \tag{7}$$

holds, where the complex exponential is defined by the power series

$$\mathrm{e}^z := \sum_{k=0}^{\infty} \frac{1}{k!} z^k. \tag{8}$$

This satisfies for instance $\mathrm{e}^{w+z} = \mathrm{e}^w \mathrm{e}^z$.

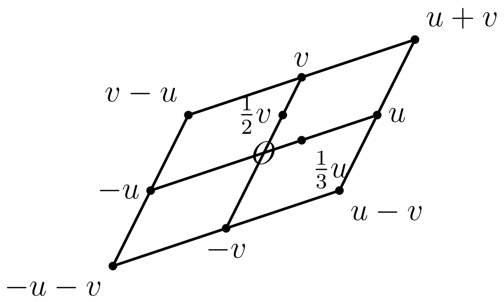


Figure 8: Schematic picture of vector operations.

2. Vector spaces

Vectors $x = (x_1, x_2, x_3, \cdots, x_n)$ are "everywhere": numbers x_k are typically results of systematic measurements related to various phenomena. To understand real vectors, we need complex numbers (see Gauss' Fundamental Theorem of Algebra, on page 42; also Fourier transform easily takes real vectors to complex). From now on, let $\mathbb{K} := \mathbb{R}$ or $\mathbb{K} := \mathbb{C}$ (real or complex number field).

Definition. *Vector space* $\mathbb{K}^n$ consists of *points*

$$x = (x_1, \cdots, x_n),$$

where $x_k \in \mathbb{K}$ is the kth *(Cartesian) coordinate* of $x \in \mathbb{K}^n$. Point

$$O := (0, \cdots, 0) \in \mathbb{K}^n$$

is the *origin*. If $\lambda \in \mathbb{K}$ and $x, y \in \mathbb{K}^n$, let

$$\begin{aligned} x + y &:= (x_1 + y_1, \cdots, x_n + y_n), \\ x - y &:= (x_1 - y_1, \cdots, x_n - y_n), \\ \lambda x &:= (\lambda x_1, \cdots, \lambda x_n), \\ -x &:= (-x_1, \cdots, -x_n). \end{aligned}$$

Let $a, b, x, y \in \mathbb{K}^n$; we identify vectors $\vec{ab}$ and $\vec{xy}$ if $b - a = y - x$.

Remark! We identify vector $\vec{xy}$ with point $y - x \in \mathbb{K}^n$.

How do the vector operations look like? Illustrate these examples:

Example. Points λx, when $x = (3, 1)$, $-\pi \le \lambda \le \pi \approx 3.14159...$

Example. $\lambda x + \mu y$, when $\lambda, \mu \in \{-1, 0, 1\}$, $x = (3, 1)$, $y = (1, 2)$.

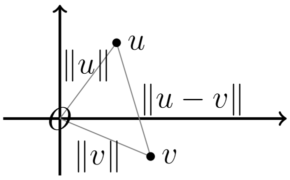


Figure 9: Norms, distances.

Geometry of vectors

Definition. The *inner product* (or *dot product*) of $u, v \in \mathbb{C}^n$ is

$$\langle u, v\rangle := \sum_{k=1}^{n} u_k\,\overline{v_k} = u_1\,\overline{v_1} + \cdots + u_n\,\overline{v_n} \quad \in \quad \mathbb{C}. \tag{9}$$

Notation $u \cdot v = \langle u, v\rangle$ is used for $u, v \in \mathbb{R}^n$. The *norm* $\|u\| \geq 0$ of $u \in \mathbb{C}^n$ is

$$\|u\| := \langle u, u\rangle^{1/2} = \left(\sum_{k=1}^{n} |u_k|^2\right)^{1/2} = (|u_1|^2 + \cdots + |u_n|^2)^{1/2}. \tag{10}$$

$\|u\|^2 = \langle u, u\rangle$ can be thought as the "*energy*" of vector $u \in \mathbb{C}^n$. Here, the number $|u_k| \geq 0$ is the "distance from the equilibrium" for each index $k \in \{1, \cdots, n\}$, with the "energy" proportional to the number $|u_k|^2$. Then the sum of such "energies" is the "total energy" $\|u\|^2$.

Definition. Define the *distance* between $u, v \in \mathbb{C}^n$ to be

$$\|u - v\| \quad \geq \quad 0. \tag{11}$$

Remark. Picture a triangle with vertices at $O, u, v \in \mathbb{C}^n$; notice that

$$\|u - v\|^2 = \|u\|^2 + \|v\|^2 - 2\,\mathrm{Re}\langle u, v\rangle.$$

Definition. Vectors $u, v \in \mathbb{C}^n$ are *orthogonal* if

$$\langle u, v\rangle = 0 \tag{12}$$

(and then $\|u - v\|^2 = \|u\|^2 + \|v\|^2$; remember Pythagoras!).

Idea: $\langle u, v\rangle$ "=" $\|u\|\|v\|\cos(\alpha)$, with α the angle between u, v. Now here $\langle u, v\rangle = 0$ means $\cos(\alpha) = 0$ (when $u \neq O \neq v$), the case of the orthogonality!

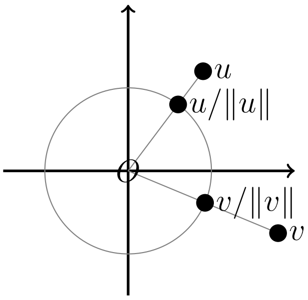


Figure 10: Unit normalizations $u/\|u\|$ and $v/\|v\|$ of vectors $u \neq O$ and $v \neq O$.

Cauchy–Schwarz inequality

Remember: Energy $\langle u, u\rangle = \|u\|^2 \geq 0$. Can we estimate $\langle u, v\rangle \in \mathbb{C}$ somehow?

Proposition (Cauchy–Schwarz inequality). *For every $u, v \in \mathbb{C}^n$,*

$$|\langle u, v\rangle| \leq \|u\|\,\|v\|. \tag{13}$$

Proof. We may assume $\|u\|\,\|v\| > 0$ (for otherwise the claim is trivial). Also, noticing that

$$\langle u, v\rangle = \|u\|\,\|v\| \left\langle \frac{u}{\|u\|}, \frac{v}{\|v\|} \right\rangle,$$

we may assume the unit normalizations $\|u\| = 1 = \|v\|$, and then just prove that $|\langle u, v\rangle| \leq 1$ (Why this is enough? Think!). Indeed, now

$$\begin{aligned}
|\langle u, v\rangle| &= \left| \sum_{k=1}^{n} u_k\, \overline{v_k} \right| \\
&\overset{(2)}{\leq} \sum_{k=1}^{n} |u_k|\,|v_k| \\
&\overset{(\star)}{\leq} \sum_{k=1}^{n} \frac{|u_k|^2 + |v_k|^2}{2} \quad = \quad \frac{\|u\|^2 + \|v\|^2}{2} \quad \overset{\|u\|=1=\|v\|}{=} \quad 1.
\end{aligned}$$

Above, inequality $(\star)$ holds because $0 \leq (|u_k| - |v_k|)^2$. **QED**

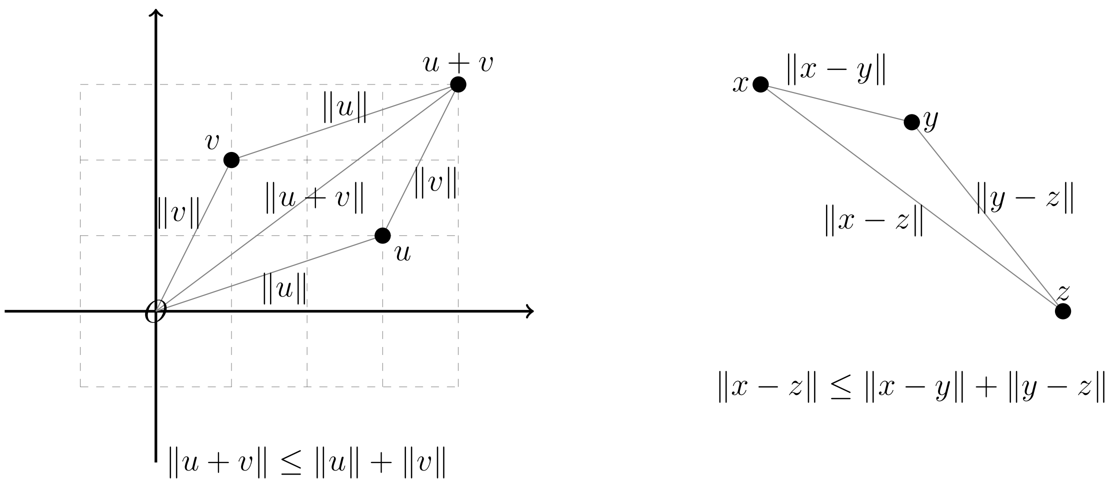


Figure 11: Triangle Inequality of vectors.

Triangle Inequality (Corollary to Cauchy–Schwarz)

Triangle Inequality. *For all* $u, v \in \mathbb{C}^n$,

$$\|u+v\| \leq \|u\| + \|v\|. \tag{14}$$

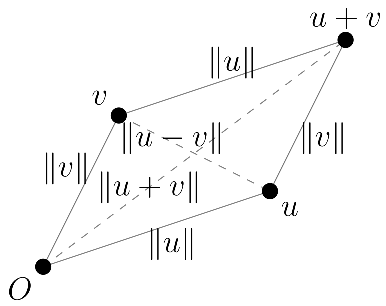
Proof. By an application of Cauchy–Schwarz (13),

$$\begin{aligned} \|u+v\|^2 &= \langle u+v, u+v\rangle \\ &= \langle u,u\rangle + \langle u,v\rangle + \langle v,u\rangle + \langle v,v\rangle \\ &= \|u\|^2 + 2\,\mathrm{Re}\langle u,v\rangle + \|v\|^2 \\ &\overset{(13)}{\leq} \|u\|^2 + 2\,\|u\|\,\|v\| + \|v\|^2 \\ &= (\|u\| + \|v\|)^2. \qquad \textbf{QED} \end{aligned}$$

Example. Let $x = London$, $y = Paris$, $z = Rome$.
For $u = x - y$ and $v = y - z$, the Triangle Inequality (14) then says:

$$\|x-z\| \leq \|x-y\| + \|y-z\|.$$

The "distance from x to z" is at most the "distance from x via y to z."



For a parallelogram in $\mathbb{C}^n$,
the sum of the squares of the diagonals
= the sum of the squares of the edges.

Figure 12: Parallelogram identity: $\|u+v\|^2 + \|u-v\|^2 = 2\|u\|^2 + 2\|v\|^2$.

Polarization and parallelogram identities

The norm was defined by the inner product. Actually, the inner product can be recovered from the norm:

Exercise. Prove the **polarization identity**

$$4\,\langle u, v\rangle = \|u+v\|^2 - \|u-v\|^2 + \mathrm{i}\|u+\mathrm{i}v\|^2 - \mathrm{i}\|u-\mathrm{i}v\|^2, \tag{15}$$

where $u, v \in \mathbb{C}^n$. Especially, notice that here

$$\begin{aligned} 4\,\mathrm{Re}\langle u, v\rangle &= \|u+v\|^2 - \|u-v\|^2, \\ \mathrm{Im}\langle u, v\rangle &= \mathrm{Re}\langle u, \mathrm{i}v\rangle. \end{aligned}$$

Exercise. Prove the **parallelogram identity** for vectors $u, v \in \mathbb{C}^n$:

$$\|u+v\|^2 + \|u-v\|^2 = 2\,\|u\|^2 + 2\,\|v\|^2. \tag{16}$$

Draw a parallelogram in a plane, with vertices at $O, u, v, u+v \in \mathbb{R}^2$: are $\|u+v\|, \|u-v\|, \|u\|, \|v\|$ connected to the distances between these vertices?

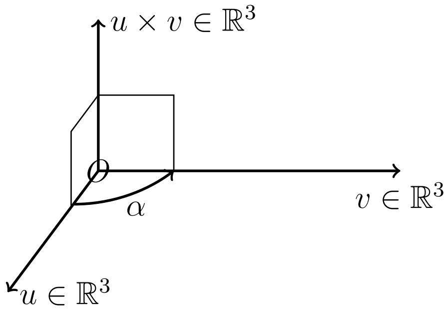


Figure 13: Cross product.

Cross product in $\mathbb{R}^3$

The *cross product* $u \times v \in \mathbb{R}^3$ of vectors $u, v \in \mathbb{R}^3$ is

$$u \times v := (u_2\, v_3, u_3\, v_1, u_1\, v_2) - (u_3\, v_2, u_1\, v_3, u_2\, v_1). \tag{17}$$

Equivalent geometric definition for the cross product:

1. $\langle u, u \times v \rangle = 0 = \langle v, u \times v \rangle$;
2. $\|u \times v\| = \|u\| \|v\| \sin(\alpha)$, with α the angle between u, v.
3. The vector triple $(u, v, u \times v)$ has right-handed orientation.

Application 1. $\|u \times v\|$ is the area of the parallelogram with vertices at $O, u, v, u + v \in \mathbb{R}^3$.

Application 2. $|\langle u \times v, w \rangle|$ is the volume of the polytope with vertices at $O, u, v, w, u + v, u + w, v + w, u + v + w \in \mathbb{R}^3$.

Application 3. $(q - p) \times (r - p)$ is a normal vector to a plane $S \subset \mathbb{R}^3$ if $p, q, r \in S$ form a non-degenerate triangle.

Nice to know: vector subspaces, dimension

Definition. Subset $V \subset \mathbb{K}^n$ is called a *vector subspace* if

(1) $O \in V$,

(2) $x + y \in V$ whenever $x, y \in V$, and

(3) $\lambda x \in V$ whenever $\lambda \in K$ and $x \in V$.

The *span* of vectors $S_1, \cdots, S_k \in \mathbb{K}^n$ is the vector subspace

$$Z_k = \mathrm{span}\{S_j\}_{j=1}^k := \left\{ \sum_{j=1}^k \lambda_j S_j \in \mathbb{K}^n : \ \lambda_1, \cdots, \lambda_k \in \mathbb{K} \right\}.$$

For example,

$$Z_1 = \mathbb{K} S_1 = \{\lambda_1 S_1 : \ \lambda \in \mathbb{K}\} \subset \mathbb{K}^n$$

(this is a line or the origin), and

$$\{O\} \subset Z_1 \subset Z_2 \subset \cdots \subset Z_{k-1} \subset Z_k \subset \mathbb{K}^n.$$

The *dimensions* are $\dim(\mathbb{K}^n) = n$ and $\dim(\{O\}) = 0$; if here $Z_{j+1} \neq Z_j$, then

$$\dim(Z_{j+1}) = 1 + \dim(Z_j).$$

Hence

$$0 \leq \dim(Z_k) \leq k.$$

Vectors $S_1, \cdots, S_k$ are *linearly independent* if $\dim(Z_k) = k$ (otherwise they are called *linearly dependent*).

Note. The dimension depends on the scalar field $\mathbb{K}$: The complex number field $\mathbb{C}$ has dimension 1 as a complex vector space. If $\mathbb{C}$ is identified as the real plane $\mathbb{R}^2$, then it has dimension 2 as a real vector space. That is,

$$\dim(\mathbb{C}) = 1 < 2 = \dim(\mathbb{R}^2).$$

Similarly,

$$\dim(\mathbb{C}^n) = n < 2n = \dim(\mathbb{R}^{2n}),$$

where we understand $\mathbb{C}^n$ to be a complex vector space, even if sometimes it might be identified with the real vector space $\mathbb{R}^{2n}$.

Nice to know: Gram–Schmidt orthonormalization

Definition. Vectors $U_1, \cdots, U_k \in \mathbb{K}^n$ are *orthonormal* if for all $j, \ell \in \{1, \cdots, k\}$

$$\langle U_j, U_\ell \rangle = \begin{cases} 1 & \text{when } j = \ell, \\ 0 & \text{when } j \neq \ell. \end{cases}$$

Gram–Schmidt algorithm. Let vectors $S_1, \cdots, S_k \in \mathbb{K}^n$ be linearly independent. The *Gram–Schmidt process* finds orthonormal vectors $U_1, \cdots, U_k \in \mathbb{K}^n$ as follows: Let

$$U_1 := S_1/\|S_1\|, \tag{18}$$

and for $j \geq 2$ then

$$U_j := \frac{V_j}{\|V_j\|}, \tag{19}$$

where

$$V_j := S_j - \sum_{\ell=1}^{j-1} \langle S_j, U_\ell \rangle \, U_\ell. \tag{20}$$

Notice that for all $d \in \{1, \cdots, k\}$ here

$$\operatorname{span}\{S_1, \cdots, S_d\} = \operatorname{span}\{U_1, \cdots, U_d\}.$$

Remark. This Gram–Schmidt process is numerically unstable (round-off errors accumulate), but there are ways to stabilize the process.

Example. Vectors $S_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ and $S_2 = \begin{bmatrix} 10 \\ -5 \end{bmatrix}$ give

$$U_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \frac{1}{\sqrt{3^2 + 4^2}} = \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix},$$

and then

$$V_2 = \begin{bmatrix} 10 \\ -5 \end{bmatrix} - \langle \begin{bmatrix} 10 \\ -5 \end{bmatrix}, \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix} \rangle \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix} = \begin{bmatrix} 10 \\ -5 \end{bmatrix} - 2 \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix} = \begin{bmatrix} 44/5 \\ -33/5 \end{bmatrix},$$

so that

$$U_2 = \frac{V_2}{\|V_2\|} = \begin{bmatrix} 4/5 \\ -3/5 \end{bmatrix}.$$

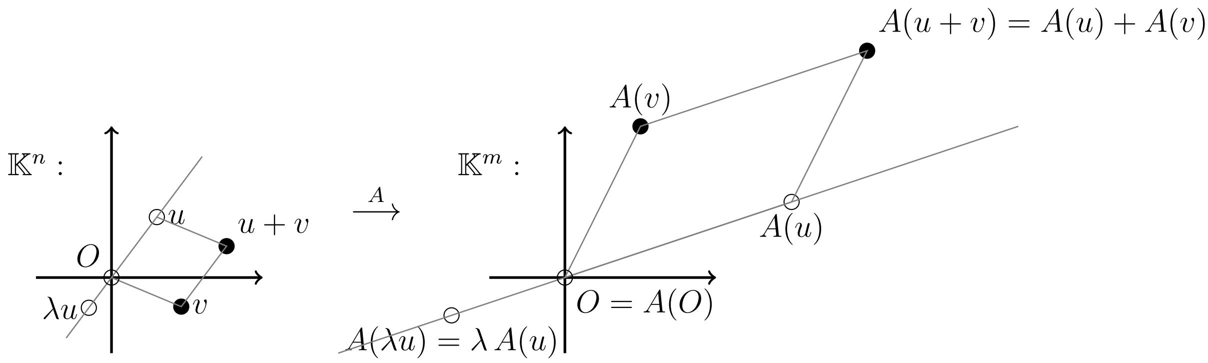


Figure 14: Idea of a linear mapping $A : \mathbb{K}^n \to \mathbb{K}^m$.

3. Linear functions, linear equations

Function $A : \mathbb{K}^n \to \mathbb{K}^m$ is *linear* if

$$\begin{aligned} A(u+v) &= A(u) + A(v), \\ A(\lambda u) &= \lambda\, A(u) \end{aligned}$$

for all $\lambda \in \mathbb{K}$ and $u, v \in \mathbb{K}^n$. Then we often write

$$Au := A(u).$$

Equivalently, function $A : \mathbb{K}^n \to \mathbb{K}^m$ is linear if and only if its *graph* $G(A)$ is a vector subspace of $\mathbb{K}^n \times \mathbb{K}^m \cong \mathbb{K}^{n+m}$, where

$$G(A) := \{(u, A(u)) : \; u \in \mathbb{K}^n\}.$$

Generic example. Define function $A : \mathbb{K}^n \to \mathbb{K}^m$ by

$$(A(u))_j := \sum_{k=1}^{n} A_{jk}\, u_k, \tag{21}$$

where *matrix elements* $A_{jk} \in \mathbb{K}$ for each $j \in \{1, \cdots, m\}$ and $k \in \{1, \cdots, n\}$. Then A is linear. Actually, this example is surprisingly(?) general, as we shall soon see.

Trivial example. We identify numbers $x \in \mathbb{R}$ with vectors $(x) \in \mathbb{R}^1$. In this sense, function $f : \mathbb{R} \to \mathbb{R}$ is linear if and only if $f(x) = ax$ for some $a \in \mathbb{R}$. Then the graph $G(f) = \{(x, f(x)) : \; x \in \mathbb{R}\}$ described by the equation $y = f(x)$ is a line through the origin $(0,0)$ of the (x,y)-plane, and this line has the slope a.

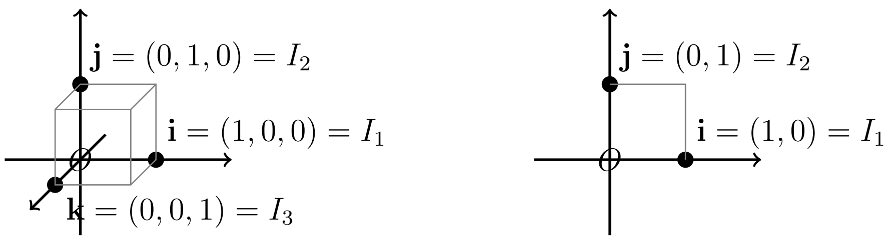


Figure 15: Standard basis vectors of $\mathbb{R}^3$ and $\mathbb{R}^2$, respectively.

Norm. The *norm* of linear $A : \mathbb{K}^n \to \mathbb{K}^m$ is the number

$$\|A\| := \max_{u \in \mathbb{K}^n : \|u\| \leq 1} \|Au\| = \max_{u \in \mathbb{K}^n, w \in \mathbb{K}^m : \|u\|, \|w\| \leq 1} |\langle Au, w \rangle|. \tag{22}$$

Then clearly $\|Av\| \leq \|A\| \, \|v\|$ for all $v \in \mathbb{K}^n$.

Idea here: Linear functions treat vector operations nicely. Norm measures the size of the mapping A (that is, how much A can stretch vectors in the extreme case).

Standard basis vectors

Definition. *Standard basis vectors* $I_1, \cdots, I_n \in \mathbb{K}^n$ are defined by

$$I_{jk} := \begin{cases} 1 & \text{if} \quad j = k, \\ 0 & \text{if} \quad j \neq k. \end{cases}$$

For example, $I_1, I_2, I_3 \in \mathbb{K}^3$ are

$$\begin{cases} I_1 = (1,0,0) = \mathbf{i}, \\ I_2 = (0,1,0) = \mathbf{j}, \\ I_3 = (0,0,1) = \mathbf{k}. \end{cases}$$

Vectors $u = (u_1, \cdots, u_n) \in \mathbb{K}^n$ can then be written

$$u = \sum_{k=1}^{n} u_k \, I_k,$$

which we shall use in the sequel. Example:

$$(9,7,5) = 9(1,0,0) + 7(0,1,0) + 5(0,0,1).$$

Matrix $[A]$ of linear function A

Let $A : \mathbb{K}^n \to \mathbb{K}^m$ be linear. Let $I_1, \cdots, I_n \in \mathbb{K}^n$ be the standard basis vectors. If $u = (u_1, \cdots, u_n) \in \mathbb{K}^n$ then $Au \in \mathbb{K}^m$, and

$$\begin{aligned} u &= \sum_{k=1}^{n} u_k\, I_k, \\ Au &\overset{A \text{ linear}}{=} \sum_{k=1}^{n} u_k\, A(I_k), \end{aligned}$$

where $A(I_k) \in \mathbb{K}^m$. Let $A_{jk} := (A(I_k))_j \in \mathbb{K}$. Then *matrix*

$$[A] := \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{bmatrix} \in \mathbb{K}^{m \times n}$$

(with m rows, n columns) has all the information about linear function

$$A : \mathbb{K}^n \to \mathbb{K}^m,$$

as

$$(Au)_j = \sum_{k=1}^{n} A_{jk}\, u_k. \tag{23}$$

In the sequel, we shall always identify

$$\text{vector} \quad u = (u_1, \cdots, u_n) \in \mathbb{K}^n \quad \text{with} \quad \text{matrix} \quad [u] = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \in \mathbb{K}^{n \times 1}.$$

We shall identify linear mapping $A : \mathbb{K}^n \to \mathbb{K}^m$ with its matrix $[A] \in \mathbb{K}^{m \times n}$. Notice that the kth column $A_k \in \mathbb{K}^{m \times 1}$ of matrix $[A] = [A_1 \ \cdots \ A_n] \in \mathbb{K}^{m \times n}$ is then

$$A_k = \begin{bmatrix} A_{1k} \\ A_{2k} \\ \vdots \\ A_{mk} \end{bmatrix} = [A(I_k)] \in \mathbb{K}^{m \times 1},$$

which is identified with the vector $A(I_k) = (A_{1k}, A_{2k}, \cdots, A_{mk}) \in \mathbb{K}^m$. Especially, the standard basis vectors $I_1, \cdots, I_n \in \mathbb{K}^n$ for the so-called *identity matrix* $[I] = [I_1 \ \cdots \ I_n] \in \mathbb{K}^{n \times n}$, for which $Iu = u$ for all $u \in \mathbb{K}^n$.

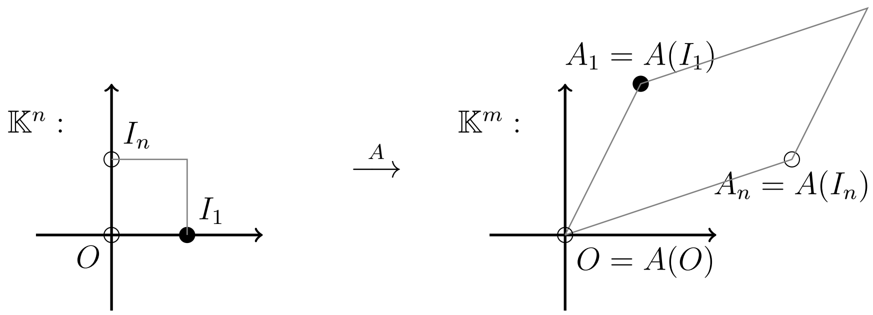


Figure 16: Matrix $[A] = [A_1 \cdots A_n] \in \mathbb{K}^{m\times n}$ of linear mapping $A : \mathbb{K}^n \to \mathbb{K}^m$.

Example. Let $A : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by

$$A(u) := (2u_1 + 3u_2 + 4u_3, 5u_1 + 6u_2 + 7u_3). \tag{24}$$

Then A is linear, and its matrix $[A] \in \mathbb{R}^{2\times 3}$ is

$$[A] = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \end{bmatrix}.$$

The matrix notation for (24) would be

$$\begin{aligned} \big[Au\big] \;=\; \big[A\big]\,\big[u\big] \;&=\; \begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \\ &=\; \begin{bmatrix} 2u_1 + 3u_2 + 4u_3 \\ 5u_1 + 6u_2 + 7u_3 \end{bmatrix}. \end{aligned}$$

For instance, if $u = (-1, -2, -3) \in \mathbb{R}^3$, then

$$\begin{aligned} [Au] \;=\; [A][u] \;&=\; \begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \end{bmatrix} \begin{bmatrix} -1 \\ -2 \\ -3 \end{bmatrix} \\ &=\; \begin{bmatrix} 2(-1) + 3(-2) + 4(-3) \\ 5(-1) + 6(-2) + 7(-3) \end{bmatrix} = \begin{bmatrix} -20 \\ -38 \end{bmatrix}, \end{aligned}$$

meaning that

$$Au = (-20, -38) \in \mathbb{R}^2.$$

We shall calculate the norm $\|A\|$ later.

Matrices can be read from inner products: If $A : \mathbb{K}^n \to \mathbb{K}^m$ is linear and $v \in \mathbb{K}^m$, then

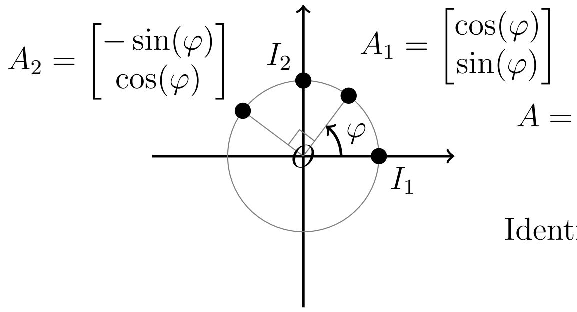
$$\langle Au, v\rangle = \sum_{j=1}^{m} (Au)_j \, \overline{v_j} = \sum_{j=1}^{m} \sum_{k=1}^{n} A_{jk} \, u_k \, \overline{v_j}.$$

Thus if $I = [I_1 \;\; I_2 \;\; I_3 \cdots]$ denotes the identity matrix of any dimension, if $u = I_k \in \mathbb{K}^n$ and $v = I_j \in \mathbb{K}^m$, then

$$\langle Au, v\rangle = A_{jk}. \tag{25}$$

So, if we know all the inner products $\langle Au, v\rangle \in \mathbb{K}$ for all $u \in \mathbb{K}^n$ and $v \in \mathbb{K}^m$, we know all the matrix elements $A_{jk} \in \mathbb{K}$.

Remark. Next we see that already the inner products $\langle Au, u\rangle$ are enough to determine $A \in \mathbb{K}^{n\times n}$ when $u \in \mathbb{C}^n$ (but not when $u \in \mathbb{R}^n$).



$$A = [A_1\ A_2] = \begin{bmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{bmatrix}.$$

$$\text{Identity matrix } I = [I_1\ I_2] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Figure 17: Rotation $A \in \mathbb{R}^{2\times 2}$ by angle φ around the origin.

Weak formulation of linear mappings

Weak Formulation Theorem. *Let $A, B \in \mathbb{C}^{n\times n}$. Then $A = B$ if*

$$\langle Au, u\rangle = \langle Bu, u\rangle \quad \text{for all } u \in \mathbb{C}^n. \tag{26}$$

Proof. Let $u, v, w \in \mathbb{C}^n$, $\lambda \in \mathbb{C}$, and $Cw = (A - B)w = Aw - Bw$. Here

$$\langle Cw, w\rangle = \langle (A-B)w, w\rangle = \langle Aw - Bw, w\rangle = \langle Aw, w\rangle - \langle Bw, w\rangle = 0$$

so that

$$\begin{aligned} 0 &\overset{0=\langle Cw,w\rangle}{=} \lambda \langle C(u+\lambda v), u + \lambda v\rangle \\ &\overset{0=\langle Cw,w\rangle}{=} |\lambda|^2 \langle Cu, v\rangle + \lambda^2 \langle Cv, u\rangle. \end{aligned}$$

Plug in $\lambda \in \{1, \mathrm{i}\}$ to get the following pair of equations:

$$\begin{cases} 0 = \langle Cu, v\rangle + \langle Cv, u\rangle, \\ 0 = \langle Cu, v\rangle - \langle Cv, u\rangle. \end{cases}$$

Clearly, $\langle Cu, v\rangle = 0$ for all $u, v \in \mathbb{C}^n$. So $Aw - Bw = Cw = O$ for all $w \in \mathbb{C}^n$. Thus $A = B$. **QED**

Remark. Complex numbers are needed in (26). If we only had $\langle Au, u\rangle = \langle Bu, u\rangle$ for all $u \in \mathbb{R}^n$, we still could have $A \neq B$. For example,

$$A = \begin{bmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{bmatrix}$$

rotates $\mathbb{R}^2$ by angle $\varphi \in \mathbb{R}$ around the origin. Then for all $u \in \mathbb{R}^2$

$$\langle Au, u\rangle = \cos(\varphi)\, \|u\|^2,$$

which does not identify A when $|\cos(\varphi)| < 1$.

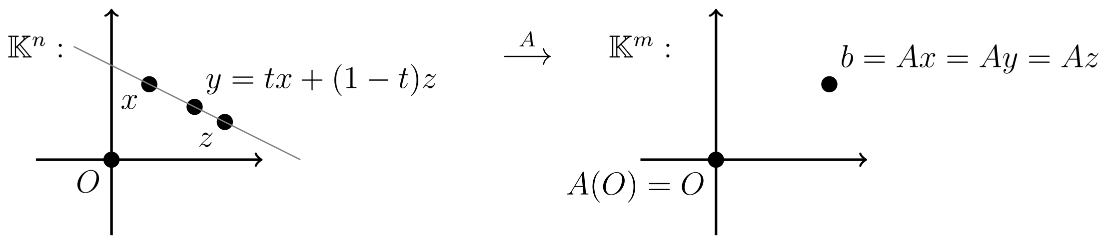


Figure 18: Linear equation $Ax = b$ may have infinitely many solutions: If $Ax = b = Az$ and $x \neq z$ then all the points y in the line passing through x and z are also solutions, as $A(tx + (1-t)z) = \ldots = b$ for all $t \in \mathbb{K}$.

4. Linear equations, Gauss elimination

For $b \in \mathbb{K}^m$ and linear $A : \mathbb{K}^n \to \mathbb{K}^m$, the *linear equation*

$$Ax = b, \tag{27}$$

can have $\underline{\text{0 or 1 or infinitely many}}$ solutions $x \in \mathbb{K}^n$.

Remark: While the real-world problems are typically non-linear, they often can be locally approximated by linear equations with a good accuracy.

Matrix formulation. Linear equation $Ax = b$ can be written by matrices:

$$[A][x] = [b], \tag{28}$$

$$\text{that is} \quad \begin{bmatrix} A_{11} & A_{12} & \ldots & A_{1n} \\ A_{21} & A_{22} & \ldots & A_{2n} \\ \vdots & \vdots & \ldots & \vdots \\ A_{m1} & A_{m2} & \ldots & A_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}, \tag{29}$$

$$\text{equivalently} \quad \begin{cases} A_{11}x_1 + A_{12}x_2 + \ldots + A_{1n}x_n &= b_1, \\ A_{21}x_1 + A_{22}x_2 + \ldots + A_{2n}x_n &= b_2, \\ & \vdots \\ A_{m1}x_1 + A_{m2}x_2 + \ldots + A_{mn}x_n &= b_m. \end{cases} \tag{30}$$

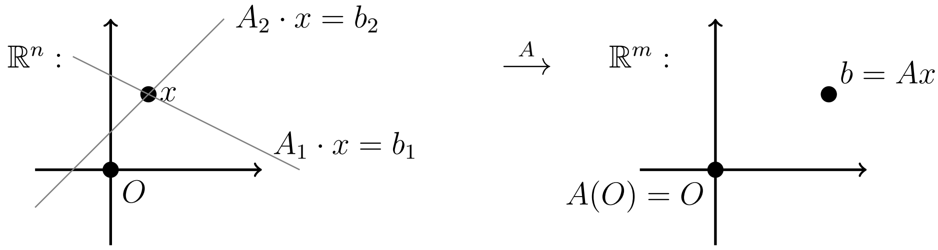


Figure 19: Linear equation $Ax = b$, with solutions x at the intersection of hypersurfaces $A_1 \cdot x = b_1$ and $A_2 \cdot x = b_2$.

Geometric interpretation

Let us consider geometric interpretation of linear equations in real vector spaces. In Problem (30), the equation

$$A_{j1}x_1 + A_{j2}x_2 + \ldots + A_{jn}x_n = b_j$$

can be written as the hypersurface equation

$$A_j \cdot x = b_j$$

in $\mathbb{R}^n$, where $A_j = (A_{jk})_{k=1}^n = (A_{j1}, A_{j2}, \cdots, A_{jm}) \in \mathbb{R}^n$. In other words, such a hypersurface is the subset $S_j \subset \mathbb{R}^n$, where

$$S_j := \{x \in \mathbb{R}^n : \ A_j \cdot x = b_j\}.$$

That is, the problem $Ax = b$ has solutions in the intersection $S \subset \mathbb{R}^n$ of hypersurfaces $S_1, S_1, \cdots, S_m$, i.e.

$$S := \{x \in \mathbb{R}^n : \quad A_j \cdot x = b_j \text{ for all } j \in \{1, \cdots, m\}\}.$$

So the linear problem has either 0 or 1 or infinitely many solutions:
(0) zero solutions if S is empty,
(1) one solution x if $S = \{x\}$ (S has just one point x),
(∞) infinitely many solutions if S contains at least two different points x and z (and then S contains the infinite line passing through x and z).

Remark: Without harm, the linear problem $Ax = b$ can be abbreviated by

$$\begin{bmatrix} A & | & b \end{bmatrix}, \quad \text{that is} \quad \begin{bmatrix} A_{11} & A_{12} & \ldots & A_{1n} & | & b_1 \\ A_{21} & A_{22} & \ldots & A_{2n} & | & b_2 \\ \vdots & \vdots & \ldots & \vdots & | & \vdots \\ A_{m1} & A_{m2} & \ldots & A_{mn} & | & b_m \end{bmatrix}.$$

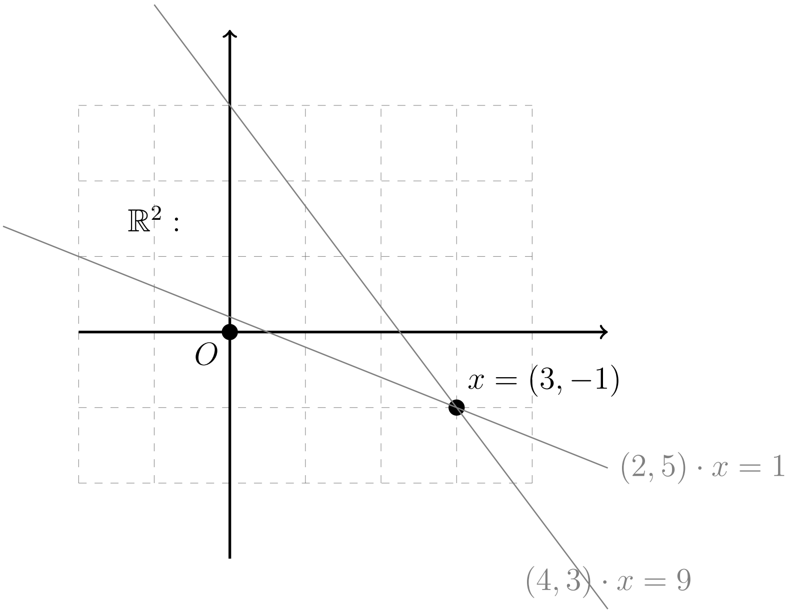


Figure 20: In the example here, the linear equation $[A][x] = [b]$ has a unique solution $x = (3, -1)$ at the intersection of hyperspaces $(2, 5) \cdot x = 1$ and $(4, 3) \cdot x = 9$ (which are lines in the plane, in this case).

Gauss eliminations

Example. [Clumsy version on the left; matrix version on the right:]

$$\begin{cases} 2x_1 + 5x_2 & = & 1, \\ 4x_1 + 3x_2 & = & 9 \end{cases} \quad \textit{or} \quad \left[\begin{array}{cc|c} 2 & 5 & 1 \\ 4 & 3 & 9 \end{array}\right]$$

$$\begin{cases} 2x_1 + 5x_2 & = & 1, \\ \quad -7x_2 & = & 7 \end{cases} \quad \textit{or} \quad \sim \left[\begin{array}{cc|c} 2 & 5 & 1 \\ 0 & -7 & 7 \end{array}\right]$$

$$\begin{cases} 2x_1 & = & 6, \\ \quad -7x_2 & = & 7 \end{cases} \quad \textit{or} \quad \sim \left[\begin{array}{cc|c} 2 & 0 & 6 \\ 0 & -7 & 7 \end{array}\right]$$

$$\begin{cases} x_1 & = & 3, \\ \quad x_2 & = & -1 \end{cases} \quad \textit{or} \quad \sim \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -1 \end{array}\right].$$

So the solution is $x = (x_1, x_2) = (3, -1)$. (**Check!**)
Geometric interpretation: this is the intersection of lines!

Sometimes no solutions

Example.

$$\begin{cases} 2x_1 + 3x_2 + 4x_3 &= 2, \\ 4x_1 + 3x_2 + 2x_3 &= 6, \\ 6x_1 + 6x_2 + 6x_3 &= 9 \end{cases} \qquad or \qquad \begin{bmatrix} 2 & 3 & 4 & | & 2 \\ 4 & 3 & 2 & | & 6 \\ 6 & 6 & 6 & | & 9 \end{bmatrix}$$

$$\begin{cases} 2x_1 + 3x_2 + 4x_3 &= 2, \\ \quad -3x_2 - 6x_3 &= 2, \\ \quad -3x_2 - 6x_3 &= 3 \end{cases} \qquad or \qquad \sim \begin{bmatrix} 2 & 3 & 4 & | & 2 \\ 0 & -3 & -6 & | & 2 \\ 0 & -3 & -6 & | & 3 \end{bmatrix}$$

$$\begin{cases} 2x_1 + 3x_2 + 4x_3 &= 2, \\ \quad -3x_2 - 6x_3 &= 2, \\ \quad 0 &= 1 \end{cases} \qquad or \qquad \sim \begin{bmatrix} 2 & 3 & 4 & | & 2 \\ 0 & -3 & -6 & | & 2 \\ 0 & 0 & 0 & | & 1 \end{bmatrix};$$

equation $0 = 1$ is false; more precisely, the last row in the last matrix reads

$$0\,x_1 + 0\,x_2 + 0\,x_3 = 1,$$

which is a contradiction! So the original problem

$$\begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 2 \\ 6 & 6 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 9 \end{bmatrix}$$

has no solution! Geometric interpretation:
the original three (hyper)planes had empty intersection!
In this example, each pair of the original (hyper)planes has an infinite line intersection (check this!).

Sometimes infinitely many solutions

Example. Ethanol C_2H_5OH burns (combusts with oxygen O_2), producing carbon dioxide CO_2 and water vapor H_2O: the reaction is

$$(x_1)\ C_2H_5OH + (x_2)\ O_2 \quad \to \quad (x_3)\ CO_2 + \ (x_4)\ H_2O,$$

where $x_1, x_2, x_3, x_4 \in \mathbb{Z}^+$. Conservation of atoms C, H, O:

$$\begin{cases} 2x_1 \qquad\quad - 1x_3 \qquad\quad = 0, \\ 6x_1 \qquad\qquad\qquad - 2x_4 = 0, \\ 1x_1 + 2x_2 - 2x_3 - 1x_4 = 0 \end{cases} \quad \textit{or} \quad \begin{bmatrix} 2 & 0 & -1 & 0 & | & 0 \\ 6 & 0 & 0 & -2 & | & 0 \\ 1 & 2 & -2 & -1 & | & 0 \end{bmatrix}$$

$$\begin{cases} 2x_1 \qquad - 1x_3 \qquad\quad = 0, \\ \qquad\quad +3x_3 - 2x_4 = 0, \\ \quad 2x_2 - \frac{3}{2}x_3 - 1x_4 = 0 \end{cases} \quad \sim \begin{bmatrix} 2 & 0 & -1 & 0 & | & 0 \\ 0 & 0 & 3 & -2 & | & 0 \\ 0 & 2 & -3/2 & -1 & | & 0 \end{bmatrix}$$

$$\begin{cases} 2x_1 \qquad - 1x_3 \qquad\quad = 0, \\ \quad 2x_2 - \frac{3}{2}x_3 - 1x_4 = 0, \\ \qquad\quad +3x_3 - 2x_4 = 0 \end{cases} \quad \sim \begin{bmatrix} 2 & 0 & -1 & 0 & | & 0 \\ 0 & 2 & -3/2 & -1 & | & 0 \\ 0 & 0 & 3 & -2 & | & 0 \end{bmatrix}$$

$$\begin{cases} 2x_1 \qquad\qquad\quad - \frac{2}{3}x_4 = 0, \\ \quad +2x_2 \qquad - 2x_4 = 0, \\ \qquad\quad +3x_3 - 2x_4 = 0 \end{cases} \quad \sim \begin{bmatrix} 2 & 0 & 0 & -\frac{2}{3} & | & 0 \\ 0 & 2 & 0 & -2 & | & 0 \\ 0 & 0 & 3 & -2 & | & 0 \end{bmatrix}$$

We obtain

$$\begin{cases} 1x_1 \qquad\qquad\quad - \frac{1}{3}x_4 = 0, \\ \quad +1x_2 \qquad - 1x_4 = 0, \\ \qquad\quad +1x_3 - \frac{2}{3}x_4 = 0 \end{cases} \quad \text{or} \quad \begin{bmatrix} 1 & 0 & 0 & -\frac{1}{3} & | & 0 \\ 0 & 1 & 0 & -1 & | & 0 \\ 0 & 0 & 1 & -\frac{2}{3} & | & 0 \end{bmatrix}.$$

When $x_4 = 3t \in \mathbb{R}$, we get

$$\begin{aligned} x &= (x_1, x_2, x_3, x_4) \\ &= (t, 3t, 2t, 3t) \in \mathbb{R}^4, \end{aligned}$$

where $t \in \mathbb{R}$ (infinitely many solutions!). The smallest $x_j \in \mathbb{Z}^+$ are

$$x = (1, 3, 2, 3),$$

so that the reaction formula is

$$C_2H_5OH + 3\ O_2 \quad \to \quad 2\ CO_2 + \ 3\ H_2O.$$

(**Check!**) Geometric interpretation:
The intersection of the original hypersurfaces in $\mathbb{R}^4$ is the line

$$\left\{ (t, 3t, 2t, 3t) \in \mathbb{R}^4 : \ t \in \mathbb{R} \right\}.$$

Example. Let us find solutions $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ to

$$\begin{cases} 2x_1 + 2x_2 + 2x_3 = 2 \\ 2x_1 + 3x_2 + 4x_3 = 2 \\ 3x_1 + 6x_2 + 9x_3 = 3 \end{cases} . \tag{31}$$

For instance, the Gauss elimination could be

$$\begin{aligned} \begin{bmatrix} 2 & 2 & 2 & | & 2 \\ 2 & 3 & 4 & | & 2 \\ 3 & 6 & 9 & | & 3 \end{bmatrix} &\sim \begin{bmatrix} 2 & 2 & 2 & | & 2 \\ 0 & 1 & 2 & | & 0 \\ 0 & 3 & 6 & | & 0 \end{bmatrix} \\ &\sim \begin{bmatrix} 2 & 2 & 2 & | & 2 \\ 0 & 1 & 2 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \\ &\sim \begin{bmatrix} 2 & 0 & -2 & | & 2 \\ 0 & 1 & 2 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & 0 & -1 & | & 1 \\ 0 & 1 & 2 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} . \end{aligned}$$

Here, if we put $x_3 = t \in \mathbb{R}$, then

$$x = (t+1, -2t, t) \in \mathbb{R}^3. \tag{32}$$

Hence the solutions form a 1-dimensional real line in the 3-dimensional real vector space $\mathbb{R}^3$.

Example. Let us find solutions $x = (x_1, x_2, x_3) \in \mathbb{C}^3$ to (31) in the previous example. Nothing changes in the Gauss elimination. If we put $x_3 = t \in \mathbb{C}$, then we obtain

$$x = (t+1, -2t, t) \in \mathbb{C}^3. \tag{33}$$

Hence the solutions form a 1-dimensional "complex line" in the 3-dimensional complex vector space $\mathbb{C}^3$.

Remark: Apparently, solving linear equations is pretty similar in the cases of real or complex scalars. This is obvious by the previous two examples above. We may always think that the real case is just a special instance of the complex case. Indeed, later during this course, when we consider eigenvalue equations $Au = \lambda u$ for real matrices $A \in \mathbb{R}^{n \times n}$, it often turns out that we actually need to treat also complex scalars $\lambda \in \mathbb{C}$ and complex vectors $u \in \mathbb{C}^n$! Likewise, a linear mapping $B : \mathbb{R}^n \to \mathbb{R}^m$ could uniquely be interpreted as a linear mapping $B : \mathbb{C}^n \to \mathbb{C}^m$, and sometimes this helps!

Gauss elimination process in nutshell

What happens in the Gauss elimination? Write linear equation $Ax = b$ as the matrix

$$\begin{bmatrix} A & | & b \end{bmatrix},$$

on which we apply *row operations* (here *row* = *equation*):

1. add weighted row to another row;
2. exchange order of two rows;
3. multiply a row by a constant $\lambda \neq 0$;

If we get from $Ax = b$ another linear problem $Cx = d$, we denote

$$\begin{bmatrix} A & | & b \end{bmatrix} \quad \sim \quad \begin{bmatrix} C & | & d \end{bmatrix}.$$

Our goal is a matrix $[C]$, where

1. next row should not have less initial zeros than the row above;
2. each column has at most one non-zero entry μ
(and you may still normalize the row with division by μ).

Write solution(s) in form

$$x = (x_1, \dots, x_n) = \dots$$

(or show that there are no solutions!)

Remark! When finding solution candidates x,
it is best to check that really

$$Ax = b.$$

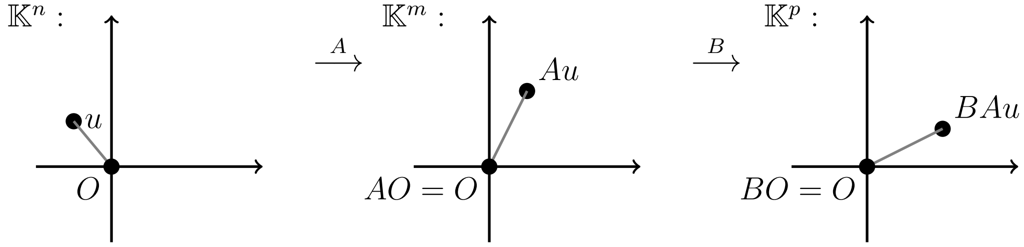


Figure 21: Composing linear mappings $A : \mathbb{K}^n \to \mathbb{K}^m$ and $B : \mathbb{K}^m \to \mathbb{K}^p$ gives a linear mapping $BA := B \circ A : \mathbb{K}^m \to \mathbb{K}^p$.

5. Matrix product $[B][A] = [BA]$, matrix inversion $[A]^{-1} = [A^{-1}]$

Let A, B be linear such that

$$\mathbb{K}^n \overset{A}{\to} \mathbb{K}^m \overset{B}{\to} \mathbb{K}^p.$$

Then $\mathbb{K}^n \overset{BA}{\to} \mathbb{K}^p$ is linear, and we define *matrix product*

$$[B]\,[A] := [BA] \in \mathbb{K}^{p\times n}.$$

Now

$$(BA)_{ij} = \sum_{k=1}^{m} B_{ik}\, A_{kj}, \tag{34}$$

because

$$(BAu)_i = \sum_{k=1}^{m} B_{ik}\,(Au)_k = \sum_{k=1}^{m} B_{ik} \sum_{j=1}^{n} A_{kj}\, u_j = \sum_{j=1}^{n} \left(\sum_{k=1}^{m} B_{ik}\, A_{kj} \right) u_j.$$

Remark. Let A, B be linear, where

$$\begin{cases} A : \mathbb{K}^{n_1} \to \mathbb{K}^{m_1}, \\ B : \mathbb{K}^{n_2} \to \mathbb{K}^{m_2}. \end{cases}$$

For BA to be defined, we must have

$$m_1 = n_2.$$

Thus $[B][A] := [BA]$ is defined only if the number of columns in $[B]$ is the number of rows in $[A]$. Then we have the norm estimate

$$\|BA\| \le \|B\|\,\|A\|, \tag{35}$$

because $\|BAu\| \le \|B\|\,\|Au\| \le \|B\|\,\|A\|\,\|u\|$.

Example. Let

$$[B] = \begin{bmatrix} 9 & 8 \\ 7 & 6 \\ 5 & 4 \end{bmatrix} \quad \text{and} \quad [A] = \begin{bmatrix} -1 & -2 \\ -3 & -4 \end{bmatrix}.$$

Now $A : \mathbb{R}^2 \to \mathbb{R}^2$ and $B : \mathbb{R}^2 \to \mathbb{R}^3$,
so $A + B$, AB and BB cannot be defined. Similarly,
we cannot compute $[A] + [B] = [A + B]$,
we cannot compute $[A][B] = [AB]$,
we cannot compute $[B][B] = [BB]$!
But $BA : \mathbb{R}^2 \to \mathbb{R}^3$ and $A^2 = AA : \mathbb{R}^2 \to \mathbb{R}^2$ can be defined, and

$$[B][A] \;=\; \begin{bmatrix} 9(-1) + 8(-3) & 9(-2) + 8(-4) \\ 7(-1) + 6(-3) & 7(-2) + 6(-4) \\ 5(-1) + 4(-3) & 5(-2) + 4(-4) \end{bmatrix} \;=\; \begin{bmatrix} -33 & -50 \\ -25 & -38 \\ -17 & -26 \end{bmatrix},$$

$$\begin{aligned} [A]^2 := [A][A] &= \begin{bmatrix} -1 & -2 \\ -3 & -4 \end{bmatrix} \begin{bmatrix} -1 & -2 \\ -3 & -4 \end{bmatrix} \\ &= \begin{bmatrix} (-1)(-1) + (-2)(-3) & (-1)(-2) + (-2)(-4) \\ (-3)(-1) + (-4)(-3) & (-3)(-2) + (-4)(-4) \end{bmatrix} \\ &= \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix}. \end{aligned}$$

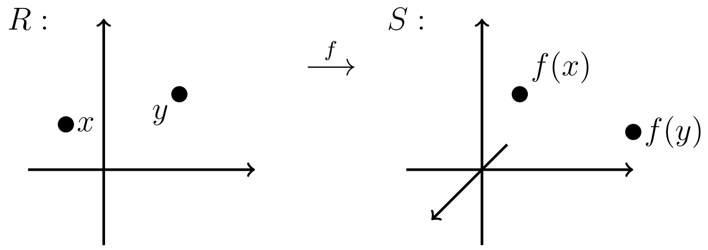


Figure 22: Injective function $f : R \to S$ does not destroy information! If $x \neq y$ for $x, y \in R$ and $f : R \to S$ injective then $f(x) \neq f(y)$.

Properties of functions

Informally, *function* $f : R \to S$ between sets R, S is a rule that on each input $x \in R$ produces an output $f(x) \in S$.

The f-*image* $f(Q) \subset S$ of a subset $Q \subset R$ is

$$f(Q) = \{f(x) : \ x \in Q\}.$$

Function $f : R \to S$ is

- *injective* provided that $f(x) = f(y)$ only if $x = y \in R$;
- *surjective* if $f(R) = S$ (that is, the f-image covers S completely);
- *bijective* if it is both injective and surjective. In this case, $f : R \to S$ has the inverse function $f^{-1} : S \to R$, where

$$\begin{aligned} & f(x) = u \\ \iff & x = f^{-1}(u). \end{aligned}$$

What if here R, S are vector spaces and $f : R \to S$ is linear?

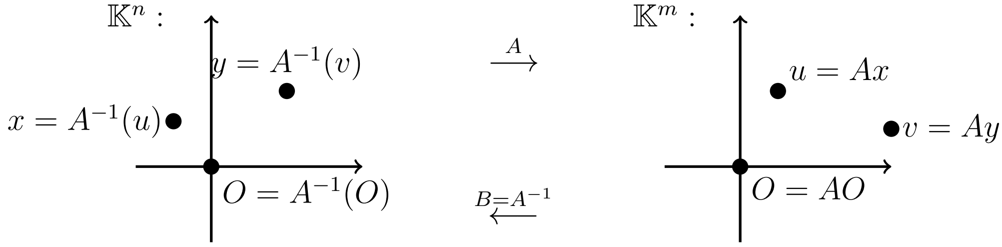


Figure 23: Inverse function $B = A^{-1}$ of a linear bijection $A : \mathbb{K}^n \to \mathbb{K}^m$ is automatically linear, and then also $m = n$.

Bijective linear functions

Theorem. *Let $A : \mathbb{K}^n \to \mathbb{K}^m$ be a linear bijection. Then*

$$\begin{aligned} &A^{-1} : \mathbb{K}^m \to \mathbb{K}^n \text{ is linear,} \\ &\text{and } m = n. \end{aligned}$$

Proof. Let us denote $B = A^{-1}$, and let $u, v \in \mathbb{K}^m$, $\lambda \in \mathbb{K}$. Then $u = A(x)$ and $v = A(y)$ for some $x, y \in \mathbb{K}^n$, and linearity of B follows:

$$\begin{aligned} B(u+v) &= B(A(x)+A(y)) \\ &= B(A(x+y)) \\ &= x+y \\ &= B(u)+B(v), \\ B(\lambda u) &= B(\lambda\, A(x)) \\ &= B(A(\lambda x)) \\ &= \lambda x \\ &= \lambda\, B(u). \end{aligned}$$

Notice that the origin $x = O \in \mathbb{K}^n$ is the unique solution to $A(x) = O \in \mathbb{K}^m$; the solution could not be unique if $m < n$ (here $A(x) = O$ is a system of m equations with n unknown variables $x_1, \cdots, x_n \in \mathbb{K}$). Hence

$$m \geq n.$$

But symmetrically $n \geq m$: here $u = O \in \mathbb{K}^m$ is the unique solution to

$$A^{-1}(u) = O \in \mathbb{K}^n.$$

QED

Inverse matrix $[A]^{-1} = [A^{-1}]$

Let $I = (x \mapsto x) : \mathbb{K}^n \to \mathbb{K}^n$ be the identity function. If $A, B, C : \mathbb{K}^n \to \mathbb{K}^n$ are linear and $BA = I = AC$ then $B = C = A^{-1}$, because

$$B = BI = B(AC) = (BA)C = IC = C.$$

Let $A : \mathbb{K}^n \to \mathbb{K}^n$ be a linear bijection; we know that then $A^{-1} : \mathbb{K}^n \to \mathbb{K}^n$ is also a linear bijection. Then the *inverse matrix* of A is defined to be the matrix of A^{-1}, that is

$$[A]^{-1} := [A^{-1}],$$

and we say that $[A]$ is *invertible*: here

$$\|A\|^{-1} \leq \|A^{-1}\|,$$

because $1 = \|I\| = \|A^{-1}A\| \leq \|A^{-1}\| \, \|A\|$.

Finding $X = A^{-1}$ means solving a linear equation

$$AX = I,$$

which can be done by the Gauss elimination

$$[A|I] \sim \ldots \sim [I|X] = [I|A^{-1}].$$

Example. $[I]^{-1} = [I^{-1}] = [I]$, that is:

$$\begin{bmatrix} 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ \ldots$$

Example.

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 5 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & -\frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{5} \end{bmatrix}.$$

Example. If $A = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$ then $B = A^{-1}$:

$$\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix} = \begin{bmatrix} 2(3)+1(-5) & 2(-1)+1(2) \\ 5(3)+3(-5) & 5(-1)+3(2) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I,$$

$$\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} 3(2)-1(5) & 3(1)-1(3) \\ -5(2)+2(5) & -5(1)+2(3) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$$

Example. Let $[A] = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$. Then

$$\begin{aligned} \begin{bmatrix} A & | & I \end{bmatrix} &= \begin{bmatrix} 2 & 1 & | & 1 & 0 \\ 5 & 3 & | & 0 & 1 \end{bmatrix} \\ &\sim \begin{bmatrix} 2 & 1 & | & 1 & 0 \\ 0 & \frac{1}{2} & | & -\frac{5}{2} & 1 \end{bmatrix} \\ &\sim \begin{bmatrix} 2 & 0 & | & 6 & -2 \\ 0 & \frac{1}{2} & | & -\frac{5}{2} & 1 \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & 0 & | & 3 & -1 \\ 0 & 1 & | & -5 & 2 \end{bmatrix} \quad = \quad \begin{bmatrix} I & | & X \end{bmatrix}. \end{aligned}$$

Thus $[A]^{-1} = [X] = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$.

Remark! It is good to check that $[X]$ is the inverse matrix to $[A]$:

$$[A][X] = \ldots = [I], \qquad [X][A] = \ldots = [I].$$

(It is enough to check only $[A][X]$ or $[X][A]$ — why?).

Example. Let us try to find the inverse to $[A] = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$:

$$\begin{aligned} \begin{bmatrix} A & | & I \end{bmatrix} &= \begin{bmatrix} 1 & 2 & 3 & | & 1 & 0 & 0 \\ 4 & 5 & 6 & | & 0 & 1 & 0 \\ 7 & 8 & 9 & | & 0 & 0 & 1 \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & 2 & 3 & | & 1 & 0 & 0 \\ 0 & -3 & -6 & | & -4 & 1 & 0 \\ 0 & -6 & -12 & | & -7 & 0 & 1 \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & 2 & 3 & | & 1 & 0 & 0 \\ 0 & -3 & -6 & | & -4 & 1 & 0 \\ 0 & 0 & 0 & | & 1 & -2 & 1 \end{bmatrix}. \end{aligned}$$

The equations in the last row are now

$$0 = 1, \quad 0 = -2, \quad 0 = 1$$

which is clearly false! The reason is that here matrix $[A] \in \mathbb{R}^{3\times 3}$ is not invertible, i.e. linear function $A : \mathbb{R}^3 \to \mathbb{R}^3$ is not bijective!

Exercise. Let $A, B, C, D, E \in \mathbb{C}^{n\times n}$. Let $I \in \mathbb{C}^{n\times n}$ be the identity matrix (that is, $Iv = v$ for all $v \in \mathbb{C}^{n\times 1}$).

(a) Assume that $AB = I = CA$. Show that $B = C$.
(Of course, this means that $B = C = A^{-1}$.)

(b) Assume now that $I - DE$ is invertible. Show that then matrix $I - ED$ has inverse $F := I + E(I - DE)^{-1}D$.
(Hint: calculate that $(I - ED)F = \cdots = I$, so that by (a) we have...)

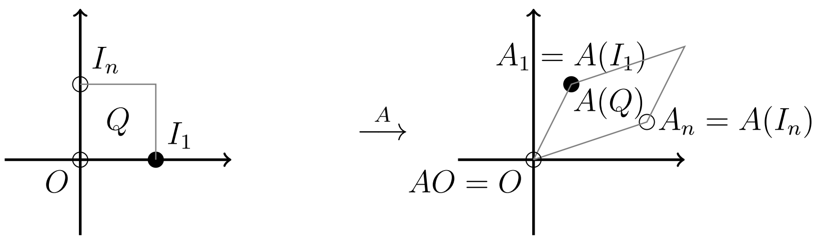


Figure 24: Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be linear. Let $Q = [0,1]^n \subset \mathbb{R}^n$ be the unit cube. Then $|\det[A]| \geq 0$ is the volume of the polyhedron $A(Q) \subset \mathbb{R}^n$, and $\det[A] < 0$ would mean flipping the orientation.

6. Determinant

The determinant

$$\det[A] \in \mathbb{K}$$

of a matrix $[A] \in \mathbb{K}^{n \times n}$ describes geometric information about the linear function $A : \mathbb{K}^n \to \mathbb{K}^n$. If a subset $S \subset \mathbb{K}^n$ has n-dimensional volume 1 then:

- $A(S) \subset \mathbb{K}^n$ has n-dimensional volume $|\det[A]|$.
- When $\mathbb{K} = \mathbb{R}$: if $\det[A] < 0$ then $A(S)$ is a "mirror image" of S; if $\det[A] > 0$ then $A(S)$ and S have the same orientation.

We shall need four laws (L1,L2,L3,L4) to find determinants:

(L1) $\det[I] = 1$ for identity matrix $[I] \in \mathbb{K}^{n \times n}$.

(L2) If one column is multiplied by $\lambda \in \mathbb{K}$, then det is multiplied by λ.

(L3) If matrix has two same columns then determinant is 0.

(L4) Let X_j be the jth column of $X = [X_1 \ \cdots \ X_n] \in \mathbb{K}^{n \times n}$.
If $C_k = A_k + B_k$ and $C_j = A_j = B_j$ whenever $j \neq k$
then $\det[C] = \det[A] + \det[B]$.

Example. For $[A] = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$, laws (L1,L2) yield

$$\det[A] = \det \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix} \overset{(L2)}{=} 2 \cdot 3 \cdot 5 \cdot \det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \overset{(L1)}{=} 2 \cdot 3 \cdot 5 = 30.$$

(This matrix $[A]$ here stretches the standard basis by the respective factors $2, 3, 5$, thus streching the 3-dimensional volumes by the factor $2 \cdot 3 \cdot 5 = 30$.)

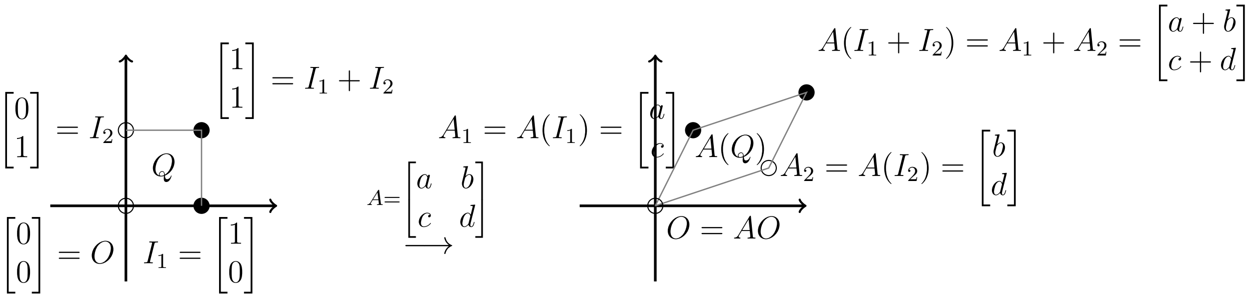


Figure 25: Matrix $[A] \in \mathbb{R}^{2\times 2}$ maps the unit cube (the usual unit *square*) $Q = [0,1]^2 = \{x = (x_1, x_2) \in \mathbb{R}^2 : \ x_1, x_2 \in [0,1]\}$ to the parallelogram $A(Q) \subset \mathbb{R}^2$, which has the 2-dimensional volume (or the usual *area*) $|\det[A]| \geq 0$.

Law (L5) (Corollary to (L3,L4)). *Sign of* det *changes by interchanging two columns!*

Proof: It is enough to check the 2-dimensional case:

$$\begin{aligned} 0 &\overset{(L3)}{=} \det \begin{bmatrix} a+b & a+b \\ c+d & c+d \end{bmatrix} \overset{(L4)}{=} \det \begin{bmatrix} a & a+b \\ c & c+d \end{bmatrix} + \det \begin{bmatrix} b & a+b \\ d & c+d \end{bmatrix} \\ &\overset{(L4)}{=} \det \begin{bmatrix} a & a \\ c & c \end{bmatrix} + \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \det \begin{bmatrix} b & a \\ d & c \end{bmatrix} + \det \begin{bmatrix} b & b \\ d & d \end{bmatrix} \\ &\overset{(L3)}{=} +\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \det \begin{bmatrix} b & a \\ d & c \end{bmatrix}. \qquad \textbf{QED} \end{aligned}$$

Example: The general computation for the 2-dimensional case is

$$\begin{aligned} \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} &\overset{(L4)}{=} \det \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} + \det \begin{bmatrix} 0 & b \\ c & d \end{bmatrix} \\ &\overset{(L4)}{=} \det \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} + \det \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} + \det \begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix} + \det \begin{bmatrix} 0 & 0 \\ c & d \end{bmatrix} \\ &\overset{(L2)}{=} ab \det \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + ad \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + bc \det \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + cd \det \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \\ &\overset{(L3,L5)}{=} (ad - bc) \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &\overset{(L1)}{=} ad - bc. \end{aligned}$$

More generally: ... Similarly for any $[A] \in \mathbb{K}^{n\times n}$,

$$\det[A] = \sum_{[P]\ permutation} a_P \det[P], \tag{36}$$

where the permutation matrices $[P] \in \mathbb{R}^{n\times n}$ have elements $P_{jk} \in \{0,1\}$ with exactly one 1 in each row and in each column; $a_P \in \mathbb{K}$ is the product of those A_{jk} for which $P_{jk} = 1$. For dimension n, there are exactly $n!$ permutation matrices. For instance, for $n = 2$ we have $n! = 2$ permutations, and

$$\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \det\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} + \det\begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix} = ad\det\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + bc\det\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = ad - bc.$$

Example. For $n = 3$, there are $n! = 6$ permutation matrices $[P] \in \mathbb{R}^{3\times 3}$, which are

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Find the determinants of these permutation matrices! Notice that $\det\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} =$

$$\begin{aligned} &= \det\begin{bmatrix} a & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & i \end{bmatrix} + \det\begin{bmatrix} a & 0 & 0 \\ 0 & 0 & f \\ 0 & h & 0 \end{bmatrix} + \det\begin{bmatrix} 0 & b & 0 \\ 0 & 0 & f \\ g & 0 & 0 \end{bmatrix} + \det\begin{bmatrix} 0 & b & 0 \\ d & 0 & 0 \\ 0 & 0 & i \end{bmatrix} + \det\begin{bmatrix} 0 & 0 & c \\ d & 0 & 0 \\ 0 & h & 0 \end{bmatrix} + \det\begin{bmatrix} 0 & 0 & c \\ 0 & e & 0 \\ g & 0 & 0 \end{bmatrix} \\ &= +aei - afh + bfg - bdi + cdh - ceg. \end{aligned}$$

Remark! By (36), Laws (L1,L2,L3,L4,L5) are ok when the word "**column**" is changed to the word "**row**". Thus if we obtain $[B] \in \mathbb{K}^{n\times n}$ from $[A] \in \mathbb{K}^{n\times n}$ by the Gauss elimination (not multiplying single rows) then

$$\det[A] = (-1)^k \det[B],$$

where k is the number of the row interchanging operations.

Example: $\det[A] = 2 \cdot 3 \cdot 7 = 42$, where

$$[A] = \begin{bmatrix} 2 & -1 & 3 \\ 4 & 1 & 2 \\ 6 & 3 & 8 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 & 3 \\ 0 & 3 & -4 \\ 0 & 6 & -1 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 & 3 \\ 0 & 3 & -4 \\ 0 & 0 & 7 \end{bmatrix} \sim \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 7 \end{bmatrix}.$$

Remark! Clearly, $A \in \mathbb{K}^{n\times n}$ is invertible if and only if it can be transformed to the identity matrix $I \in \mathbb{K}^{n\times n}$ by the Gauss row operations:

$$[A|I] \overset{\text{Gauss}}{\sim} [I|A^{-1}].$$

On the other hand, invertibility is equivalent to $\det[A] \neq 0$, since the determinant can be calculated by the Gauss row operations.

Sub-determinants. For finding determinants, the Gauss row operations are most useful if we know the exact numerical values of the matrix elements. For more general determinant formulas, application of (36) might be better. Related to this, we may go from dimension n to dimension $n-1$ by sub-determinants: for $[A] \in \mathbb{K}^{n\times n}$, let

$$[B^{(j,k)}] = [\text{omit row } j \text{ and column } k \text{ from } A] \in \mathbb{K}^{(n-1)\times(n-1)},$$

$$\begin{aligned} \det[A] &= \sum_{j=1}^{n} (-1)^{j+k} A_{jk} \det[B^{(j,k)}] \\ &= \sum_{k=1}^{n} (-1)^{j+k} A_{jk} \det[B^{(j,k)}]. \end{aligned}$$

Here the signs have the pattern $\left[(-1)^{j+k}\right]_{j,k=1}^{n} = \begin{bmatrix} +1 & -1 & +1 & -1 & \cdots \\ -1 & +1 & -1 & +1 & \cdots \\ +1 & -1 & +1 & -1 & \cdots \\ -1 & +1 & -1 & +1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$.

Example. Let us first develop the sub-determinants with respect to the first row $\begin{bmatrix} a & b & c \end{bmatrix}$, and then with respect to the second column $\begin{bmatrix} b \\ e \\ h \end{bmatrix}$:

$$\begin{aligned} \det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} &= +a \det \begin{bmatrix} e & f \\ h & i \end{bmatrix} - b \det \begin{bmatrix} d & f \\ g & i \end{bmatrix} + c \det \begin{bmatrix} d & e \\ g & h \end{bmatrix} \\ &= -b \det \begin{bmatrix} d & f \\ g & i \end{bmatrix} + e \det \begin{bmatrix} a & c \\ g & i \end{bmatrix} - h \det \begin{bmatrix} a & c \\ d & f \end{bmatrix}. \end{aligned}$$

Row operations as matrix products

Any simple Gauss row operation $B \mapsto EB$ on matrix $B \in \mathbb{K}^{n\times n}$ can be written as an elementary matrix product $EB \in \mathbb{K}^{n\times n}$: for instance,

$$\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix}a & b & c\\ d & e & f\\ g & h & i\end{bmatrix} = \begin{bmatrix}a & b & c\\ g & h & i\\ d & e & f\end{bmatrix}$$

corresponds to interchanging rows (2nd and 3rd),

$$\begin{bmatrix}1 & 0 & 0\\ 0 & \lambda & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}a & b & c\\ d & e & f\\ g & h & i\end{bmatrix} = \begin{bmatrix}a & b & c\\ \lambda d & \lambda e & \lambda f\\ g & h & i\end{bmatrix}$$

corresponds to multiplying a row (2nd) by constant $\lambda \in \mathbb{K}$, and

$$\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & \lambda\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}a & b & c\\ d & e & f\\ g & h & i\end{bmatrix} = \begin{bmatrix}a & b & c\\ d+\lambda g & e+\lambda h & f+\lambda i\\ g & h & i\end{bmatrix}$$

corresponds to adding a λ-weighted row (3rd) to another row (2nd).

Theorem. *Determinant is* multiplicative*: for all $A, B \in \mathbb{K}^{n\times n}$*

$$\det[AB] = \det[A]\det[B]. \tag{37}$$

Proof. For non-invertible A this is trivial. An invertible A is of the form

$$A = E_1\, E_2 \cdots E_k, \tag{38}$$

where each $E_j \in \mathbb{K}^{n\times n}$ corresponds to a Gauss row operation as above. Here

$$\det[E_j\, M] = \det[E_j]\det[M] \tag{39}$$

for all $M \in \mathbb{K}^{n\times n}$. Thereby

$$\begin{aligned}
\det[AB] &\overset{(38)}{=} \det[E_1\, E_2 \cdots E_k\, B]\\
&\overset{(39)}{=} \det[E_1]\det[E_2\, \cdots\, E_k\, B]\\
\cdots &\overset{(39)}{=} \det[E_1]\det[E_2]\, \cdots\, \det[E_k]\det[B]\\
\cdots &\overset{(39)}{=} \det[E_1\, E_2\, \cdots\, E_k]\det[B]\\
&\overset{(38)}{=} \det[A]\det[B]. \qquad\qquad \textbf{QED}
\end{aligned}$$

Corollary: *If $A \in \mathbb{K}^{n\times n}$ is invertible then*

$$\det[A^{-1}] = 1/\det[A], \tag{40}$$

because $1 = \det[I] = \det[AA^{-1}] = \det[A]\det[A^{-1}]$.

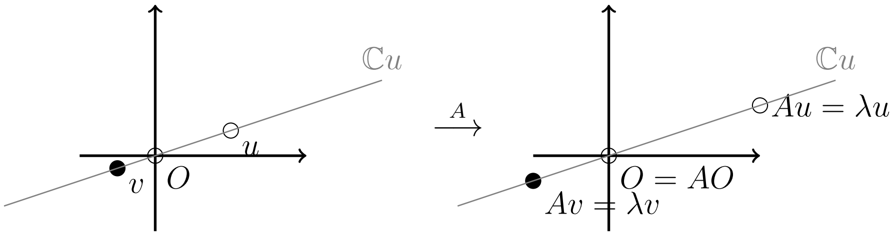


Figure 26: Eigenvectors $u, v \in \mathbb{C}^n$ on the same eigenline $\mathbb{C}u$ of $A \in \mathbb{C}^{n\times n}$.

7. Eigenvalues and eigenvectors

Linear $A : \mathbb{C}^n \to \mathbb{C}^n$ (and corresponding matrix $[A] \in \mathbb{C}^{n\times n}$) has *eigenvector* $u \in \mathbb{C}^n$ (or $[u] \in \mathbb{C}^{n\times 1}$) and *eigenvalue* $\lambda \in \mathbb{C}$ if

$$Au = \lambda u,$$

where eigenvector $u \neq O$ (of course, trivially $AO = O = \lambda O$, where $O \in \mathbb{C}^n$ is the origin). The *spectrum* $\sigma(A) \subset \mathbb{C}$ is the set of all eigenvalues of A.

Remark. Notice that if $t \in \mathbb{C}$ then here $A(tu) = tA(u) = \lambda(tu)$: thus it would be appropriate to talk about the *eigenline*

$$\mathbb{C}u = \{tu \in \mathbb{C}^n : \ t \in \mathbb{C}\},$$

where $tu \in \mathbb{C}^n$ is another eigenvector whenever $t \neq 0$.

How to find all the eigenvalues? Notice that

$$\begin{aligned} Au = \lambda u \quad &\Longleftrightarrow \quad (A - \lambda I)u = O \\ &\overset{A-\lambda I \text{ linear}, \ u\neq O}{\Longleftrightarrow} \quad A - \lambda I \text{ not invertible,} \end{aligned}$$

which means $\det[A - \lambda I] = 0$, so you may find the eigenvalues $\lambda \in \mathbb{C}$ by solving this so-called *characteristic equation*.

How to find the eigenvectors? Let $\lambda \in \mathbb{C}$ be an eigenvalue of a square matrix $A \in \mathbb{C}^{n\times n}$. The corresponding eigenvectors $u \in \mathbb{C}^n$ satisfy $Au = \lambda u$ (equivalently $(A - \lambda I)u = O$), and we find them by the Gauss elimination on

$$[A - \lambda I|O].$$

Just remember that the origin $O \in \mathbb{C}^n$ is never an eigenvector.

Eigenvalues as roots of characteristic polynomial

Definition. The *characteristic polynomial* of $[A] \in \mathbb{C}^{n\times n}$ is $p_A : \mathbb{C} \to \mathbb{C}$, where

$$p_A(z) := \det[A - zI]. \tag{41}$$

By the **Fundamental Theorem of Algebra** [Gauss], polynomials split uniquely in $\mathbb{C}$ into product of first order terms. Thus

$$p_A(z) = (-1)^n\,(z-\lambda_1)\cdots(z-\lambda_n), \tag{42}$$

where $\lambda_1,\cdots,\lambda_n \in \mathbb{C}$ are the eigenvalues of A. Hence here the spectrum $\sigma(A) = \{\lambda_1,\cdots,\lambda_n\} \subset \mathbb{C}$ has at most n different elements $\lambda \in \mathbb{C}$.
The *algebraic multiplicity*

$$d = m_a(\lambda)$$

of an eigenvalue $\lambda \in \sigma(A)$ is the degree d of the factor $(z-\lambda)^d$ in p_A.
For instance, if $[A] \in \mathbb{R}^{10\times 10}$ with $p_A(z) = (z-2)^3(z+5)^7$,
then $m_a(2) = 3$ and $m_a(-5) = 7$.
The *geometric multiplicity*

$$m_g(\lambda)$$

of an eigenvalue $\lambda \in \sigma(A)$ is the dimension of the vector subspace spanned by the corresponding eigenvectors of A. For $[A] \in \mathbb{C}^{n\times n}$ always

$$1 \le m_g(\lambda) \le m_a(\lambda) \le n, \quad \text{and} \quad \sum_{\lambda\in\sigma(A)} m_a(\lambda) = n. \tag{43}$$

Notice also that

$$p_A(z) = \sum_{k=0}^{n} c_k\, z^k = c_n\, z^n + c_{n-1}\, z^{n-1} + \cdots + c_1\, z + c_0,$$

where $c_n = (-1)^n$, $c_0 = p_A(0)$, and

$$\begin{aligned} \lambda_1\cdots\lambda_n &\overset{(42)}{=} c_0 \overset{(41)}{=} \det[A], \\ -\sum_{k=1}^{n}\lambda_k &\overset{(42)}{=} c_{n-1} \overset{(41)}{=} -\mathrm{tr}[A], \end{aligned}$$

where the *trace* tr[A] $\in \mathbb{C}$ is the sum of the diagonal elements of $[A]$,

$$\mathrm{tr}[A] := \sum_{k=1}^{n} A_{kk}. \tag{44}$$

In other words:
the determinant is the product of the eigenvalues, and
the trace is the sum of the eigenvalues
when we take the algebraic multiplicities into account!

Example. If $[A] = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \in \mathbb{C}^{3\times 3}$ then

$$\begin{aligned} p_A(z) &= \det[A - zI] \\ &= \det \begin{bmatrix} a-z & 0 & 0 \\ 0 & b-z & 0 \\ 0 & 0 & c-z \end{bmatrix} \\ &= (-1)^3\,(z-a)(z-b)(z-c), \end{aligned}$$

giving eigenvalues $a, b, c \in \mathbb{C}$. Moreover,

$$p_A(z) = (-1)^3\left(z^3 - (a+b+c)z^2 + (ab+ac+bc)z - abc\right),$$

where

$$\begin{aligned} \mathrm{tr}[A] &= a+b+c \;\in\; \mathbb{C}, \\ \det[A] &= abc \;\in\; \mathbb{C}. \end{aligned}$$

Example. If $[A] = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{C}^{2\times 2}$ then

$$\begin{aligned} p_A(z) &= \det[A - zI] \\ &= \det \begin{bmatrix} a-z & b \\ c & d-z \end{bmatrix} \\ &= (a-z)(d-z) - bc \\ &= (-1)^2(z^2 - (a+d)z + (ad-bc)) \\ &= (-1)^2(z-\lambda_1)(z-\lambda_2), \end{aligned}$$

where eigenvalues $\lambda_1, \lambda_2 \in \mathbb{C}$ can be found easily. Notice that here

$$\begin{aligned} \mathrm{tr}[A] &= a+d \;\in\; \mathbb{C}, \\ \det[A] &= ad-bc \;\in\; \mathbb{C}. \end{aligned}$$

Example. Let $R : \mathbb{R}^2 \to \mathbb{R}^2$, for which $R(x) = (x_2, x_1)$, that is

$$[R] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Here $\det[R] = -1$, and R reflects the real plane $\mathbb{R}^2$ with respect to the hyperplane (or line) $\{x \in \mathbb{R}^2 : \ x_1 = x_2\}$. Now

$$\begin{aligned} p_R(z) &= \det[R - zI] \\ &= \det \begin{bmatrix} 0 - z & 1 \\ 1 & 0 - z \end{bmatrix} \\ &= z^2 - 1 \\ &= (z-1)(z+1), \end{aligned}$$

giving eigenvalues $\lambda = \pm 1$. Corresponding eigenvectors x? $R(x) = \lambda x$? We solve $(R - \lambda I)(x) = O$ by the Gauss elimination:

$$[R - \lambda I | O] = \left[\begin{array}{cc|c} -\lambda & 1 & 0 \\ 1 & -\lambda & 0 \end{array}\right] \overset{Gauss}{\sim} \left[\begin{array}{cc|c} 0 & 1-\lambda^2 & 0 \\ 1 & -\lambda & 0 \end{array}\right] \overset{\lambda=\pm 1}{=} \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & -\lambda & 0 \end{array}\right].$$

Eigenvectors for $\lambda = +1$: $x = (t, t) \in \mathbb{C}^2$ for $0 \neq t \in \mathbb{C}$.
Eigenvectors for $\lambda = -1$: $x = (t, -t) \in \mathbb{C}^2$ for $0 \neq t \in \mathbb{C}$.

Example. Let $Q : \mathbb{R}^2 \to \mathbb{R}^2$, for which $Q(x) = (x_2, -x_1)$, that is

$$[Q] = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Here $\det[Q] = +1$, and Q rotates the real plane $\mathbb{R}^2$ around the origin by $\pi/2$ radians in the negative direction (i.e. by 90 degrees clockwise). Now

$$\begin{aligned} p_Q(z) &= \det[Q - zI] \\ &= \det \begin{bmatrix} 0 - z & 1 \\ -1 & 0 - z \end{bmatrix} \\ &= z^2 + 1 \\ &= (z - \mathrm{i})(z + \mathrm{i}), \end{aligned}$$

giving eigenvalues $\lambda = \pm\mathrm{i}$. Corresponding eigenvectors x? $Q(x) = \lambda x$? We solve $(Q - \lambda I)(x) = O$ by the Gauss elimination:

$$[Q - \lambda I | O] = \left[\begin{array}{cc|c} -\lambda & 1 & 0 \\ -1 & -\lambda & 0 \end{array}\right] \overset{Gauss}{\sim} \left[\begin{array}{cc|c} 0 & 1+\lambda^2 & 0 \\ -1 & -\lambda & 0 \end{array}\right] \overset{\lambda=\pm\mathrm{i}}{=} \left[\begin{array}{cc|c} 0 & 0 & 0 \\ -1 & -\lambda & 0 \end{array}\right].$$

Eigenvectors for $\lambda = +\mathrm{i}$: $x = (t, +\mathrm{i}t) \in \mathbb{C}^2$ for $0 \neq t \in \mathbb{C}$.
Eigenvectors for $\lambda = -\mathrm{i}$: $x = (t, -\mathrm{i}t) \in \mathbb{C}^2$ for $0 \neq t \in \mathbb{C}$.

Example. Let us find the eigenvalues and the eigenvectors for

$$A = SDS^{-1} = \begin{bmatrix} 5 & 2 \\ 2 & 8 \end{bmatrix}, \quad \text{where} \quad D = \begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix} \quad \text{and} \quad S = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}. \tag{45}$$

The reader might want to find the eigenvalues by factorizing the characteristic polynomial $z \mapsto \det[A - zI]$, and then finding the corresponding eigenvectors by the Gauss elimination from $[A - \lambda I | O]$, as in the previous cases. Be my guest, do that for your personal exercise.

However, we are going to find them easier. Namely, we claim that the eigenvalues are 9 and 4, obtained from the diagonal elements of the **diagonal matrix** D; and that then the corresponding eigenvectors are the non-zero scalar multiples of the columns of matrix S:

$$s \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{and} \quad t \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \tag{46}$$

where constants $s \neq 0 \neq t$. Why? What is the "magic" here? Well, let

$$A = SDS^{-1}, \quad \text{where} \quad D = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix} \quad \text{and} \quad S = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}.$$

Then

$$A \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} = AS = SDS^{-1}S = SD = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix} = \begin{bmatrix} \lambda a_1 & \mu b_1 \\ \lambda a_2 & \mu b_2 \end{bmatrix},$$

so that

$$A \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \lambda \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \quad \text{and} \quad A \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \mu \begin{bmatrix} b_1 \\ b_2 \end{bmatrix},$$

And similar thinking works for all matrices of the form

$$A = SDS^{-1} \in \mathbb{C}^{n \times n},$$

where $S \in \mathbb{C}^{n \times n}$ is invertible and $D \in \mathbb{C}^{n \times n}$ is **diagonal**, meaning that $D_{jk} = 0$ whenever $j \neq k$: then the eigenvalues of A are just the diagonal elements $\lambda_k := D_{kk} \in \mathbb{C}$ (where $k \in \{1, \cdots, n\}$), and a corresponding eigenvector is given by the kth column $S_k \in \mathbb{C}^{n \times 1}$ of matrix $S = [S_1 \cdots S_n]$:

$$A[S_1 \cdots S_n] = AS = SDS^{-1}S = SD = [\lambda_1 S_1 \cdots \lambda_n S_n].$$

This is related to so-called "diagonalization", which is the topic of the next Chapter.

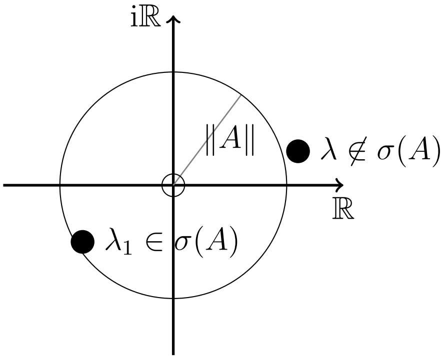


Figure 27: $|\lambda_1| \le \|A\|$ if $\lambda_1 \in \mathbb{C}$ is an eigenvalue of $A \in \mathbb{C}^{n\times n}$.

Eigenvalues/matrix norm. Recall that the norm of matrix $A \in \mathbb{C}^{n\times n}$ is

$$\begin{aligned}\|A\| &:= \max_{u\in\mathbb{C}^n:\|u\|\le 1} \|Au\| \\ &= \max_{u,v\in\mathbb{C}^n:\|u\|,\|v\|\le 1} |\langle Au, v\rangle| .\end{aligned}$$

If $\lambda_1 \in \sigma(A)$ then $|\lambda_1| \le \|A\|$, because if $Au = \lambda_1 u$ then

$$|\lambda_1|\,\|u\| = \|\lambda_1 u\| = \|Au\| \le \|A\|\,\|u\|.$$

So, if $|\lambda| > \|A\|$, then $\lambda \notin \sigma(A)$, and

$$\begin{aligned}&(\lambda I - A)\sum_{k=0}^{n}(A/\lambda)^k \\ = \;& (\lambda I - A)\left(I + (A/\lambda)^1 + (A/\lambda)^2 + \ldots + (A/\lambda)^n\right) \\ = \;& \lambda I I - AI + \lambda I(A/\lambda)^1 - A(A/\lambda)^1 + \lambda I(A/\lambda)^2 - \ldots - A(A/\lambda)^n \\ = \;& \lambda I - \lambda(A/\lambda)^{n+1} \quad \overset{n\to\infty}{\longrightarrow} \quad \lambda I,\end{aligned}$$

where the limit exists, because

$$\|(A/\lambda)^{n+1}\| = \|A^{n+1}\|/|\lambda|^{n+1} \le (\|A\|/|\lambda|)^{n+1} \overset{n\to\infty}{\longrightarrow} 0,$$

since here $\|A\|/|\lambda| < 1$. Hence

$$(\lambda I - A)^{-1} = \lambda^{-1}\sum_{k=0}^{\infty}(A/\lambda)^k.$$

Moreover, by applying the geometric series, we obtain the norm estimate

$$\begin{aligned}\left\|(\lambda I - A)^{-1}\right\| \quad &\le \quad |\lambda|^{-1}\sum_{k=0}^{\infty}\|A\|^k/|\lambda|^k \\ &\overset{|\lambda|\ge\|A\|}{=} \quad \frac{1}{|\lambda| - \|A\|}.\end{aligned}$$

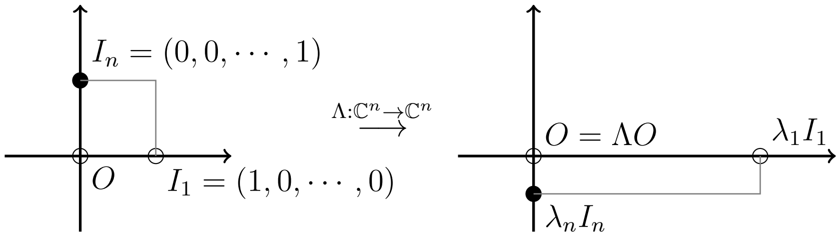


Figure 28: Action of a diagonal matrix $\Lambda \in \mathbb{C}^{n\times n}$ with $\Lambda_{kk} = \lambda_k$.

8. Diagonalization

Matrix $\Lambda = [\Lambda_{jk}]_{j,k=1}^n \in \mathbb{C}^{n\times n}$ is *diagonal* if $\Lambda_{jk} = 0$ whenever $j \neq k$. That is, with $\lambda_k := \Lambda_{kk} \in \mathbb{C}$ we have

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}.$$

Let $I_1, I_2, \cdots, I_n \in \mathbb{R}^{n\times 1}$ be the standard basis vectors. Here obviously $\Lambda(I_k) = \lambda_k I_k$, meaning that I_k is an eigenvector corresponding to the eigenvalue $\lambda_k \in \mathbb{C}$. Diagonal matrices are easy to sum and to multiply, e.g. here

$$\Lambda^{999} = \begin{bmatrix} (\lambda_1)^{999} & 0 & \cdots & 0 \\ 0 & (\lambda_2)^{999} & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & (\lambda_n)^{999} \end{bmatrix}.$$

Also, here Λ is invertible if and only if $\lambda_1\lambda_2\lambda_3\cdots\lambda_n \neq 0$, and then

$$\Lambda^{-1} = \begin{bmatrix} (\lambda_1)^{-1} & 0 & \cdots & 0 \\ 0 & (\lambda_2)^{-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & (\lambda_n)^{-1} \end{bmatrix}.$$

All this is to say: diagonal matrices are really easy to handle. It turns out that many other matrices can be *diagonalized*, making them easy to treat when seen from a suitable viewpoint.

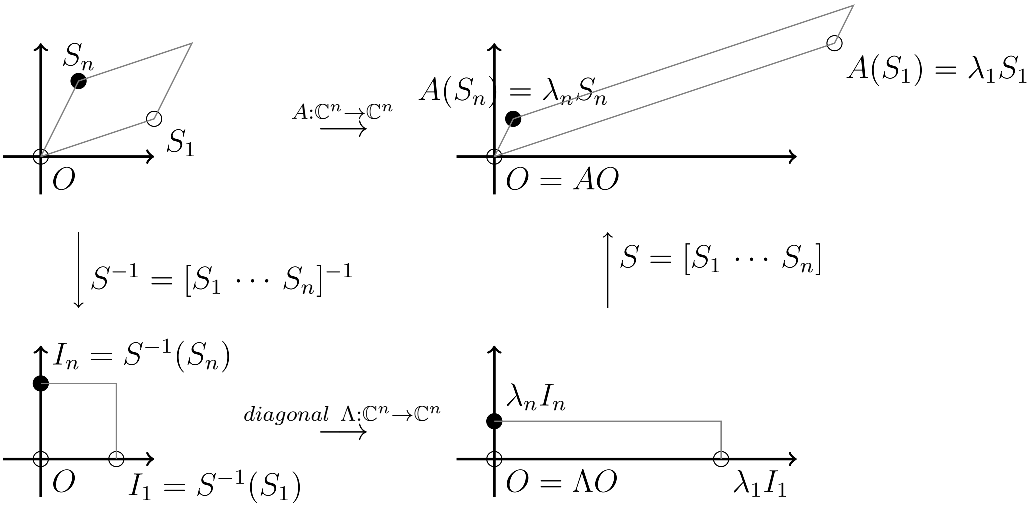


Figure 29: Diagonalization $\Lambda = S^{-1}AS$. In other words, $A = S\Lambda S^{-1}$.

Diagonalization. Matrix $A \in \mathbb{C}^{n\times n}$ can be *diagonalized* if there is invertible $S \in \mathbb{C}^{n\times n}$ for which

$$\Lambda = S^{-1}AS \in \mathbb{C}^{n\times n}$$

is diagonal, i.e. $\Lambda_{jk} = 0$ whenever $j \neq k$: then

$$A = S\Lambda S^{-1},$$

and (writing $\lambda_k := \Lambda_{kk} \in \mathbb{C}$) we have

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix},$$

where $\lambda_1, \lambda_2, \cdots, \lambda_n \in \mathbb{C}$ are the eigenvalues of A. The kth column S_k of $S = [S_1 \cdots S_n]$ is the eigenvector of A corresponding to the eigenvalue $\lambda_k \in \mathbb{C}$:

$$(AS)_{jk} = (S\Lambda S^{-1}S)_{jk} = (S\Lambda)_{jk} = \sum_{\ell=1}^{n} S_{j\ell}\,\Lambda_{\ell k} = \lambda_k\, S_{jk}.$$

We may say that a diagonalizable matrix $A = S\Lambda S^{-1}$ "looks like a diagonal matrix Λ after changing the coordinate point of view by S".

Remark: Such diagonalization $A = S\Lambda S^{-1}$, if it exists, may not be unique, as it changes if the choose different eigenvectors, or if we permute the order of the eigenvalues (and the corresponding order of the eigenvectors). For instance,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \begin{bmatrix} t_2 b & t_1 a \\ t_2 d & t_1 c \end{bmatrix} \begin{bmatrix} \lambda_2 & 0 \\ 0 & \lambda_1 \end{bmatrix} \begin{bmatrix} t_2 b & t_1 a \\ t_2 d & t_1 c \end{bmatrix}^{-1}$$

whenever $ad - bc \neq 0 \neq t_1 t_2$. Here, eigenvectors

$$t_1 \begin{bmatrix} a \\ c \end{bmatrix}, \quad t_2 \begin{bmatrix} b \\ d \end{bmatrix}$$

respectively correspond to the eigenvalues $\lambda_1, \lambda_2 \in \mathbb{C}$. Nevertheless, we have:

Theorem. *Matrix $A \in \mathbb{C}^{n\times n}$ can be diagonalized if and only if $m_g(\lambda) = m_a(\lambda)$ for all the eigenvalues λ of A.*

Remark: For a matrix $A \in \mathbb{C}^{n\times n}$, remember that $1 \leq m_g(\lambda) \leq m_a(\lambda) \leq n$ always. Matrix $A \in \mathbb{C}^{n\times n}$ cannot be diagonalized if $m_g(\lambda) < m_a(\lambda)$ for just one eigenvalue λ of A, as then we cannot find enough eigenvectors to achieve the invertible eigenvector matrix S eventually! In the last two examples in the previous Chapter (Eigenvalues and eigenvectors), the eigenvalues of the diagonalizable matrices $R, Q : \mathbb{C}^2 \to \mathbb{C}^2$ have all multiplicities 1.

Example. Let $[A] = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$, where $b \neq 0$ (so $[A]$ is not diagonal). Now

$$\begin{aligned} p_A(z) &= \det[A - zI] \\ &= \det \begin{bmatrix} 1-z & b \\ 0 & 1-z \end{bmatrix} \\ &= (1-z)^2, \end{aligned}$$

giving algebraic multiplicity $m_a(\lambda) = 2$ for eigenvalue $\lambda = 1$. As $b \neq 0$, then

$$(A - \lambda I)(x) = O$$

has only solution $x = (t, 0)$ for constants $t \in \mathbb{C}$. So the geometric multiplicity

$$m_g(\lambda) = 1 < m_a(\lambda).$$

This is an example of a non-diagonalizable matrix! In other words, here $S^{-1}AS$ is not diagonal no matter which invertible S we choose.

$$L = \begin{bmatrix} L_{11} & 0 & 0 & \cdots & 0 \\ L_{21} & L_{22} & 0 & \cdots & 0 \\ L_{31} & L_{32} & L_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ L_{n1} & L_{n2} & L_{n3} & \cdots & L_{nn} \end{bmatrix}, \quad U = \begin{bmatrix} U_{11} & U_{12} & U_{13} & \cdots & U_{1n} \\ 0 & U_{22} & U_{23} & \cdots & U_{2n} \\ 0 & 0 & U_{33} & \cdots & U_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & U_{nn} \end{bmatrix}.$$

Figure 30: Lower triangular $L \in \mathbb{C}^{n \times n}$, upper triangular $U \in \mathbb{C}^{n \times n}$.

Example. Let $\lambda \in \mathbb{C}$ be the only eigenvalue of diagonalizable $A \in \mathbb{C}^{n \times n}$. Then

$$A = S(\lambda I)S^{-1} = \lambda SS^{-1} = \lambda I.$$

That is, $A =$ *constant times identity*.

Definition. Matrix $M \in \mathbb{C}^{n \times n}$ is *triangular* if it is
lower triangular ($M_{jk} = 0$ whenever $j < k$) or
upper triangular ($M_{jk} = 0$ whenever $j > k$).
Especially, if M is both upper and lower triangular then it is diagonal.

Example. If triangular A has zero diagonal, then $\lambda = 0$ is its only eigenvalue; by the previous example, such A can be diagonalized only if $A = O$.

Example. $AS = S\Lambda$, if

$$A = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}, \quad S = \begin{bmatrix} 1 & -b \\ 0 & a \end{bmatrix}, \quad \Lambda = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}.$$

Thus if $a \neq 0$ then we have $A = S\Lambda S^{-1}$.
(If $a = 0$ then S is not invertible.)

Remark. On page 61 we learn that matrix is *normal* (see page 53) if and only if it has a unitary diagonalization. In the previous example above, $A = S\Lambda S^{-1}$ is a normal matrix if and only if $b = 0$: thus some non-normal matrices can be diagonalized (yet not unitarily diagonalized).

Functions of square matrices

Analytic function $f : \mathbb{C} \to \mathbb{C}$ can be presented as power series

$$f(z) = \sum_{k=0}^{\infty} c_k \, z^k. \tag{47}$$

E.g. functions $\exp, \cos, \sin$ are analytic. Define $f : \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}$ by

$$f(A) = \sum_{k=0}^{\infty} c_k \, A^k. \tag{48}$$

Here $f(A)_{ij} \neq f(A_{ij})$ often! But with diagonalization $A = S\Lambda S^{-1}$,

$$f(A) = \sum_{k=0}^{\infty} c_k \, (S\Lambda S^{-1})^k = \cdots = S \left(\sum_{k=0}^{\infty} c_k \, \Lambda^k \right) S^{-1} = S \; f(\Lambda) \; S^{-1},$$

where $f(\Lambda) \in \mathbb{C}^{n\times n}$ is diagonal with $f(\Lambda)_{jj} = f(\Lambda_{jj}) \in \mathbb{C}$. Nice!

Example. Let $A = S\Lambda S^{-1}$, where

$$\Lambda = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}.$$

Then

$$\begin{aligned} \exp(A) &= S \, \exp(\Lambda) \, S^{-1}, \\ |A|^{1/2} &= S|\Lambda|^{1/2} \, S^{-1}, \end{aligned}$$

where

$$\exp(\Lambda) = \begin{bmatrix} \mathrm{e}^a & 0 & 0 \\ 0 & \mathrm{e}^b & 0 \\ 0 & 0 & \mathrm{e}^c \end{bmatrix}, \quad |\Lambda|^{1/2} = \begin{bmatrix} \sqrt{|a|} & 0 & 0 \\ 0 & \sqrt{|b|} & 0 \\ 0 & 0 & \sqrt{|c|} \end{bmatrix}.$$

So, what would be $|A|$ then?

Application to differential equations: Let $A \in \mathbb{C}^{n\times n}$. Suppose that smooth functions $x_1, \cdots, x_n : \mathbb{R} \to \mathbb{C}$ satisfy

$$x'(t) = A \, x(t)$$

(where $(x'(t))_j = (x_j)'(t) = \frac{\mathrm{d}}{\mathrm{d}t} x_j(t)$, naturally), then

$$x(t) = \exp(tA) \, x(0).$$

If here $A = S\Lambda S^{-1}$ then $\exp(tA) = S \exp(t\Lambda) S^{-1}$, which is easy to find.

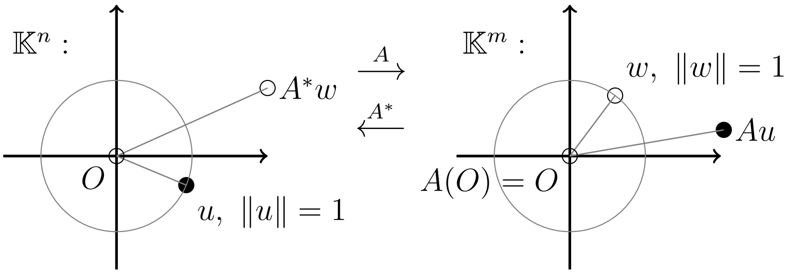


Figure 31: Adjoint $A^* : \mathbb{K}^m \to \mathbb{K}^n$ is the mirror image of linear $A : \mathbb{K}^n \to \mathbb{K}^m$ in the sense that $\langle u, A^*w\rangle = \langle Au, w\rangle$ for all $u \in \mathbb{K}^n$ and $w \in \mathbb{K}^m$.

9. Adjoint matrices

For a linear mapping $A : \mathbb{K}^n \to \mathbb{K}^m$, the *adjoint* is the linear mapping $A^* : \mathbb{K}^m \to \mathbb{K}^n$ defined by the inner product duality

$$\langle Au, w\rangle =: \langle u, A^*w\rangle \tag{49}$$

for all $u \in \mathbb{K}^n$ and $w \in \mathbb{K}^m$. Why this definition works? Indeed,

$$\langle Au, w\rangle \quad = \quad \sum_{j=1}^{m} (Au)_j \, \overline{w_j} = \sum_{j=1}^{m}\sum_{k=1}^{n} A_{jk}\, u_k\, \overline{w_j} = \sum_{k=1}^{n} u_k \left(\sum_{j=1}^{m} \overline{A_{jk}} w_j \right)^{*},$$

which shows that the adjoint A^* has the matrix elements

$$(A^*)_{kj} = \overline{A_{jk}} \in \mathbb{K}.$$

Clearly $(A^*)^* = A$. There is the norm equality $\|A^*\| = \|A\|$, because

$$\begin{aligned} \|A\| \quad &= \max_{u\in\mathbb{C}^n, v\in\mathbb{C}^m : \|u\|,\|v\|\le 1} |\langle Au, v\rangle| \\ &= \max_{u\in\mathbb{C}^n, v\in\mathbb{C}^m : \|u\|,\|v\|\le 1} |\langle u, A^*v\rangle| \quad = \quad \|A^*\|. \end{aligned}$$

Example. The adjoint of $A = \begin{bmatrix} 1+2\mathrm{i} & 3+4\mathrm{i} & 5+6\mathrm{i} \\ 7+8\mathrm{i} & 9+10\mathrm{i} & 11+12\mathrm{i} \end{bmatrix} \in \mathbb{C}^{2\times 3}$ is

$$A^* = \begin{bmatrix} 1-2\mathrm{i} & 7-8\mathrm{i} \\ 3-4\mathrm{i} & 9-10\mathrm{i} \\ 5-6\mathrm{i} & 11-12\mathrm{i} \end{bmatrix} \in \mathbb{C}^{3\times 2}.$$

Important classes of matrices

Definition. Let us introduce the following classes of square matrices:
Matrix $P \in \mathbb{C}^{n\times n}$ is *positive* if $\langle Px, x\rangle \geq 0$ for all $x \in \mathbb{C}^n$.
Matrix $S \in \mathbb{C}^{n\times n}$ is *symmetric* (or *self-adjoint*) if $S^* = S$.
Matrix $N \in \mathbb{C}^{n\times n}$ is *normal* if $N^*N = NN^*$.
Matrix $U \in \mathbb{C}^{n\times n}$ is *unitary* if $U^* = U^{-1}$: then

$$\langle Ux, Uy\rangle = \langle x, y\rangle$$

for all $x, y \in \mathbb{C}^n$; such U preserves distances and angles in $\mathbb{C}^n$!

Definition. For $A \in \mathbb{R}^{m\times n}$, the adjoint is the transpose $A^{\mathrm{T}} = A^*$. A real unitary matrix $A \in \mathbb{R}^{n\times n}$ is called *orthogonal*: this means $A^{\mathrm{T}}A = I = A\,A^{\mathrm{T}}$, in other words $A^{\mathrm{T}} = A^{-1}$ here.

Remark. These classes of matrices have some connections:
Above, $Positive \Rightarrow Symmetric$, and $Symmetric \Rightarrow Normal$,
but $Normal \not\Rightarrow Symmetric$, and $Symmetric \not\Rightarrow Positive$.
Also, $Unitary \Rightarrow Normal$, but $Normal \not\Rightarrow Unitary$.
Term "*positive*" is often more accurately "*positive semi-definite*".

Exercise. Let us study a diagonal matrix $\Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \in \mathbb{C}^{n\times n}$.
For which $\lambda_k \in \mathbb{C}$ the following properties hold?
(a) Λ is positive.
(b) Λ is symmetric.
(c) Λ is normal.
(d) Λ is unitary.

Remark. Condition $U^*U = I$ means that the columns $U_k \in \mathbb{C}^{n\times 1}$ of unitary $U = [U_1 \cdots U_n] \in \mathbb{C}^{n\times n}$ are mutually orthonormal to each other, as

$$I_{jk} = (U^*U)_{jk} = {U_j}^* U_k = \langle U_k, U_j\rangle,$$

where $I_{kk} = 1$ and $I_{jk} = 0$ when $j \neq k$. Moreover, if $A \in \mathbb{C}^{n\times n}$ is

$$positive/symmetric/normal/unitary$$

then also U^*AU has the same property.

Eigenproperties for symmetric / positive / unitary

Claim: *The eigenvalues of a symmetric matrix are real.*

Proof. Let $\lambda \in \mathbb{C}$ be an eigenvalue of a symmetric matrix $S = S^* \in \mathbb{C}^{n\times n}$ with eigenvector $x \in \mathbb{C}^{n\times 1}$. Then

$$\begin{array}{rclcl} \lambda\langle x, x\rangle & = & \langle \lambda x, x\rangle & & \\ & \stackrel{Sx=\lambda x}{=} & \langle Sx, x\rangle & = & \langle x, S^* x\rangle \\ & \stackrel{S^*=S}{=} & \langle x, Sx\rangle & & \\ & \stackrel{Sx=\lambda x}{=} & \langle x, \lambda x\rangle & = & \overline{\lambda}\langle x, x\rangle. \end{array}$$

Thus $\lambda = \overline{\lambda}$ (i.e. $\lambda \in \mathbb{R}$), because $\langle x, x\rangle = \|x\|^2 \neq 0$. **QED**

Claim: *Positive $P \in \mathbb{C}^{n\times n}$ is symmetric, and its eigenvalues are ≥ 0.*

Proof. By the Weak Formulation Theorem we have $P = P^*$, as

$$\langle Px, x\rangle = \langle x, Px\rangle^* \stackrel{real}{=} \langle x, Px\rangle = \langle P^* x, x\rangle$$

for any $x \in \mathbb{C}^n$. Let $Pu = \lambda u$, where $u \neq O$ eigenvector with eigenvalue $\lambda \in \mathbb{C}$. Then

$$0 \leq \langle Pu, u\rangle = \langle \lambda u, u\rangle = \lambda\langle u, u\rangle = \lambda\|u\|^2,$$

so that $0 \leq \lambda$. **QED**

Remark: If $A^* = A$ such that $Au = \lambda u$ and $Av = \mu v$ for $\lambda, \mu \in \mathbb{R}$ then

$$\lambda\langle u, v\rangle = \langle \lambda u, v\rangle = \langle Au, v\rangle \stackrel{A^*=A}{=} \langle u, Av\rangle = \langle u, \mu v\rangle \stackrel{\mu\in\mathbb{R}}{=} \mu\langle u, v\rangle.$$

Hence if here $\lambda \neq \mu$, then we must have $\langle u, v\rangle = 0$. Think about this! (Eigenvectors corresponding to different eigenvalues of a symmetric matrix are automatically orthogonal.)

Claim: *The eigenvalues $\lambda \in \mathbb{C}$ of unitary $U \in \mathbb{C}^{n\times n}$ satisfy $|\lambda| = 1$.*

Proof. Let $Ux = \lambda x$, where $\langle x, x\rangle = \|x\|^2 = 1$. Then

$$|\lambda|^2 = \lambda\lambda^*\langle x, x\rangle = \langle \lambda x, \lambda x\rangle = \langle Ux, Ux\rangle = \langle x, x\rangle = 1.$$

QED

Example. The *Discrete Fourier Transform* (DFT) of vector $v \in \mathbb{C}^n$ is $\widehat{v} \in \mathbb{C}^n$, where

$$\widehat{v}_j := \sum_{k=1}^{n} \mathrm{e}^{-\mathrm{i}2\pi jk/n}\, v_k. \tag{50}$$

Let us define $U : \mathbb{C}^n \to \mathbb{C}^n$ by $Uv := \widehat{v}/\sqrt{n}$, that is

$$U_{jk} = \frac{1}{\sqrt{n}}\, \mathrm{e}^{-\mathrm{i}2\pi jk/n}.$$

Then U is unitary:

$$\begin{aligned} (U^*U)_{jk} &= \sum_{\ell=1}^{n} (U^*)_{j\ell}\, U_{\ell k} \\ &= \sum_{k=1}^{n} \overline{U_{\ell j}}\, U_{\ell k} \\ &= n^{-1} \sum_{k=1}^{n} \mathrm{e}^{\mathrm{i}2\pi(j-k)\ell/n} \\ &= \begin{cases} 1, & \text{if } j = k, \\ 0, & \text{if } j \neq k \end{cases}, \end{aligned}$$

where in the last step we applied the geometric sum formula

$$\sum_{k=1}^{n} q^k \overset{q \neq 1}{=} q\, \frac{1-q^n}{1-q}.$$

Remark. Invertibility of matrix $S = [S_1 \cdots S_n] \in \mathbb{C}^{n\times n}$ means that the column vectors $S_1, \cdots, S_n$ are linearly independent; out of these vectors the Gram–Schmidt algorithm gives orthonormal vectors $U_1, \cdots, U_n$, and then $U = [U_1 \cdots U_n] \in \mathbb{C}^{n\times n}$ is unitary. Moreover, here

$$\operatorname{span}\{S_1, \cdots, S_k\} = \operatorname{span}\{U_1, \cdots, U_k\}$$

for each $k \in \{1, \cdots, n\}$. We return to this remark in the next Chapter.

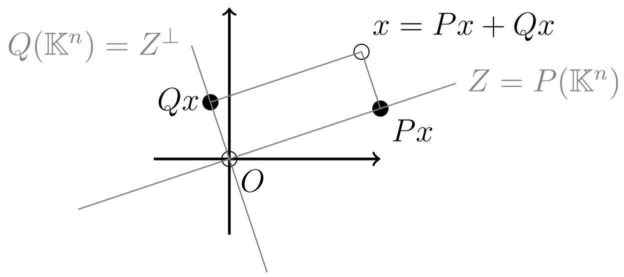


Figure 32: Orthogonal projection $P \in \mathbb{K}^{n\times n}$ onto vector subspace $Z \subset \mathbb{K}^n$. Then the linear mapping $Q = I - P$ is the orthogonal projection onto the orthogonal vector subspace $Z^\perp = \{u \in \mathbb{K}^n : \langle u, z\rangle = 0 \text{ for all } z \in Z\}$.

Nice to know: orthogonal projections

For a vector subspace $Z \subset \mathbb{K}^n$ of dimension $d = \dim(Z)$ there are linearly independent vectors $S_1, \cdots, S_d \in \mathbb{K}^n$ such that $Z = \operatorname{span}\{S_1, \cdots, S_d\}$. From the sequence $(S_1, \cdots, S_d, I_1, I_2, \cdots, I_n)$, after throwing away the early vectors that violate the linear independence, the Gram–Schmidt process finds the orthonormal vectors $U_1, \cdots, U_n \in \mathbb{K}^n$ such that $Z = \operatorname{span}\{U_1, \cdots, U_d\}$. If we define $P : \mathbb{K}^n \to \mathbb{K}^n$ by

$$Px := \sum_{k=1}^{d} \langle x, U_k\rangle\, U_k, \tag{51}$$

then P is the *orthogonal projection onto* Z, satisfying $P = P^* = P^2$.

Remark: If $P \in \mathbb{K}^{n\times n}$ satisfies $P = P^* = P^2$ then $P = [P_1 \cdots P_n]$ is the orthogonal projection onto $Z := P(\mathbb{K}^n) = \operatorname{span}\{P_1, \cdots, P_n\}$.

The idea: $Px \in Z$ is the closest point in Z to $x \in \mathbb{K}^n$. In other words, Px is the "perpendicular shadow of point x on subspace Z". Now let $Q = I - P$, that is $x = Px + Qx$ (for all $x \in \mathbb{K}^n$). Then we have the Pythagorean identity

$$\|x\|^2 = \|Px\|^2 + \|Qx\|^2.$$

Here, Q is the orthogonal projection onto the orthogonal vector subspace $Z^\perp = \{u \in \mathbb{K}^n : \langle u, z\rangle = 0 \text{ for all } z \in Z\}$. Notice that $PQ = O = QP$, and

$$\begin{array}{rcccl} P(\mathbb{K}^n) & = & Z & = & \{z \in \mathbb{K}^n : Qz = O\}, \\ Q(\mathbb{K}^n) & = & Z^\perp & = & \{z \in \mathbb{K}^n : Pz = O\}. \end{array}$$

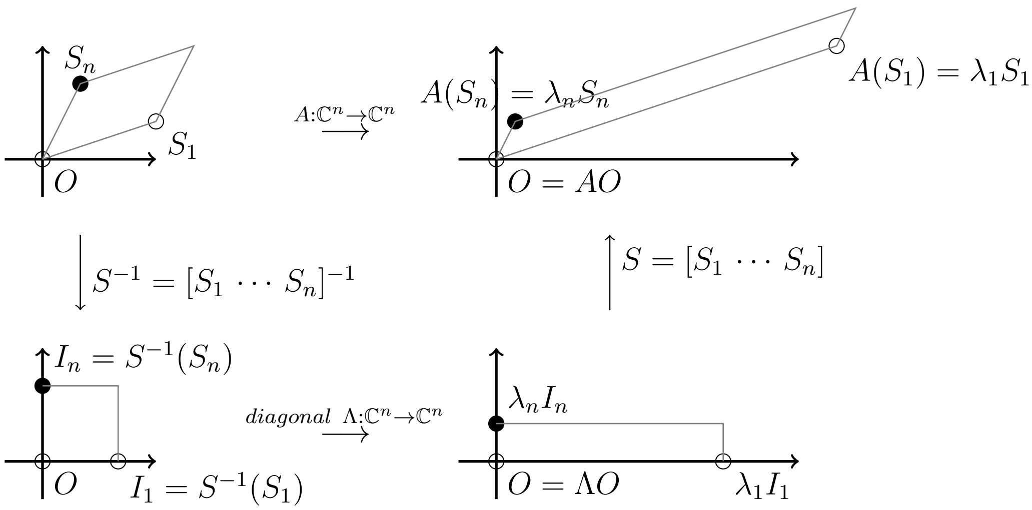


Figure 33: Diagonalization $\Lambda = S^{-1}AS$. In other words, $A = S\Lambda S^{-1}$.

10. Unitary diagonalization

Remember the following matrix properties:

- $M \in \mathbb{C}^{n\times n}$ is *symmetric* (or *self-adjoint*) if $M^* = M$.
- $B \in \mathbb{C}^{n\times n}$ is *normal* if $B^*B = B\,B^*$.
- $U \in \mathbb{C}^{n\times n}$ is *unitary* if $U^* = U^{-1}$.

Also, remember what the diagonalization $\Lambda = S^{-1}AS$ for a matrix $A \in \mathbb{C}^{n\times n}$ means: previously we just required $S \in \mathbb{C}^{n\times n}$ to be invertible. In *unitary* diagonalization, we want S *unitary*: remember that unitary matrices preserve inner products and norms! In order for this to be possible, something has to be assumed from A, but fortunately this assumption turns out to be quite natural: namely, A should be *normal*!

Example. Let $A = \begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix}$, where $b, c \in \mathbb{C}$. Then $A^* = \begin{bmatrix} 0 & c^* \\ b^* & 0 \end{bmatrix}$. Thus here A is symmetric if and only if $c = b^*$. Moreover,

$$\begin{aligned} A^*A &= \begin{bmatrix} 0 & c^* \\ b^* & 0 \end{bmatrix}\begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix} &= \begin{bmatrix} |c|^2 & 0 \\ 0 & |b|^2 \end{bmatrix}, \\ AA^* &= \begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix}\begin{bmatrix} 0 & c^* \\ b^* & 0 \end{bmatrix} &= \begin{bmatrix} |b|^2 & 0 \\ 0 & |c|^2 \end{bmatrix}. \end{aligned}$$

Hence here A is normal if and only if $|b| = |c|$.

Gram–Schmidt process reformulated

Let matrix $S = [S_1 \cdots S_n] \in \mathbb{C}^{n\times n}$ be invertible, with columns $S_k \in \mathbb{C}^{n\times 1}$. The *Gram–Schmidt process* gives unitary $U = [U_1 \cdots U_n] \in \mathbb{C}^{n\times n}$ as follows: Let

$$U_1 := S_1/\|S_1\|, \tag{52}$$

and for $k \geq 2$ then

$$U_k := \frac{V_k}{\|V_k\|}, \tag{53}$$

where

$$V_k := S_k - \sum_{j=1}^{k-1} \langle S_k, U_j\rangle\, U_j. \tag{54}$$

It is trivial that $\langle U_k, U_k\rangle = 1$. Suppose we know that $U_1, \cdots, U_{k-1}$ are orthonormal. Then $U_1, \cdots, U_k$ are also orthonormal, as

$$\langle V_k, U_\ell\rangle = \langle S_k, U_\ell\rangle - \sum_{j=1}^{k-1} \langle S_k, U_j\rangle\langle U_j, U_\ell\rangle = \langle S_k, U_\ell\rangle - \langle S_k, U_\ell\rangle = 0.$$

Hence $U_1, \cdots, U_n$ are orthonormal, so $U^* = U^{-1}$. Also notice that for all $k \in \{1, \cdots, n\}$ here

$$\mathrm{span}\{S_1, \cdots, S_k\} = \mathrm{span}\{U_1, \cdots, U_k\}.$$

This Gram–Schmidt process is numerically unstable (round-off errors accumulate), but there are ways to stabilize the process.

Example. Let $S = \begin{bmatrix} 4 & 6 \\ 3 & 2 \end{bmatrix}$. Then

$$U_1 = S_1/\|S_1\| = \begin{bmatrix} 4 \\ 3 \end{bmatrix} /\sqrt{4^2 + 3^2} = \begin{bmatrix} 4/5 \\ 3/5 \end{bmatrix},$$

and

$$\begin{aligned} V_2 &= S_2 - \langle S_1, U_1\rangle\, U_1 \\ &= \begin{bmatrix} 6 \\ 2 \end{bmatrix} - \langle \begin{bmatrix} 6 \\ 2 \end{bmatrix}, \begin{bmatrix} 4/5 \\ 3/5 \end{bmatrix} \rangle \begin{bmatrix} 4/5 \\ 3/5 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \end{bmatrix} - 6 \begin{bmatrix} 4/5 \\ 3/5 \end{bmatrix} = \begin{bmatrix} 6/5 \\ -8/5 \end{bmatrix}, \end{aligned}$$

so $U_2 = \frac{V_2}{\|V_2\|} = \begin{bmatrix} 3/5 \\ -4/5 \end{bmatrix}$. We have

$$U = [U_1\ U_2] = \begin{bmatrix} 4/5 & 3/5 \\ 3/5 & -4/5 \end{bmatrix}.$$

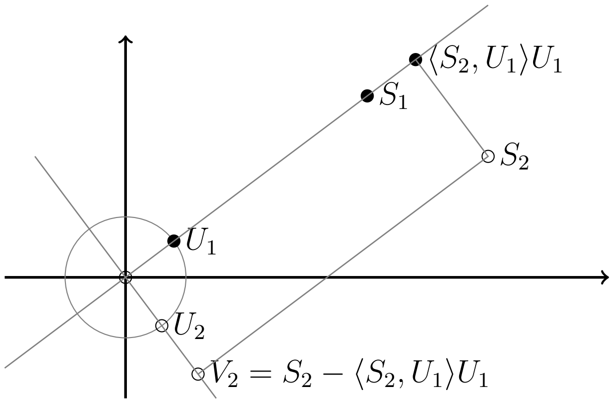


Figure 34: Gram–Schmidt: from invertible $S = [S_1 \cdots S_n] \in \mathbb{C}^{n\times n}$ to unitary $U = [U_1 \cdots U_n] \in \mathbb{C}^{n\times n}$ such that $\mathrm{span}\{S_1, \cdots, S_k\} = \mathrm{span}\{U_1, \cdots, U_k\}$.

Unitary triangulation of square matrices

Unitary triangulation of a matrix $A \in \mathbb{C}^{n\times n}$ means that

$$A = U\Lambda U^* \tag{55}$$

for a unitary matrix $U \in \mathbb{C}^{n\times n}$ and an upper triangular matrix $\Lambda \in \mathbb{C}^{n\times n}$. We prove (55) by induction, by reducing case $n > 1$ to case $n-1$ (case $n = 1$ is trivial). Take an eigenvalue $\lambda \in \mathbb{C}$ with a normalized eigenvector $v \in \mathbb{C}^{n\times 1}$:

$$Av = \lambda v, \quad \|v\| = 1.$$

By Gram–Schmidt, find unitary $V \in \mathbb{C}^{n\times n}$ with the first column v. So

$$A = V \begin{bmatrix} \lambda & w \\ O & \widetilde{A} \end{bmatrix} V^* = V \begin{bmatrix} \lambda & w \\ O & \widetilde{U}\widetilde{\Lambda}\widetilde{U}^* \end{bmatrix} V^*$$

for some $\widetilde{A} \in \mathbb{C}^{(n-1)\times(n-1)}$ and $w \in \mathbb{C}^{1\times(n-1)}$, where $O \in \mathbb{R}^{(n-1)\times 1}$ is the zero column vector: due to the case $n-1$ of unitary triangulation, $\widetilde{A} = \widetilde{U}\,\widetilde{\Lambda}\widetilde{U}^*$, where $\widetilde{U} \in \mathbb{C}^{(n-1)\times(n-1)}$ is unitary and $\widetilde{\Lambda} \in \mathbb{C}^{(n-1)\times(n-1)}$ is upper triangular. Define unitary U and upper triangular Λ by

$$U := V \begin{bmatrix} 1 & O^* \\ O & \widetilde{U} \end{bmatrix}, \qquad \Lambda := \begin{bmatrix} \lambda & w\widetilde{U} \\ O & \widetilde{\Lambda} \end{bmatrix}$$

(check that U is unitary and Λ upper triangular). Here (55) holds, because

$$\begin{aligned} U\Lambda U^* &= V \begin{bmatrix} 1 & O^* \\ O & \widetilde{U} \end{bmatrix} \begin{bmatrix} \lambda & w\widetilde{U} \\ O & \widetilde{\Lambda} \end{bmatrix} \begin{bmatrix} 1 & O^* \\ O & \widetilde{U}^* \end{bmatrix} V^* \\ &= V \begin{bmatrix} 1 & O^* \\ O & \widetilde{U} \end{bmatrix} \begin{bmatrix} \lambda & w \\ O & \widetilde{\Lambda}\widetilde{U}^* \end{bmatrix} V^* = V \begin{bmatrix} \lambda & w \\ O & \widetilde{U}\widetilde{\Lambda}\widetilde{U}^* \end{bmatrix} V^* = A. \end{aligned}$$

$$\Lambda = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} & \Lambda_{13} & \cdots & \Lambda_{1n} \\ 0 & \Lambda_{22} & \Lambda_{23} & \cdots & \Lambda_{2n} \\ 0 & 0 & \Lambda_{33} & \cdots & \Lambda_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \Lambda_{nn} \end{bmatrix} \overset{\Lambda^*\Lambda=\Lambda\Lambda^*}{\Longrightarrow} \Lambda = \begin{bmatrix} \Lambda_{11} & 0 & 0 & \cdots & 0 \\ 0 & \Lambda_{22} & 0 & \cdots & 0 \\ 0 & 0 & \Lambda_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \Lambda_{nn} \end{bmatrix}.$$

Figure 35: Any normal (upper or lower) triangular matrix is diagonal! And what if the matrix would be symmetric and triangular?

Normal triangular matrix is diagonal!

Let $\Lambda \in \mathbb{C}^{n\times n}$. Then

$$\begin{aligned} (\Lambda^*\Lambda)_{kk} &= \sum_{\ell=1}^{n} (\Lambda^*)_{k\ell}\, \Lambda_{\ell k} &= \sum_{\ell=1}^{n} \overline{\Lambda_{\ell k}}\, \Lambda_{\ell k} &= \sum_{\ell=1}^{n} |\Lambda_{\ell k}|^2, \\ (\Lambda\Lambda^*)_{kk} &= \sum_{\ell=1}^{n} \Lambda_{k\ell}\, (\Lambda^*)_{\ell k} &= \sum_{\ell=1}^{n} \Lambda_{k\ell}\overline{\Lambda_{k\ell}} &= \sum_{\ell=1}^{n} |\Lambda_{k\ell}|^2. \end{aligned}$$

Let Λ be normal ($\Lambda^*\Lambda = \Lambda\Lambda^*$) and upper triangular ($\Lambda_{ij} = 0$ if $i > j$). Then

$$|\Lambda_{nn}|^2 = \sum_{\ell=1}^{n} |\Lambda_{n\ell}|^2 = (\Lambda\Lambda^*)_{nn} = (\Lambda^*\Lambda)_{nn} = \sum_{\ell=1}^{n} |\Lambda_{\ell n}|^2 = |\Lambda_{nn}|^2 + \sum_{\ell=1}^{n-1} |\Lambda_{\ell n}|^2,$$

which shows that $\Lambda_{\ell n} = 0$ for $\ell \in \{1, \cdots, n-1\}$. Thus

$$\Lambda = \begin{bmatrix} \widetilde{\Lambda} & O \\ O^* & \Lambda_{nn} \end{bmatrix},$$

where $\widetilde{\Lambda} \in \mathbb{C}^{(n-1)\times(n-1)}$ is a normal upper triangular matrix and $O \in \mathbb{R}^{(n-1)\times 1}$ is the zero vector. By reducing dimensions $n > n-1 > \cdots > 1$, we get:

Theorem. *Normal triangular matrices $\Lambda \in \mathbb{C}^{n\times n}$ are diagonal.*
(A trivial result: symmetric triangular matrices are diagonal and real! Think about that!)

Exercise. Now we know that any matrix $A \in \mathbb{C}^{n\times n}$ is of the form $A = U\Lambda U^*$, where $U \in \mathbb{C}^{n\times n}$ is unitary and $\Lambda \in \mathbb{C}^{n\times n}$ is upper triangular. Show that here A is normal if and only if Λ is normal (in which case Λ must be diagonal by the Theorem above!).

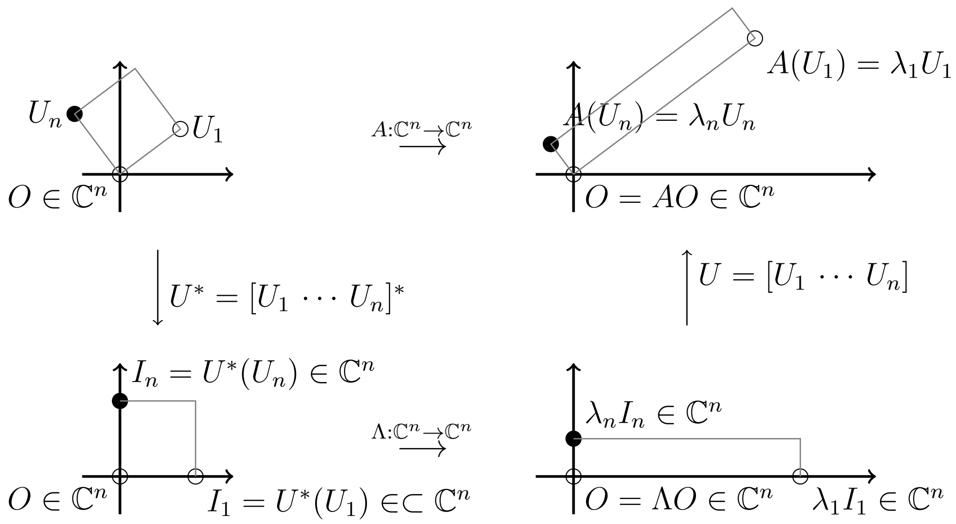


Figure 36: Idea of unitary diagonalization $\Lambda = U^*AU$ of normal $A \in \mathbb{C}^{n\times n}$. Equivalently, this means $A = U\Lambda U^*$. Unitary operations U^* and U preserve distances and angles.

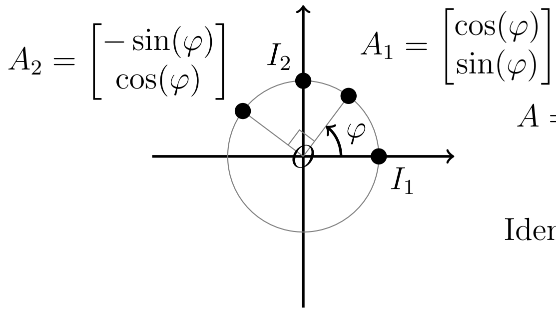
Unitary diagonalization of normal matrices

Thus, for any $A \in \mathbb{C}^{n\times n}$ there is unitary triangulation $A = U\Lambda U^*$, where $U \in \mathbb{C}^{n\times n}$ is unitary and $\Lambda \in \mathbb{C}^{n\times n}$ is upper triangular. It is easy to see that here Λ is normal if and only if A is normal. Because normal triangular matrices are diagonal (see page 60), we get:

Normal matrices can be diagonalized by unitary matrices!
More precisely:

Theorem. *Conditions (1) and (2) are equivalent:*
(1) $A \in \mathbb{C}^{n\times n}$ *is normal (that is,* $A^*A = AA^*$*).*
(2) $A = U\Lambda U^*$ *for unitary* $U \in \mathbb{C}^{n\times n}$ *and diagonal* $\Lambda \in \mathbb{C}^{n\times n}$*.*

Remark: In this result on the unitary diagonalization $A = U\Lambda U^*$, we have $A^* = A$ if and only if $\Lambda^* = \Lambda$ (Why? Compute this!). So, a normal matrix is symmetric if and only if its eigenvalues are real. However, there are non-normal diagonalizable matrices with real eigenvalues: see the example on page 50.



$$A = [A_1 \ A_2] = \begin{bmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{bmatrix}.$$

$$\text{Identity matrix } I = [I_1 \ I_2] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Figure 37: Rotation $A \in \mathbb{R}^{2\times 2}$ by angle φ around the origin.

Eigenvalues in unitary diagonalization

Let $A \in \mathbb{C}^{n\times n}$ be normal, that is $A^*A = AA^*$. We saw that this is equivalent to the existence of unitary diagonalization $A = U\Lambda U^*$. For normal $A \in \mathbb{C}^{n\times n}$ and **all** its eigenvalues $\lambda \in \mathbb{C}$, the reader can find out the following facts:

(a) $A^* = A^{-1}$ (unitary A) if and only if $|\lambda| = 1$.

(b) $A^* = A$ (symmetric A) if and only if $\lambda \in \mathbb{R}$.

(c) $\langle Ax, x\rangle \geq 0$ for all $x \in \mathbb{C}^n$ (positive A) if and only if $\lambda \geq 0$.

(d) $A^* = A = A^2$ (orthogonal projection A) if and only if $\lambda \in \{0, 1\}$.

Exercise. Prove these above-mentioned “if and only if” claims (a,b,c,d).

Example. In particular, orthogonal projections are always positive, and positive operators are always symmetric.
The only unitary positive operator is the identity $I \in \mathbb{C}^{n\times n}$.
The only unitary orthogonal projection is the identity $I \in \mathbb{C}^{n\times n}$.

Example. Let us find a unitary diagonalization for the rotation matrix

$$A = \begin{bmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{bmatrix} \in \mathbb{R}^{2\times 2}.$$

Then the characteristic polynomial satisfies

$$\det[A - zI] = z^2 - 2\cos(\varphi)z + 1 = (z - \lambda_1)(z - \lambda_2),$$

where $\lambda_k = \cos(\varphi) \pm \mathrm{i}\sin(\varphi) \overset{\text{Euler}}{=} \mathrm{e}^{\pm\mathrm{i}\varphi}$. Then $A = U\Lambda U^*$, where e.g.

$$\Lambda = \begin{bmatrix} \mathrm{e}^{\mathrm{i}\varphi} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\varphi} \end{bmatrix}, \qquad U = \frac{1}{\sqrt{2}} \begin{bmatrix} \mathrm{i} & \mathrm{i} \\ 1 & -1 \end{bmatrix}.$$

Check this!

Example. If we already know diagonalization $R = SDS^{-1}$, this sometimes helps in finding a unitary diagonalization: the eigenvectors might already be orthogonal (especially if R is symmetric with distinct eigenvalues), and then we just need to normalize the eigenvectors! For instance, consider

$$R = SDS^{-1} = \begin{bmatrix} 5 & 2 \\ 2 & 8 \end{bmatrix}, \quad \text{where} \quad D = \begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix} \quad \text{and} \quad S = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}. \tag{56}$$

Clearly, the columns S_1, S_2 of matrix $S = [S_1\ S_2]$ are eigenvectors corresponding to the respective eigenvalues are 9 and 4. Since the eigenvalues of this symmetric matrix are different, this automatically guarantees that the eigenvectors are orthogonal (but of course you can verify this by computing the inner product). By scaling the columns of S to unit vectors, we obtain a unitary matrix

$$V := \begin{bmatrix} s & -2t \\ 2s & t \end{bmatrix} \in \mathbb{C}^{2\times 2}, \quad \text{where} \quad |s| = 1/\sqrt{5} = |t|.$$

Indeed, V is unitary, as

$$VV^* = \begin{bmatrix} s & -2t \\ 2s & t \end{bmatrix} \begin{bmatrix} s^* & 2s^* \\ -2t^* & t^* \end{bmatrix} = \begin{bmatrix} |s|^2 + 4|t|^2 & 2|s|^2 - 2|t|^2 \\ 2|s|^2 - 2|t|^2 & 4|s|^2 + |t|^2 \end{bmatrix} \overset{|s|^2=1/5=|t|^2}{=} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$$

We may check that we got a unitary diagonalization of R:

$$\begin{aligned} VDV^* &= \begin{bmatrix} s & -2t \\ 2s & t \end{bmatrix} \begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} s & -2t \\ 2s & t \end{bmatrix}^* \\ &= \begin{bmatrix} 9s & -8t \\ 18s & 4t \end{bmatrix} \begin{bmatrix} s^* & 2s^* \\ -2t^* & t^* \end{bmatrix} \\ &= \begin{bmatrix} 9|s|^2 + 16|t|^2 & 18|s|^2 - 8|t|^2 \\ 18|s|^2 - 8|t|^2 & 36|s|^2 + 4|t|^2 \end{bmatrix} \\ &\overset{|s|^2=1/5=|t|^2}{=} \begin{bmatrix} 5 & 2 \\ 2 & 8 \end{bmatrix} \quad = \quad R. \end{aligned}$$

Notice that there are infinitely many different choices for s, t. Notice also that if we would have changed the order of the eigenvalues, we should have changed the order of the eigenvectors, accordingly:

$$R = \begin{bmatrix} 5 & 2 \\ 2 & 8 \end{bmatrix} = \begin{bmatrix} -2t & s \\ t & 2s \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix} \begin{bmatrix} -2t & s \\ t & 2s \end{bmatrix}^*.$$

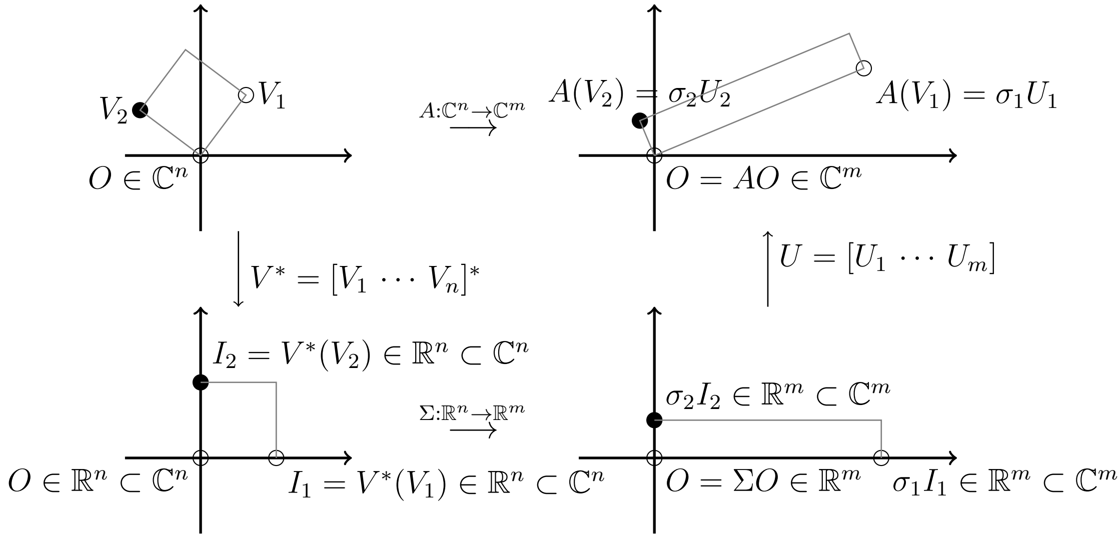


Figure 38: Idea of SVD, or Singular Value Decomposition $A = U\Sigma V^*$: here matrices V^* and U are "rotations" of the vector spaces, and matrix Σ is a "scaling/projection/embedding".

11. SVD (Singular Value Decomposition)

Definition. A *singular value decomposition* (SVD) of a matrix $A \in \mathbb{C}^{m\times n}$ is a matrix triple (U, Σ, V) such that

$$A = U\Sigma V^*,$$

where $U = [U_1 \, \cdots \, U_m] \in \mathbb{C}^{m\times m}$ and $V = [V_1 \, \cdots \, V_n] \in \mathbb{C}^{n\times n}$ are unitary, and $\Sigma \in \mathbb{R}^{m\times n}$ is the diagonal matrix of *singular values* $\Sigma_{jj} = \sigma_j \geq 0$ of A: that is $\Sigma_{jk} = 0$ when $j \neq k$. We also demand that $\Sigma_{jj} = \sigma_j \geq \sigma_{j+1}$ for all j.

Remark: If $A = U\Sigma V^*$ as above, then $A(V_j) = \sigma_j \, U_j$, and

$$\begin{aligned} A^*A &= V\,(\Sigma^*\Sigma)\,V^*, \\ AA^* &= U\,(\Sigma\Sigma^*)\,U^*, \end{aligned}$$

where $\Sigma^*\,\Sigma \in \mathbb{R}^{n\times n}$ and $\Sigma\,\Sigma^* \in \mathbb{R}^{m\times m}$ are positive diagonal matrices, where

$${\sigma_j}^2 = (\Sigma^*\,\Sigma)_{jj} = (\Sigma\,\Sigma^*)_{jj}\,.$$

This suggests that an SVD could be found by the unitary diagonalization!

How to find the SVD?

For $A \in \mathbb{C}^{m\times n}$, matrix $A^*A \in \mathbb{C}^{n\times n}$ is normal (A^*A is even positive), so the unitary diagonalization gives us

$$A^*A = V\Lambda V^*,$$

where the diagonal matrix $\Lambda \in \mathbb{C}^{n\times n}$ has the eigenvalues $\lambda_k := \Lambda_{kk}$ of A^*A, with the unitary matrix $V = [V_1 \cdots V_n] \in \mathbb{C}^{n\times n}$ having the corresponding eigenvectors $V_k \in \mathbb{C}^{n\times 1}$. Now

$$\begin{aligned}
\langle A(V_j), A(V_k)\rangle &= \langle A^*A(V_j), V_k\rangle \\
&= \langle \lambda_j V_j, V_k\rangle \\
&= \lambda_j \langle V_j, V_k\rangle \\
&\overset{V^*V=I}{=} \begin{cases} \lambda_j & \text{if } j = k, \\ 0 & \text{if } j \neq k. \end{cases}
\end{aligned}$$

So let $\sigma_j := \|A(V_j)\| = \sqrt{\lambda_j}$ (conventionally here we have already arranged the order so that $\lambda_j \geq \lambda_{j+1}$ when $1 \leq j < n$). Then find a unitary matrix $U = [U_1 \cdots U_m] \in \mathbb{C}^{m\times m}$ for which

$$A(V_j) = \sigma_j U_j.$$

More precisely: If $\sigma_j > 0$ then $U_j = A(V_j)/\sigma_j$. If $m > j > n$ or if $\sigma_j = 0$ then we have more freedom of choosing U_j. Finally, define $\Sigma \in \mathbb{R}^{m\times n}$ such that $\Sigma_{jk} = 0$ whenever $j \neq k$, and $\Sigma_{jj} := \sigma_j$ whenever $j \leq \min\{m, n\}$, i.e.

$$\Sigma \overset{m\leq n}{=} \begin{bmatrix} \sigma_1 & 0 & 0 & \cdots & 0 \\ 0 & \ddots & 0 & \cdots & 0 \\ 0 & 0 & \sigma_m & \cdots & 0 \end{bmatrix}, \quad \Sigma \overset{m=n}{=} \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \sigma_n \end{bmatrix}, \quad \Sigma \overset{m\geq n}{=} \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \sigma_n \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus (U, Σ, V) is an SVD for $A \in \mathbb{C}^{m\times n}$, because clearly

$$\begin{aligned}
AV &= U\Sigma, \\
A &= U\Sigma V^*.
\end{aligned}$$

Moreover, since here

$$A^* = V\Sigma^* U^*,$$

we notice that (V, Σ^*, U) is an SVD for $A^* \in \mathbb{C}^{n\times m}$.

A boring example. Unitary $A \in \mathbb{C}^{n\times n}$ has an SVD (A, I, I), as $A = AII^*$. Another SVD for unitary A here is (I, I, A^*), since $A = II(A^*)^*$.

Example. Let $A = \begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{C}^{2\times 1}$. Then

$$A^*A = \begin{bmatrix} \overline{a} & \overline{b} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} |a|^2 + |b|^2 \end{bmatrix} \in \mathbb{C}^{1\times 1},$$

and clearly $A^*A = V\Lambda V^*$ for

$$V = \begin{bmatrix} V_1 \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} \lambda_1 \end{bmatrix} = \begin{bmatrix} |a|^2 + |b|^2 \end{bmatrix}.$$

Now

$$\Sigma = \begin{bmatrix} \sigma_1 \\ 0 \end{bmatrix} \in \mathbb{R}^{2\times 1},$$

with the singular value $\sigma_1 = \sqrt{\lambda_1} = \sqrt{|a|^2 + |b|^2}$. As $\sigma_1 U_1 = A(V_1) = \begin{bmatrix} a \\ b \end{bmatrix}$, we have $U_1 = \begin{bmatrix} a/\sigma_1 \\ b/\sigma_1 \end{bmatrix}$. We can choose e.g. $U_2 = \begin{bmatrix} -\overline{b}/\sigma_1 \\ \overline{a}/\sigma_1 \end{bmatrix}$. Thus $A = U\Sigma V^*$ reads now

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a/\sigma_1 & -\overline{b}/\sigma_1 \\ b/\sigma_1 & \overline{a}/\sigma_1 \end{bmatrix} \begin{bmatrix} \sigma_1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix}.$$

Another example. In the previous example. $A^* = V\Sigma^*U^*$ means

$$\begin{bmatrix} \overline{a} & \overline{b} \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 \end{bmatrix} \begin{bmatrix} \overline{a}/\sigma_1 & \overline{b}/\sigma_1 \\ -b/\sigma_1 & a/\sigma_1 \end{bmatrix}.$$

Naturally, this (V, Σ^*, U) is an SVD for A^*.

Example. Let $A \in \mathbb{C}^{n\times n}$ be positive, i.e. $\langle Au, u\rangle \geq 0$ for all $u \in \mathbb{C}^n$. By unitary diagonalization, $A = U\Lambda U^*$ for some unitary $U \in \mathbb{C}^{n\times n}$ and a positive diagonal matrix $\Lambda \in \mathbb{R}^{n\times n}$. In this case, (U, Λ, U) is an SVD of A, if the eigenvalues $\lambda_j = \Lambda_{jj} \geq 0$ are in decreasing order $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_n$.

Example. Let $A = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$, where $a, d \in \mathbb{C}$ such that $|a| \geq |d| > 0$. Then

$$A = U\Sigma V^* = \begin{bmatrix} a/|a| & 0 \\ 0 & d/|d| \end{bmatrix} \begin{bmatrix} |a| & 0 \\ 0 & |d| \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^*,$$

giving an SVD (U, Σ, V) of A. An SVD is not unique: if $1 = |\lambda| = |\mu|$, here

$$A = \begin{bmatrix} \lambda a/|a| & 0 \\ 0 & \mu d/|d| \end{bmatrix} \begin{bmatrix} |a| & 0 \\ 0 & |d| \end{bmatrix} \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}^*$$

Example. Let $A = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$, where $a, d \in \mathbb{C}$ such that $|d| \geq |a| > 0$. Then

$$A = U\Sigma V^* = \begin{bmatrix} 0 & a/|a| \\ d/|d| & 0 \end{bmatrix} \begin{bmatrix} |d| & 0 \\ 0 & |a| \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^*$$

gives an SVD (U, Σ, V) of $A \in \mathbb{C}^{2\times 2}$.

SVD in nutshell: An SVD (U, Σ, V) can be found for any $A \in \mathbb{C}^{m\times n}$. Here

$$A = U\Sigma V^*.$$

$\Sigma \in \mathbb{R}^{m\times n}$ is unique (when we demand $\Sigma_{jj} = \sigma_j \geq \sigma_{j+1}$),
but there is some freedom in choosing the unitary matrices U, V.
Here $\lambda_j = {\sigma_j}^2$ are the common eigenvalues of A^*A and AA^*.
$V_j \in \mathbb{C}^{n\times 1}$ are eigenvectors of symmetric $A^*A \in \mathbb{C}^{n\times n}$,
$U_j \in \mathbb{C}^{m\times 1}$ are eigenvectors of symmetric $AA^* \in \mathbb{C}^{m\times m}$, and

$$A^* = V\Sigma^* U^*$$

(of course, (V, Σ^*, U) is an SVD for $A^* \in \mathbb{C}^{n\times m}$).
In case of $\sigma_j = 0$, it does not matter
how vectors $U_j \in \mathbb{C}^{m\times 1}$ and $V_j \in \mathbb{C}^{n\times 1}$ are chosen.

Remark. Above, $A = U\Sigma V^* \in \mathbb{C}^{m\times n}$ and $A^* = V\Sigma^* U^* \in \mathbb{C}^{n\times m}$.
Is there some essential difference in finding these SVDs?
Well, we first diagonalize either $A^*A \in \mathbb{C}^{n\times n}$ or $AA^* \in \mathbb{C}^{m\times m}$
... which of the dimensions m, n is smaller... ?

Remark. SVD helps us understanding the image

$$A(\mathbb{C}^n) := \{A(x) \in \mathbb{C}^m : \ x \in \mathbb{C}^n\}$$

and the zero set (or kernel)

$$\mathrm{Ker}(A) := \{x \in \mathbb{C}^n : A(x) = O \in \mathbb{C}^m\}.$$

Here

$$A(\mathbb{C}^n) = \mathrm{span}\,\{U_j : \ \sigma_j \neq 0\}\,,$$
$$\mathrm{Ker}(A) = \mathrm{span}\,\{V_j : \ \sigma_j = 0\}\,.$$

Example. Let us find a singular value decomposition (U, Σ, V) for matrix

$$A = \begin{bmatrix} 2 & 0 \\ 1 & 2 \\ 0 & 2 \end{bmatrix} = U\Sigma V^*.$$

Now $A^*A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 1 & 2 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 2 & 8 \end{bmatrix}$, meaning that $A^*A = R$ from (56) on page 63. There we saw that

$$R = V\Lambda V^*, \quad \Lambda = \begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix}, \quad V = \begin{bmatrix} s & -2t \\ 2s & t \end{bmatrix}, \quad |s| = 1/\sqrt{5} = |t|.$$

So, the singular values of A are $\sigma_1 = \sqrt{\lambda_1} = \sqrt{9} = 3$, $\sigma_2 = \sqrt{\lambda_2} = \sqrt{4} = 2$. Hence $\Sigma = \begin{bmatrix} 3 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}$. Due to $A = U\Sigma V^*$, we have $AV = U\Sigma$, and here

$$\begin{aligned} AV &= \begin{bmatrix} 2 & 0 \\ 1 & 2 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} s & -2t \\ 2s & t \end{bmatrix} = \begin{bmatrix} 2s & -4t \\ 5s & 0 \\ 4s & 2t \end{bmatrix}, \\ U\Sigma &= [U_1\ U_2\ U_3]\ \Sigma = [\sigma_1 U_1\ \sigma_2 U_2] = [3U_1\ 2U_2]. \end{aligned}$$

Therefore $U_1 = \begin{bmatrix} 2s/3 \\ 5s/3 \\ 4s/3 \end{bmatrix}$ and $U_2 = \begin{bmatrix} -2t \\ 0 \\ t \end{bmatrix}$. For $U_3 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ we can choose any unit vector ($1 = \|U_3\|^2 = |a|^2 + |b|^2 + |c|^2$) which is orthogonal to vectors U_1, U_2: for instance, $(a, b, c) = (1, -2, 2)/3$ is fine. Let us check the answer:

$$\begin{aligned} U\Sigma V^* &= \begin{bmatrix} 2s/3 & -2t & a \\ 5s/3 & 0 & b \\ 4s/3 & t & c \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} s & -2t \\ 2s & t \end{bmatrix}^* \\ &= \begin{bmatrix} 2s & -4t \\ 5s & 0 \\ 4s & 2t \end{bmatrix} \begin{bmatrix} s^* & 2s^* \\ -2t^* & t^* \end{bmatrix} \\ &= \begin{bmatrix} 2|s|^2 + 8|t|^2 & 4|s|^2 - 4|t|^2 \\ 5|s|^2 & 10|s|^2 \\ 4|s|^2 - 4|t|^2 & 8|s|^2 + 2|t|^2 \end{bmatrix} \\ &\overset{|s|^2 = 1/5 = |t|^2}{=} \begin{bmatrix} 2 & 0 \\ 1 & 2 \\ 0 & 2 \end{bmatrix} = A. \end{aligned}$$

Matrix norm and singular values. The norm of a matrix $A \in \mathbb{C}^{m \times n}$ is

$$\|A\| = \max_{u \in \mathbb{C}^n : \|u\| \leq 1} \|Au\|.$$

Here $\|A\| = \sigma_1$, the largest singular value of A: this follows from

$$\|Au\|^2 = \langle Au, Au \rangle = \langle A^* Au, u \rangle = \langle u, AA^* u \rangle,$$

because by the unitary diagonalization it is clear that

$$\|A^* A\| = {\sigma_1}^2 = \|AA^*\|.$$

Example. If

$$[A] = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \end{bmatrix}$$

then

$$[AA^*] = \begin{bmatrix} 29 & 56 \\ 56 & 110 \end{bmatrix}$$

has the characteristic polynomial

$$\begin{aligned} 0 &= \det[AA^* - zI] \\ &= z^2 - 139z + 54 \\ &= (z - {\sigma_1}^2)(z - {\sigma_2}^2), \end{aligned}$$

where $\sigma_1, \sigma_2 > 0$ are the singular values of A, and $\sigma_1 > \sigma_2 > 0$. Especially,

$$\|A\| = \sigma_1 = \left(\frac{139 + \sqrt{139^2 - 4 \cdot 54}}{2} \right)^{1/2} \approx 11.7732961.$$

Who would have guessed that just by looking at $[A]$? ;)

Rhetoric question: What is the smallest constant C for which

$$\|Au\| \leq C \|u\|$$

for all $u \in \mathbb{C}^n$, when $A : \mathbb{C}^n \to \mathbb{C}^m$ is linear?
(Answer: $C = \|A\| = \sigma_1$.)

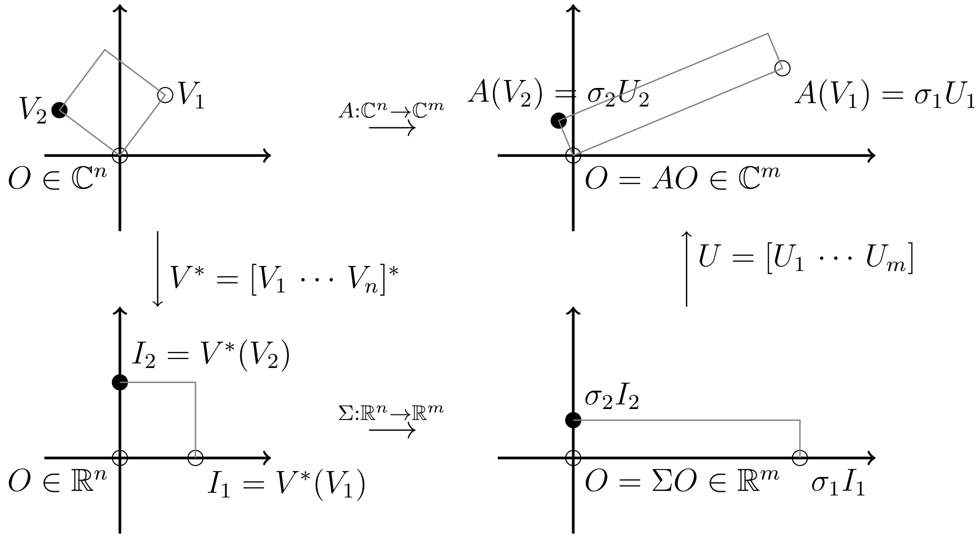


Figure 39: Idea of SVD, or Singular Value Decomposition $A = U\Sigma V^*$.

12. Applications of SVD

Recall: An SVD (U, Σ, V) can be found for any matrix $A \in \mathbb{C}^{m\times n}$. Here

$$A = U\Sigma V^*,$$

where the positive diagonal matrix $\Sigma \in \mathbb{R}^{m\times n}$ is unique,
but there is some freedom in choosing unitary $U \in \mathbb{C}^{m\times m}$ and $V \in \mathbb{C}^{n\times n}$.
$\lambda_j = {\sigma_j}^2 = {\Sigma_{jj}}^2 \geq \lambda_{j+1}$ are the common eigenvalues of A^*A and AA^*.
$V_j \in \mathbb{C}^{n\times 1}$ are eigenvectors of symmetric $A^*A \in \mathbb{C}^{n\times n}$,
$U_j \in \mathbb{C}^{m\times 1}$ are eigenvectors of symmetric $AA^* \in \mathbb{C}^{m\times m}$,
$A(V_j) = \sigma_j\, U_j$, and

$$A^* = V\Sigma^* U^*$$

(of course, (V, Σ^*, U) is an SVD for A^*).

Remark. Often in real-life applications, the singular values

$$\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq \cdots \geq 0$$

of the data matrix $A \in \mathbb{R}^{m\times n}$ decay rapidly to zero, and this gives tools for compressing and denoising the data, when we only take into account the contribution of the largest singular values! Moreover, Gauss' powerful method of the *least squares solution* can be viewed as an application of an SVD to find the *pseudo-inverse* $A^+ \in \mathbb{C}^{n\times m}$ of a matrix $A \in \mathbb{C}^{m\times n}$.

Remark! Define $\tilde{U}, \tilde{\Sigma}, \tilde{V}$ by putting O to the columns $k+1, k+2, k+3, \cdots$ of the corresponding SVD-matrices U, Σ, V. Then

$$\tilde{A} = \tilde{U}\tilde{\Sigma}\tilde{V}^*$$

is the best kth rank approximation to $A \in \mathbb{C}^{m\times n}$.

Example. If $U = [U_1\ U_2\ U_3] \in \mathbb{C}^{3\times 3}$ and $V = [V_1\ V_2\ V_3\ V_4] \in \mathbb{C}^{4\times 4}$ are unitary, then

$$A = U \begin{bmatrix} 7 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 5 & 0 \end{bmatrix} V^* \in \mathbb{C}^{3\times 4}$$

has the best 1st rank approximation

$$U \begin{bmatrix} 7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} V^* = [U_1\ O\ O] \begin{bmatrix} 7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} [V_1\ O\ O\ O]^*,$$

and the best 2nd rank approximation of A would be

$$U \begin{bmatrix} 7 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} V^* = [U_1\ U_2\ O] \begin{bmatrix} 7 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} [V_1\ V_2\ O\ O]^*.$$

Here matrices $O \in \mathbb{C}^{\ell\times 1}$ are the zero columns in appropriate dimensions ℓ.

Application. A grey-scale $m \times n$-pixel image is a matrix $A \in \mathbb{R}^{m\times n}$, where $A_{jk} \in [0, 1]$ is intensity at (j, k):
for instance, 0 is black, 0.5 is middle grey, 1 is white.
Storing image A takes mn numbers A_{jk},
but the kth rank approximation $\tilde{A}$ takes only

$$m\,k + k + n\,k = (m + n + 1)\,k$$

numbers, where often $k \ll m$, $k \ll n$.
In this fashion, SVD can also be used to remove noise from photographs (noise typically contributes to smaller singular values).

Pseudo-inverse

For $A \in \mathbb{C}^{m\times n}$, equation $A(x) = b$ may have a unique solution, or infinitely many solutions, or it may have no solutions at all! However, $A(x) = b$ has always the "best SVD-solution"

$$\tilde{x} := A^+(b),$$

where

$$A^+ = V\Sigma^+U^* \in \mathbb{C}^{n\times m}$$

is the *pseudo-inverse* of $A = U\Sigma V^*$; here (U, Σ, V) is an SVD of A, and $\Sigma^+ \in \mathbb{R}^{n\times m}$ is the diagonal matrix, where the non-zero diagonal elements are $1/\sigma_j$ for the positive singular values $\sigma_j > 0$. If $Ax = b$ has no solutions, this *least squares "solution"* $\tilde{x} \in \mathbb{C}^n$ is the best "solution" in sense that

$$\|A(\tilde{x}) - b\| \leq \|A(x) - b\|$$

for all $x \in \mathbb{C}^n$. And if $Ax = b$ has infinitely many solutions, then $\tilde{x} \in \mathbb{C}^n$ is the best solution in sense that it has the minimal norm among all the solutions.

Example. From

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix} = A$$

we see that $U\Sigma V^* = A$ has the pseudo-inverse

$$A^+ = V\Sigma^+U^* = \begin{bmatrix} 0 & 0 \\ 1/3 & 0 \end{bmatrix},$$

where

$$V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ \Sigma^+ = \begin{bmatrix} 1/3 & 0 \\ 0 & 0 \end{bmatrix}, \ U^* = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then

$$A^+A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad A\,A^+ = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Notice that A is not surjective:

$$Ax = \begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3x_2 \\ 0 \end{bmatrix}.$$

For $b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \in \mathbb{C}^{2\times 1}$, the "least squares solution" to $Ax = b$ is

$$\widetilde{x} = A^+(b) = \begin{bmatrix} 0 & 0 \\ 1/3 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 0 \\ b_1/3 \end{bmatrix}.$$

Polar decomposition

The *polar decomposition* of $z \in \mathbb{C}$ is $z = \mathrm{e}^{\mathrm{i}\theta}|z|$, where $\theta = \arg(z) \in \mathbb{R}$ is the argument and $|z| \geq 0$ is the absolute value of z.

Definition. A *polar decomposition* $(E, |A|)$ of matrix $A \in \mathbb{C}^{n\times n}$ satisfies

$$A = E|A|,$$

where matrices $E, |A| \in \mathbb{C}^{n\times n}$ are obtained from an SVD:

$$\begin{aligned} A &= U\Sigma V^* = (UV^*)(V\Sigma V^*), \\ E &:= UV^*, \quad |A| := V\Sigma V^*. \end{aligned} \tag{57}$$

So $E \in \mathbb{C}^{n\times n}$ is unitary, and $|A| = (A^*A)^{1/2} \in \mathbb{C}^{n\times n}$ is positive:

$$\langle |A|u, u\rangle \geq 0$$

for all $u \in \mathbb{C}^n$. Notice that $(A^*A)^{1/2} \neq (AA^*)^{1/2}$ when A is not normal!

Example. From

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix}$$

we see that $U\Sigma V^* = A := \begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix}$ has a polar decomposition $(E, |A|)$, where

$$A = E|A| = (UV^*)(V\Sigma V^*),$$

where

$$\begin{aligned} E = UV^* &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \\ |A| = V\Sigma V^* &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}. \end{aligned}$$

Notice that

$$|A| = (A^*A)^{1/2} = \left(\begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix} \right)^{1/2} = \begin{bmatrix} 0 & 0 \\ 0 & 9 \end{bmatrix}^{1/2}.$$

Some other applications: SVD is used in the Google search algorithm (see e.g. Gilbert Strang's book on linear algebra).
SVD finds regular features in statistics (in tables of numbers;
see PCA, Principal Component Analysis).

Nice to know: what is a vector space?

Definition. A *vector space* over the scalar field $\mathbb{K}$ is a set V having an element $O \in V$ called the *origin*, and having the operations

$$\begin{aligned} ((u,v) \mapsto u+v) &\ : \ V \times V \to V, \\ ((\lambda,u) \mapsto u) &\ : \ \mathbb{K} \times V \to V \end{aligned}$$

such that for all $u, v, w \in V$ and $\lambda, \mu \in \mathbb{K}$ we have

$$\begin{aligned} &(u+v)+w = u+(v+w), \\ &u+v = v+u, \\ &u+O = u, \\ &u+(-1)u = O, \\ &1u = u, \\ &\lambda(\mu u) = (\lambda\mu)u, \\ &\lambda(u+v) = \lambda u + \lambda v, \\ &(\lambda+\mu)u = \lambda u + \mu u. \end{aligned}$$

Write $u+v+w := (u+v)+w = u+(v+w)$ and $-u := (-1)u$.

Example. During this course, we studied especially the Euclidean vector spaces $V = \mathbb{K}^n$ (of finite dimension n). With a moment of thought, we understand that the space $V = \mathbb{K}^{m\times n}$ of matrices can be considered as a vector space (of finite dimension mn), with operations

$$([A],[B]) \mapsto [A]+[B] = [A+B], \quad (\lambda,[A]) \mapsto \lambda[A] = [\lambda A].$$

Example. As a useful example of an infinite-dimensional vector space, think about the space $V = C([a,b])$ of continuous functions $u : [a,b] \to \mathbb{K}$. Here $[a,b] \subset \mathbb{R}$ is a finite closed interval with $a < b$, and the vector operations are given by

$$\begin{aligned} (u+v)(x) &:= u(x)+v(x), \\ (\lambda u)(x) &:= \lambda\, u(x) \end{aligned}$$

whenever $a \leq x \leq b$. For example, vector $u \in V$ could be a sound signal, and then at time $x \in [a,b]$ the value $u(x) \in \mathbb{R}$ is the difference of the air pressure to the normal constant ambient pressure. In this case, the energy of the signal $u \in V$ during the time interval $[a,b]$ is proportional to

$$\|u\|^2 := \int_a^b |u(x)|^2 \,\mathrm{d}x.$$

More about this topic e.g. in the course on Fourier analysis.

Nice to know: what is a linear operator?

Definition. Let V, W be vector spaces over the same scalar field $\mathbb{K}$. Function $A : V \to W$ is called *linear* (or a *linear operator*, or a *linear mapping*) if

$$\begin{aligned} A(u+v) &= A(u) + A(v), \\ A(\lambda u) &= \lambda\, A(u) \end{aligned}$$

for all $u, v \in V$ and $\lambda \in \mathbb{K}$. Then we simply write $Au := A(u)$.

Example. During this course, we saw linear operators $A : V \to W$ between the Euclidean vector spaces $V = \mathbb{K}^n$ and $W = \mathbb{K}^m$, and saw that these corresponded to matrices $[A] \in \mathbb{K}^{m \times n}$ by

$$(Au)_j = \sum_{k=1}^{n} A_{jk}\, u_k. \tag{58}$$

Example. Let $V = C([a,b])$ be the infinite-dimensional vector space of continuous functions $u : [a,b] \to \mathbb{K}$ as above. Let $K : [a,b] \times [a,b] \to \mathbb{K}$ be a continuous function of two variables. Then

$$Au(x) = \int_a^b K(x,y)\, u(y)\, \mathrm{d}y \tag{59}$$

defines a linear operator $A : V \to V$. Application here: if $u \in V$ describes a noisy input signal, then the output signal $Au \in V$ might be less noisy, if we choose the integral kernel function K nicely.

Remark. Notice the formal resemblance of the equations (58) and (59). Actually, we may approximate (59) by finite-dimensional matrices in the following fashion: if $u \in V = C([a,b])$, let $n \in \mathbb{Z}^+$ and

$$x_j := a + \frac{j}{n}(b-a),$$

i.e. we take equispaced samples $x_1, \cdots, x_n \in [a,b]$ of variable $x \in [a,b]$. Then

$$\begin{aligned} Au(x_j) &= \int_a^b K(x_j, y)\, u(y)\, \mathrm{d}y \\ &\approx \frac{b-a}{n} \sum_{k=1}^{n} K(x_j, x_k)\, u(x_k). \end{aligned}$$

This is a finite-dimensional matrix approximation to the integral operator A.

Feedback, please...

Thanks to the assistants of the course for commenting these lecture notes!
All further feedback is welcome: please contact

`ville.turunen@aalto.fi`